LINEAR PLATE BENDING

1 Linear plate bending

A plate is a body of which the material is located in a small region around a surface in the three-dimensional space. A special surface is the *mid-plane*. Measured from a point in the mid-plane, the distance to both surfaces of the plate is equal. The geometry of the plate and position of its points is described in an orthonormal coordinate system, either Cartesian (coordinates $\{x, y, z\}$) or cylindrical (coordinates $\{r, \theta, z\}$).

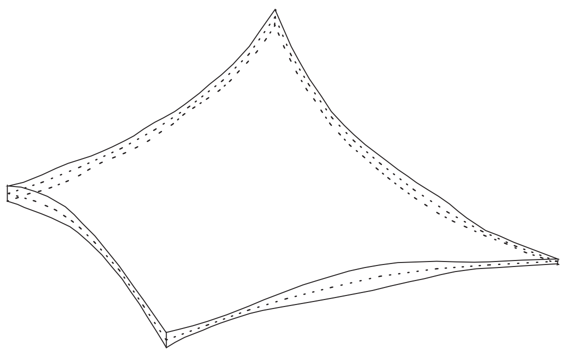


Fig. 1.1 : *Plate with curved mid-plane and variable thickness*

1.1 Geometry

The following assumptions about the geometry are supposed to hold here :

- the mid-plane is planar in the undeformed state,
- the mid-plane coincides with the global coordinate plane $z = 0$,
- the thickness h is uniform.

Along the edge of the plate a coordinate s is used and perpendicular to this a coordinate n.

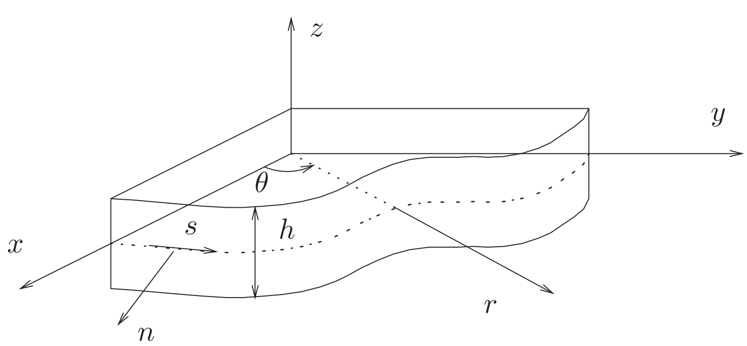


Fig. 1.2 : *Planar plate with uniform thickness*

1.2 External loads

The plate is loaded with forces and moments, which can be concentrated in one point or distributed over a surface or along a line. The following loads are defined and shown in the figure :

- force per unit of area in the mid-plane : $s_x(x, y), s_y(x, y)$ or $s_r(r, \theta),\ s_t(r, \theta)$
- force per unit of area perpendicular to the mid-plane : $p(x, y)$ or $p(r, \theta)$
- force per unit of length in the mid-plane : $f_n(s), f_s(s)$
- force per unit of length perpendicular to the mid-plane : $f_z(s)$
- bending moment per unit of length along the edge : $m_b(s)$
- torsional moment per unit of length along the edge : $m_w(s)$

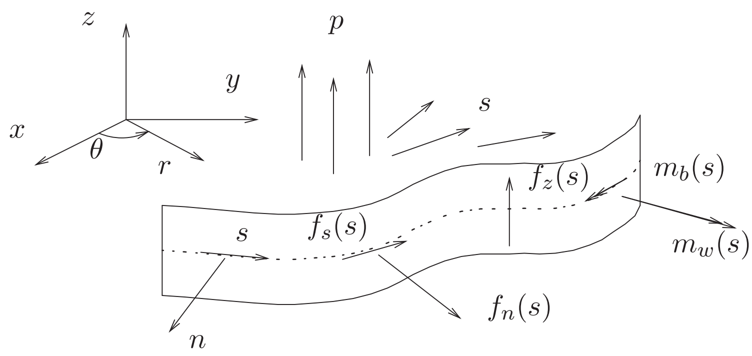


Fig. 1.3 : *External loads on a plate*

1.3 Cartesian coordinate system

In the Cartesian coordinate system, three orthogonal coordinate axes with coordinates $\{x, y, z\}$ are used to identify material and spatial points. As stated before, we assume that the mid-plane of the plate coincides with the plane $z = 0$ in the undeformed state. This is not a restrictive assumption, but allows for simplification of the mathematics.

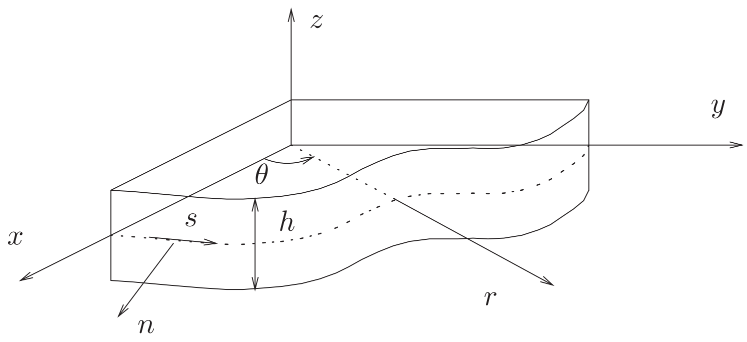


Fig. 1.4 : *Plate in a Cartesian coordinate system*

1.3.1 Displacements

The figure shows two points P and Q in the undeformed and in the deformed state. In the undeformed state the point Q is in the mid-plane and has coordinates $(x, y, 0)$. The out-of-plane point P has coordinates (x, y, z). As a result of deformation, the displacement of the mid-plane point Q in the (x, y, z)-coordinate directions are u, v and w, respectively. The displacement components of point P, indicated as u_x, u_y and u_z, can be related to those of point Q.

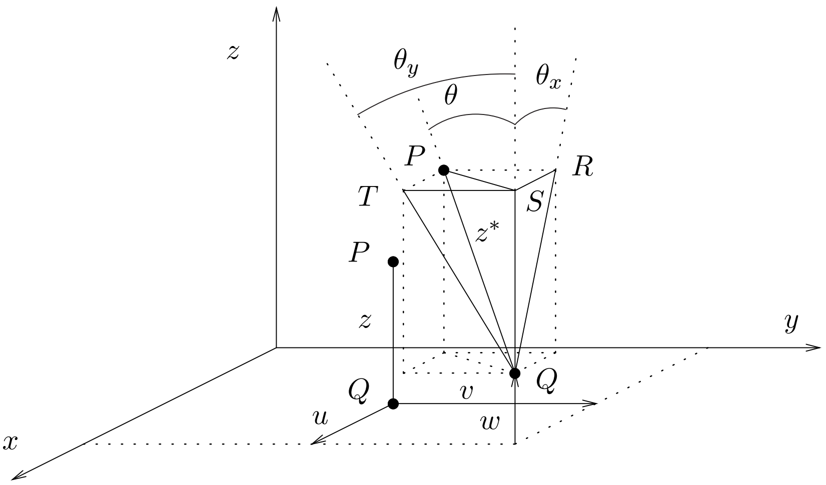


Fig. 1.5 : *Displacement of a out-of-midplane point due to bending*

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
u_x = u - SR = u - QR\sin(\theta_x) \\
u_y = v - ST = v - QT\sin(\theta_y) \\
u_z = w + SQ - z
\end{array}\right\} \\[2ex]
\left.\begin{array}{l}
QR = z^* \cos(\theta_y) \\
QT = z^* \cos(\theta_x) \\
SQ = z^* \cos(\theta)
\end{array}\right\}
\end{array}\right\}
\rightarrow
\left.\begin{array}{l}
u_x = u - z^* \sin(\theta_x)\cos(\theta_y) \\
u_y = v - z^* \cos(\theta_x)\sin(\theta_y) \\
u_z = w + z^* \cos(\theta) - z
\end{array}\right\}
$$

No out-of-plane shear

When it is assumed that there is no out-of-plane shear deformation, the so-called Kirchhoff hypotheses hold :

- straight line elements, initially perpendicular to the mid-plane remain straight,
- straight line elements, initially perpendicular to the mid-plane remain perpendicular to the mid-plane.

The angles θ_x and θ_y can be replaced by the rotation angles ϕ_x and ϕ_y respectively. This is illustrated for one coordinate direction in the figure.

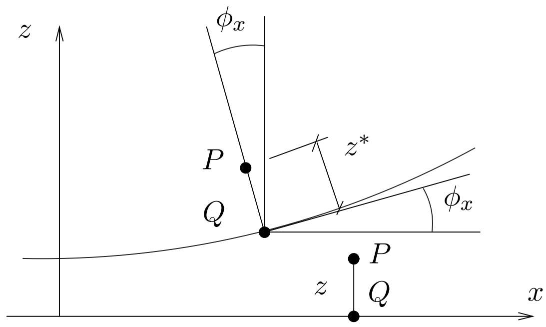


Fig. 1.6 : *Displacement of out-of-midplane point due to bending with no shear*

$$\theta_x = \phi_x \quad ; \quad \theta_y = \phi_y \quad ; \quad \theta = \phi$$

Small rotation

With the assumption that rotations are small, the cosine functions approximately have value 1 and the sine functions can be replaced by the rotation angles. These rotations can be expressed in the derivatives [1] of the z-displacement w w.r.t. the coordinates x and y.

$$\left.\begin{array}{l} \cos(\phi) = \cos(\phi_y) = \cos(\phi_x) = 1 \\ \sin(\phi_y) = \phi_y = w_{,y} \quad ; \quad \sin(\phi_x) = \phi_x = w_{,x} \end{array}\right\} \rightarrow$$

$$\left.\begin{array}{l} u_x = u - z^* w_{,x} \\ u_y = v - z^* w_{,y} \\ u_z = w + z^* - z \end{array}\right\}$$

Constant thickness

The thickness of the plate may change due to loading perpendicular to the mid-plane or due to contraction due to in-plane deformation. The difference between z and z^* is obviously a function of z. Terms of order higher than z^2 are neglected in this expression. It is now assumed that the thickness remains constant, which means that $\zeta(x,y) = 0$ has to hold.

With the assumption $z^* \approx z$, which is correct for small deformations and thin plates, the displacement components of the out-of-plane point P can be expressed in mid-plane displacements.

$$\begin{array}{l} \Delta z(x,y) = z^*(x,y) - z(x,y) = \zeta(x,y)z + \eta(x,y)z^2 + O(z^3) \\ \Delta h = \zeta(x,y)(\frac{h}{2}) - \zeta(x,y)(-\frac{h}{2}) \end{array}$$

$$\Delta h = 0 \quad \rightarrow \quad \zeta(x,y) = 0$$

[1] Derivatives are denoted as e.g. $\frac{\partial}{\partial x}(\;) = (\;)_{,x}$ and $\frac{\partial^2}{\partial x \partial y}(\;) = (\;)_{,xy}$

$$
\begin{aligned}
u_x(x,y,z) &= u(x,y) - z w_{,x} \\
u_y(x,y,z) &= v(x,y) - z w_{,y} \\
u_z(x,y,z) &= w(x,y) + \eta(x,y) z^2
\end{aligned}
$$

1.3.2 Curvatures and strains

The linear strain components in a point out of the mid-plane, can be expressed in the mid-plane strains and curvatures. A sign convention, as is shown in the figure, must be adopted and used consistently. It was assumed that straight line elements, initially perpendicular to the mid-plane, remain perpendicular to the mid-plane, so γ_{xz} and γ_{yz} should be zero. Whether this is true will be evaluated later.

$$
\left.
\begin{aligned}
\varepsilon_{xx} &= u_{x,x} &&= u_{,x} - z w_{,xx} &&= \varepsilon_{xx0} - z\kappa_{xx} \\
\varepsilon_{yy} &= u_{y,y} &&= v_{,y} - z w_{,yy} &&= \varepsilon_{yy0} - z\kappa_{yy} \\
\gamma_{xy} &= u_{x,y} + u_{y,x} &&= u_{,y} + v_{,x} - 2z w_{,xy} &&= \gamma_{xy0} - z\kappa_{xy}
\end{aligned}
\right\} \quad \rightarrow \quad \underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa}
$$

$$
\begin{aligned}
\varepsilon_{zz} &= u_{z,z} &&= 2\eta(x,y) z \\
\gamma_{xz} &= u_{x,z} + u_{z,x} &&= \eta_{,x} z^2 \\
\gamma_{yz} &= u_{y,z} + u_{z,y} &&= \eta_{,y} z^2
\end{aligned}
$$

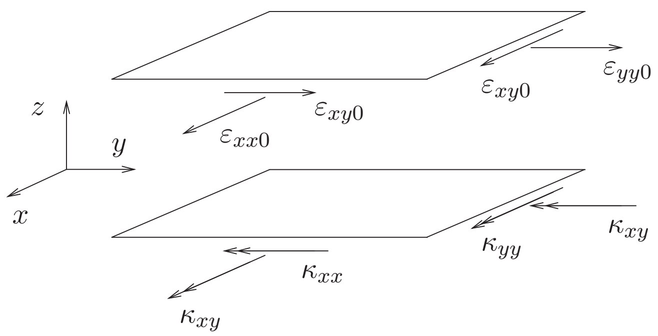


Fig. 1.7 : *Curvatures and strain definitions*

1.3.3 Stresses

It is assumed that a plane stress state exists in the plate : $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$. Because the plate may be loaded with a distributed load perpendicular to its surface, the first assumption is only approximately true. The second and third assumptions are in consistence with the Kirchhoff hypothesis, but has to be relaxed a little, due to the fact that the plate can be loaded with perpendicular edge loads. However, it will always be true that the stresses σ_{zz}, σ_{zx} and σ_{zy} are much smaller than the in-plane stresses σ_{xx}, σ_{yy}, and σ_{xy}.

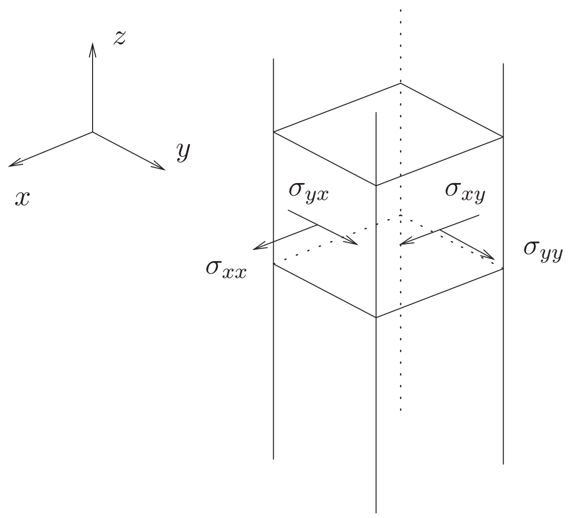


Fig. 1.8 : *In-plane stresses*

1.3.4 Isotropic elastic material behavior

For linear elastic material behavior Hooke's law relates strains to stresses. Material stiffness ($\underline{C}$) and compliance ($\underline{S}$) matrices can be derived for the plane stress state.

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \quad ; \quad \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

$$\text{short notation} \quad : \quad \underset{\sim}{\varepsilon} = \underline{S}\underset{\sim}{\sigma} \quad \rightarrow \quad \underset{\sim}{\sigma} = \underline{S}^{-1}\underset{\sim}{\varepsilon} = \underline{C}\underset{\sim}{\varepsilon} = \underline{C}(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})$$

Inconsistency with plane stress

The assumption of a plane stress state implies the shear stresses σ_{yz} and σ_{zx} to be zero. From Hooke's law it then immediately follows that the shear components γ_{yz} and γ_{zx} are also zero. However, the strain-displacement relations result in non-zero shear. For thin plates, the assumption will be more correct than for thick plates.

$$\varepsilon_{zz} = 2\eta z \quad \rightarrow \quad \eta(x,y) = \frac{\varepsilon_{zz}}{2z} = \frac{1}{2z}\frac{-\nu}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{\nu}{2(1-\nu)}(w_{,xx} + w_{,yy})$$

$$\gamma_{yz} = \frac{\nu z^2}{2(1-\nu)}(w_{,xxy} + w_{,yyy}) \neq 0$$

$$\gamma_{zx} = \frac{\nu z^2}{2(1-\nu)}\left(w_{,xxx} + w_{,xyy}\right) \neq 0$$

1.3.5 Cross-sectional forces and moments

Cross-sectional forces and moments can be calculated by integration of the in-plane stress components over the plate thickness. A sign convention, as indicated in the figure, must be adopted and used consistently.

The stress components σ_{zx} and σ_{zy} are much smaller than the relevant in-plane components. Integration over the plate thickness will however lead to shear forces, which cannot be neglected.

$$\underset{\sim}{N} = \begin{bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \underset{\sim}{\sigma}\, dz = \int_{-h/2}^{h/2} \{\underline{C}(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})\}\, dz = h\underline{C}\underset{\sim}{\varepsilon}_0$$

$$\underset{\sim}{M} = \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = -\int_{-h/2}^{h/2} \underset{\sim}{\sigma} z\, dz = -\int_{-h/2}^{h/2} \{\underline{C}(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})\} z\, dz = \frac{1}{12} h^3 \underline{C}\underset{\sim}{\kappa}$$

$$D_x = \int_{-h/2}^{h/2} \sigma_{zx}\, dz \qquad ; \qquad D_y = \int_{-h/2}^{h/2} \sigma_{zy}\, dz$$

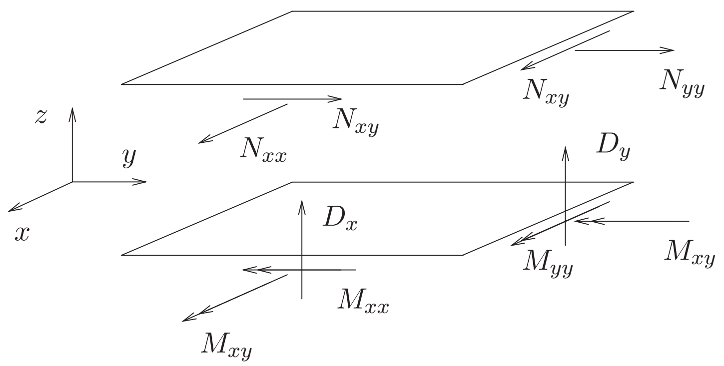


Fig. 1.9 : *Cross-sectional forces and moments*

Stiffness- and compliance matrix

Integration leads to the stiffness and compliance matrix of the plate.

$$\begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \underline{C}h & \underline{0} \\ \underline{0} & \underline{C}h^3/12 \end{bmatrix} \begin{bmatrix} \underset{\sim}{\varepsilon}_0 \\ \underset{\sim}{\kappa} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \underset{\sim}{\varepsilon}_0 \\ \underset{\sim}{\kappa} \end{bmatrix} = \begin{bmatrix} \underline{S}/h & \underline{0} \\ \underline{0} & 12\underline{S}/h^3 \end{bmatrix} \begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix}$$

1.3.6 Equilibrium equations

Consider a small "column" cut out of the plate perpendicular to its plane. Equilibrium requirements of the cross-sectional forces and moments, lead to the equilibrium equations. The cross-sectional shear forces D_x and D_y can be eliminated by combining some of the equilibrium equations.

$$N_{xx,x} + N_{xy,y} + s_x = 0$$
$$N_{yy,y} + N_{xy,x} + s_y = 0$$

$$\left.\begin{aligned} &D_{x,x} + D_{y,y} + p = 0 \\ &M_{xy,x} + M_{yy,y} - D_y = 0 \\ &M_{xy,y} + M_{xx,x} - D_x = 0 \end{aligned}\right\} \quad \rightarrow \quad M_{xx,xx} + M_{yy,yy} + 2M_{xy,xy} + p = 0$$

The equilibrium equation for the cross-sectional bending moments can be transformed into a differential equation for the displacement w. The equation is a fourth-order partial differential equation, which is referred to as a bi-potential equation.

$$\left.\begin{aligned} &\begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix} \\ &M_{xx,xx} + M_{yy,yy} + 2M_{xy,xy} + p = 0 \end{aligned}\right\} \quad \rightarrow$$

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} = (\frac{\partial^2}{x^2} + \frac{\partial^2}{\partial y^2})(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}) = \frac{p}{B}$$

$$B = \frac{Eh^3}{12(1-\nu^2)} = \text{plate modulus}$$

1.3.7 Orthotropic elastic material behavior

A material which properties are in each point symmetric w.r.t. three orthogonal planes, is called orthotropic. The three perpendicular directions in the planes are denoted as the 1-, 2- and 3-coordinate axes and are referred to as the material coordinate directions. Linear elastic behavior of an orthotropic material is characterized by 9 independent material parameters. They constitute the components of the compliance ($\underline{S}$) and stiffness ($\underline{C}$) matrix.

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} =
\begin{bmatrix}
E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\
-\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{23}^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{31}^{-1}
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}
$$

$$
\underset{\sim}{\varepsilon} = \underline{S}\underset{\sim}{\sigma} \qquad \rightarrow \qquad \underset{\sim}{\sigma} = \underline{S}^{-1}\underset{\sim}{\varepsilon} = \underline{C}\underset{\sim}{\varepsilon}
$$

$$
\underline{C} = \frac{1}{\Delta}
\begin{bmatrix}
\frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & \frac{\nu_{21}\nu_{32}+\nu_{31}}{E_2E_3} & 0 & 0 & 0 \\
\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & \frac{\nu_{12}\nu_{31}+\nu_{32}}{E_1E_3} & 0 & 0 & 0 \\
\frac{\nu_{12}\nu_{23}+\nu_{13}}{E_1E_2} & \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \Delta G_{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \Delta G_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta G_{31}
\end{bmatrix}
$$

$$
\text{with} \quad \Delta = \frac{(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13})}{E_1E_2E_3}
$$

Orthotropic plate

In an orthotropic plate the third material coordinate axis (3) is assumed to be perpendicular to the plane of the plate. The 1- and 2-directions are in its plane and rotated over an angle α (counterclockwise) w.r.t. the global x- and y-coordinate axes.

For the plane stress state the linear elastic orthotropic material behavior is characterized by 4 independent material parameters : E_1, E_2, G_{12} and ν_{12}. Due to symmetry we have $\nu_{21} = \frac{E_2}{E_1}\nu_{12}$.

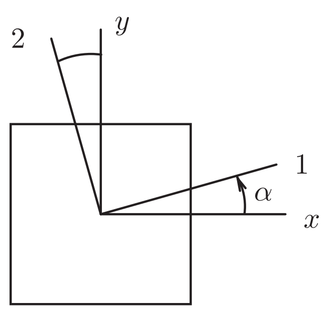


Fig. 1.10 : *Orthotropic plate with material coordinate system*

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} =
\begin{bmatrix}
E_1^{-1} & -\nu_{21}E_2^{-1} & 0 \\
-\nu_{12}E_1^{-1} & E_2^{-1} & 0 \\
0 & 0 & G_{12}^{-1}
\end{bmatrix}
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \qquad \rightarrow
$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{1-\nu_{21}\nu_{12}} \begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{21}\nu_{12})G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

Using a transformation matrices $\underline{T}_\varepsilon$ and $\underline{T}_\sigma$, for strain and stress components, respectively, the strain and stress components in the material coordinate system (index *) can be related to those in the global coordinate system.

$$c = \cos(\alpha) \qquad ; \qquad s = \sin(\alpha)$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1}\, \underset{\sim}{\varepsilon}^*$$

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{\sigma} = \underline{T}_\sigma^{-1}\, \underset{\sim}{\sigma}^*$$

$$\underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1}\underset{\sim}{\varepsilon}^* \qquad \text{idem} \qquad \underset{\sim}{\sigma} = \underline{T}_\sigma^{-1}\underset{\sim}{\sigma}^* \quad \rightarrow$$

$$\underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1}\underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon^{-1}\underline{S}^*\underset{\sim}{\sigma}^* = \underline{T}_\varepsilon^{-1}\underline{S}^*\underline{T}_\sigma\,\underset{\sim}{\sigma} = \underline{S}\,\underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \underline{T}_\sigma^{-1}\underset{\sim}{\sigma}^* = \underline{T}_\sigma^{-1}\underline{C}^*\underset{\sim}{\varepsilon}^* = \underline{T}_\sigma^{-1}\underline{C}^*\underline{T}_\varepsilon\,\underset{\sim}{\varepsilon} = \underline{C}\,\underset{\sim}{\varepsilon}$$

1.4 Cylindrical coordinate system

In the cylindrical coordinate system, three orthogonal coordinate axes with coordinates $\{r, \theta, z\}$ are used to identify material and spatial points. It is assumed that the mid-plane of the plate coincides with the plan $z = 0$ in the undeformed state.

We only consider axisymmetric geometries, loads and deformations. For such axisymmetric problems there is no dependency of the cylindrical coordinate θ. Here we take as an extra assumption that there is no tangential displacement.

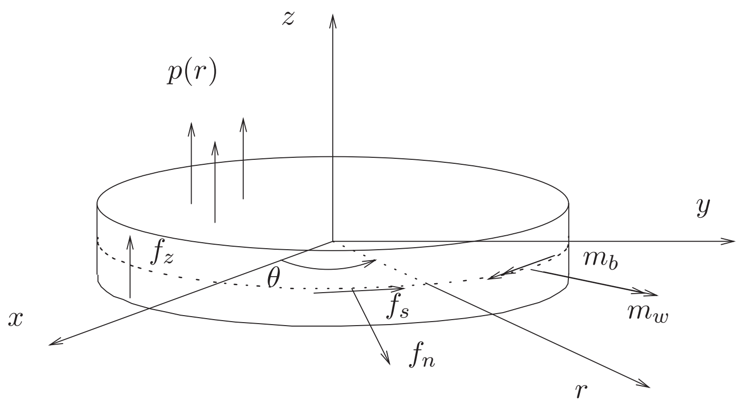


Fig. 1.11 : *Plate in a cylindrical coordinate system*

1.4.1 Displacements

All assumptions about the deformation, described in the former section, are also applied here :

- no out-of-plane shear (Kirchhoff hypotheses),
- small rotation,
- constant thickness.

The displacement components in radial (u_r) and axial (u_z) direction of an out-of-plane point P can then be expressed in the displacement components u and w of the corresponding – same x and y coordinates – point Q in the mid-plane.

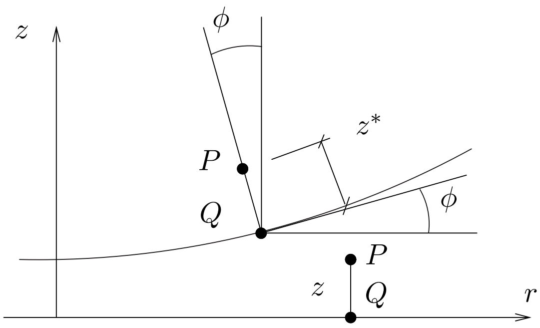


Fig. 1.12 : *Displacement of a out-of-midplane point due to bending*

$$
\begin{aligned}
u_r(r,\theta,z) &= u(r,\theta) - zw_{,r} \\
u_z(r,\theta,z) &= w(r,\theta) + \eta(r,\theta)z^2
\end{aligned}
$$

1.4.2 Curvatures and strains

The linear strain components in a point out of the mid-plane, can be expressed in the mid-plane strains and curvatures. A sign convention, as indicated in the figure, must be adopted and used consistently. It was assumed that straight line elements, initially perpendicular to the mid-plane, remain perpendicular to the mid-plane, so $\gamma_{rz} = 0$.

$$
\left.
\begin{aligned}
\varepsilon_{rr} &= u_{r,r} &&= u_{,r} - zw_{,rr} &&= \varepsilon_{rr0} - z\kappa_{rr} \\
\varepsilon_{tt} &= \frac{1}{r}u_r &&= \frac{1}{r}u - \frac{z}{r}w_{,r} &&= \varepsilon_{tt0} - z\kappa_{tt} \\
\gamma_{rt} &= u_{r,t} + u_{t,r} &&= 0
\end{aligned}
\right\} \quad \rightarrow \quad \underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa}
$$

$$
\begin{aligned}
\varepsilon_{zz} &= u_{z,z} &&= 2\eta(r,z)z \\
\varepsilon_{rz} &= u_{r,z} + u_{z,r} &&= \eta_{,r}z^2 \\
\gamma_{tz} &= u_{t,z} + u_{z,t} &&= 0
\end{aligned}
$$

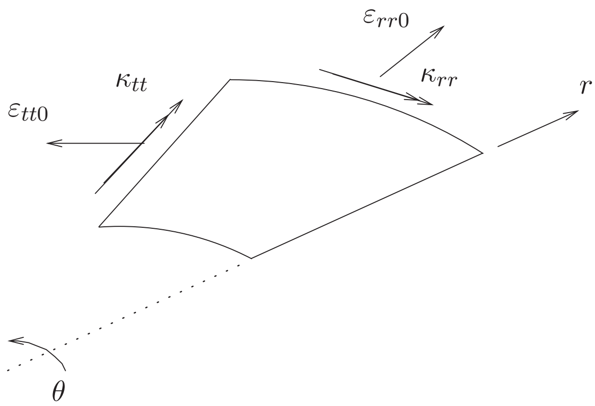


Fig. 1.13 : *Curvatures and strain definitions*

1.4.3 Stresses

A plane stress state is assumed to exist in the plate. The relevant stresses are σ_{rr} and σ_{tt}. Other stress components are much smaller or zero.

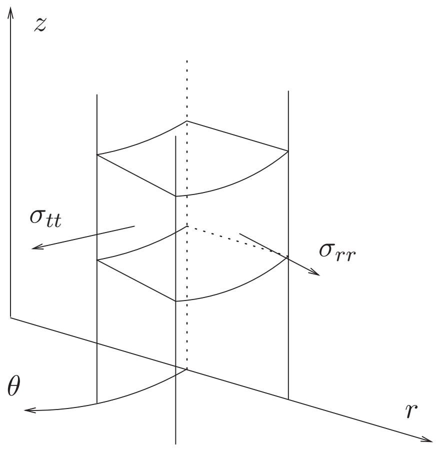


Fig. 1.14 : *In-plane stresses*

1.4.4 Isotropic elastic material behavior

For linear elastic material behavior Hooke's law relates strains to stresses. Material stiffness ($\underline{C}$) and compliance ($\underline{S}$) matrices can be derived for the plane stress state.

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{tt} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} \qquad ; \qquad \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{rr} + \sigma_{tt})$$

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{tt} \end{bmatrix}$$

$$\text{short notation} \quad : \quad \underset{\sim}{\varepsilon} = \underline{S}\,\underset{\sim}{\sigma} \quad \rightarrow \quad \underset{\sim}{\sigma} = \underline{S}^{-1}\underset{\sim}{\varepsilon} = \underline{C}\,\underset{\sim}{\varepsilon} = \underline{C}\,(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})$$

1.4.5 Cross-sectional forces and moments

Cross-sectional forces and moments can be calculated by integration of the in-plane stress components over the plate thickness. A sign convention, as indicated in the figure, must be adopted and used consistently.

The stress component σ_{zr} is much smaller than the relevant in-plane components. Integration over the plate thickness will however lead to a shear force, which cannot be neglected.

$$
\begin{aligned}
\underset{\sim}{N} &= \begin{bmatrix} N_r \\ N_t \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underline{C}\,(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})\, dz = h\underline{C}\underset{\sim}{\varepsilon}_0 \\
\underset{\sim}{M} &= \begin{bmatrix} M_r \\ M_t \end{bmatrix} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_{rr} \\ \sigma_{tt} \end{bmatrix} z\, dz = -\int_{-\frac{h}{2}}^{\frac{h}{2}} \underline{C}\,(\underset{\sim}{\varepsilon}_0 - z\underset{\sim}{\kappa})\, z\, dz = \tfrac{1}{12}\, h^3 \underline{C}\underset{\sim}{\kappa} \\
D &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{zr}\, dz
\end{aligned}
$$

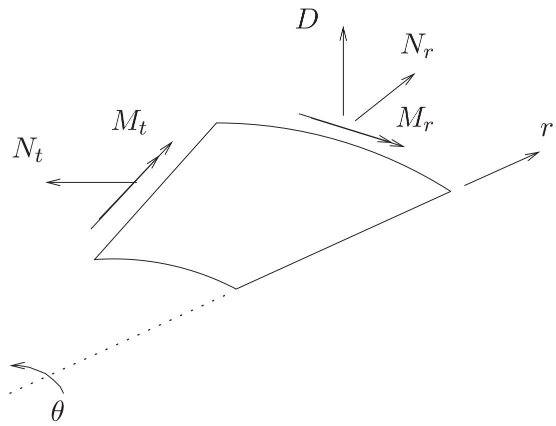


Fig. 1.15 : *Cross-sectional forces and moments*

Stiffness- and compliance matrix

Integration leads to the stiffness and compliance matrix of the plate.

$$
\begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \underline{C}h & \underline{0} \\ \underline{0} & \underline{C}h^3/12 \end{bmatrix} \begin{bmatrix} \underset{\sim}{\varepsilon}_0 \\ \underset{\sim}{\kappa} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \underset{\sim}{\varepsilon}_0 \\ \underset{\sim}{\kappa} \end{bmatrix} = \begin{bmatrix} \underline{S}/h & \underline{0} \\ \underline{0} & 12\underline{S}/h^3 \end{bmatrix} \begin{bmatrix} \underset{\sim}{N} \\ \underset{\sim}{M} \end{bmatrix}
$$

1.4.6 Equilibrium equations

Consider a small "column" cut out of the plate perpendicular to its plane. Equilibrium requirements of the cross-sectional forces and moments, lead to the equilibrium equations. The cross-sectional shear force D can be eliminated by combining some of the equilibrium equations.

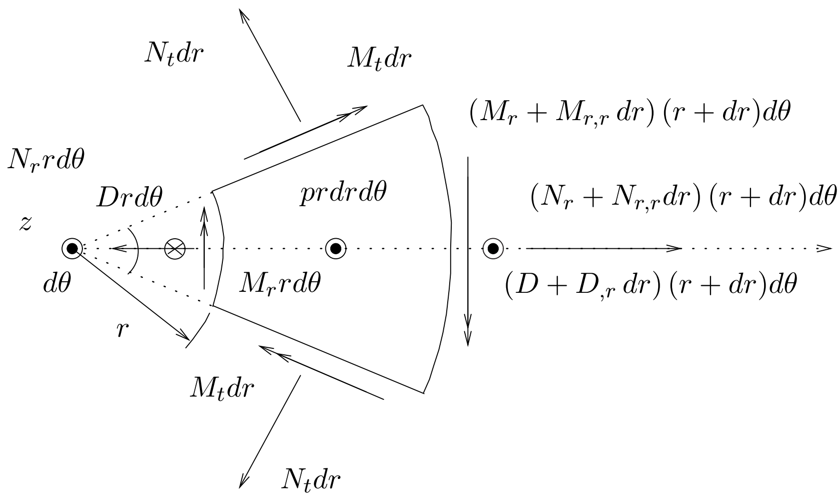


Fig. 1.16 : *Forces on infinitesimal part of the plate*

$$\left.\begin{array}{l} N_{r,r} + \dfrac{1}{r}(N_r - N_t) + s_r = 0 \\ rD_{,r} + D + rp = 0 \\ rM_{r,r} + M_r - M_t - rD = 0 \end{array}\right\} \quad \rightarrow \quad rM_{r,rr} + 2M_{r,r} - M_{t,r} + rp = 0$$

The equilibrium equation for the cross-sectional bending moments can be transformed into a differential equation for the displacement w. The equation is a fourth-order partial differential equation, which is referred to as a bi-potential equation.

$$\left.\begin{array}{l} \begin{bmatrix} M_r \\ M_t \end{bmatrix} = \dfrac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} -w_{,rr} \\ -\frac{1}{r} w_{,r} \end{bmatrix} \\ \\ rM_{r,rr} + 2M_{r,r} - M_{t,r} + rp = 0 \end{array}\right\} \quad \rightarrow$$

$$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = \frac{p}{B} \qquad \rightarrow \qquad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{p}{B}$$

$$B = \frac{Eh^3}{12(1-\nu^2)} = \text{plate modulus}$$

1.4.7 Solution

The differential equation has a general solution comprising four integration constants. Depending on the external load, a particular solution w_p also remains to be specified. The integration constants have to be determined from the available boundary conditions, i.e. from the prescribed displacements and loads.

The cross-sectional forces and moments can be expressed in the displacement w and/or the strain components.

$$w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) + w_p$$

$$\begin{aligned} M_r &= -B\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \\ M_t &= -B\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) \\ D &= \frac{dM_r}{dr} + \frac{1}{r}\left(M_r - M_t\right) = -B\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \end{aligned}$$

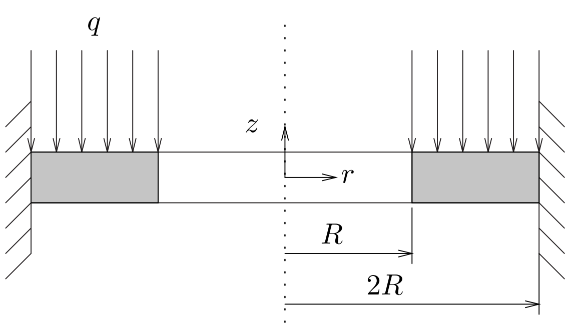
1.4.8 Examples

This section contains some examples of axisymmetric plate bending problems.

- Plate with central hole
- Solid plate with distributed load
- Solid plate with vertical point load

Plate with central hole

A circular plate with radius $2R$ has a central hole with radius R. The plate is clamped at its outer edge and is loaded by a distributed load q perpendicular to its plane in negative z-direction. The equilibrium equation can be solved with proper boundary conditions, which are listed below.



boundary conditions

- $w(r = 2R) = 0$
- $\left(\frac{dw}{dr}\right)(r = 2R) = 0$
- $m_b(r = R) = M_r(r = R) = 0$
- $f_z(r = R) = D(r = R) = 0$

differential equation	$\frac{d^4w}{dr^4} + \frac{2}{r}\frac{d^3w}{dr^3} - \frac{1}{r^2}\frac{d^2w}{dr^2} + \frac{1}{r^3}\frac{dw}{dr} = -\frac{q}{B}$
general solution	$w = a_1 + a_2r^2 + a_3\ln(r) + a_4r^2\ln(r) + w_p$
particulate solution	$w_p = \alpha r^4 \quad \rightarrow \quad \text{substitution} \quad \rightarrow$
	$24\alpha + 48\alpha - 12\alpha + 4\alpha = -\frac{q}{B} \quad \rightarrow \quad \alpha = -\frac{q}{64B}$
solution	$w = a_1 + a_2r^2 + a_3\ln(r) + a_4r^2\ln(r) - \frac{q}{64B}\,r^4$

The 4 constants in the general solution can be solved using the boundary conditions. For this purpose these have to be elaborated to formulate the proper equations.

$$\frac{dw}{dr} = 2a_2r + \frac{a_3}{r} + a_4(2r\ln(r) + r) - \frac{q}{16B}\,r^3$$

$$\begin{aligned}
M_r &= -B\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) \\
&= -B\left\{2(1+\nu)a_2 - (1-\nu)\frac{1}{r^2}\,a_3 + 2(1+\nu)\ln(r)a_4 + (3+\nu)a_4 - (3+\nu)\frac{q}{16B}\,r^2\right\} \\
M_t &= -B\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) \\
&= -B\left\{2(1+\nu)a_2 + (1-\nu)\frac{1}{r^2}a_3 + 2(1+\nu)\ln(r)a_4 + (3\nu+1)a_4 - (3\nu+1)\,\frac{q}{16B}\,r^2\right\} \\
D &= \frac{dM_r}{dr} + \frac{1}{r}\,(M_r - M_t) \\
&= B\left(-\frac{4}{r}\,a_4 + \frac{q}{2B}\,r\right)
\end{aligned}$$

The equations are summarized below. The 4 constants can be solved.

After solving the constants, the vertical displacement at the inner edge of the hole can be calculated.

$$\begin{aligned}
w(r = 2R) &= a_1 + 4a_2R^2 + a_3\ln(R) + 4a_4R^2\ln(R) - \frac{q}{4B}\,R^4 = 0 \\
\frac{dw}{dr}(r = 2R) &= 4a_2R + \frac{1}{2R}\,a_3 + 4a_4R\ln(R) + 2a_4R - \frac{q}{2B}\,R^3 = 0 \\
M_r(r = R) &= -B\left\{2(1+\nu)a_2 - (1-\nu)\,\frac{1}{R^2}\,a_3 + \right.
\end{aligned}$$

$$\left.(2(1+\nu)\ln(R) + (3+\nu))\, a_4 - (3+\nu)\,\frac{q}{16B}\, R^2 \right\} = 0$$

$$D(r=R) = B\left(-\frac{4}{R}\, a_4 + \frac{q}{2B}\, R\right\} = 0$$

$$w(r=R) = -\frac{qR^4}{64B}\left\{\frac{-9(5+\nu) + 48(3+\nu)\ln(2) - 64(1+\nu)(\ln(2))^2}{5-3\nu}\right\}$$

For the values listed in the table, calculated values are shown as a function of the radial distance r.

inner radius	R	0.1	m
outer radius	$2R$	0.2	m
thickness	h	0.01	m
Young's modulus	E	100	GPa
Poisson's ratio	ν	0.3	-
global load	q	-100	Pa

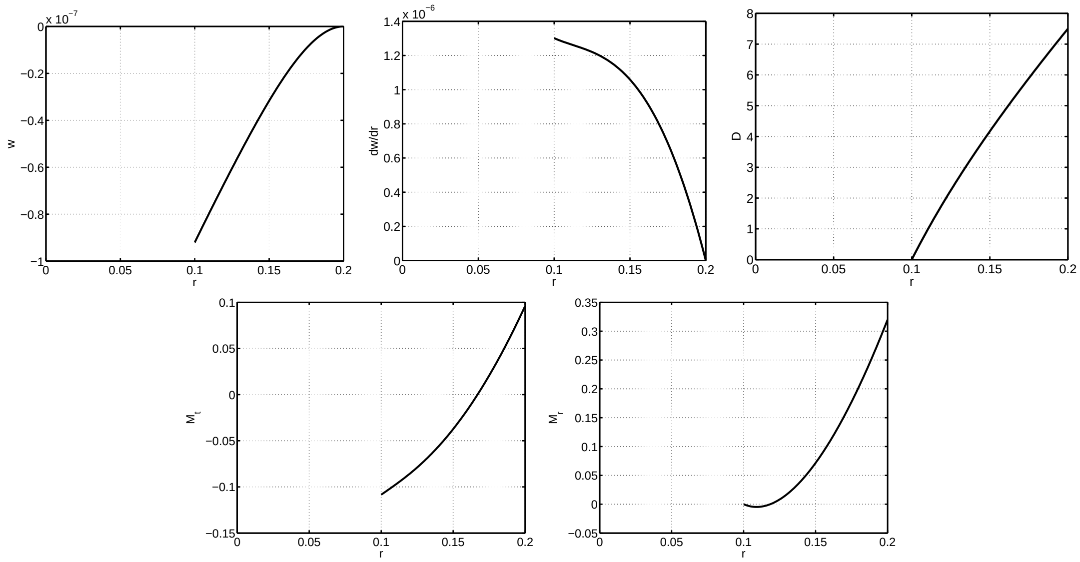
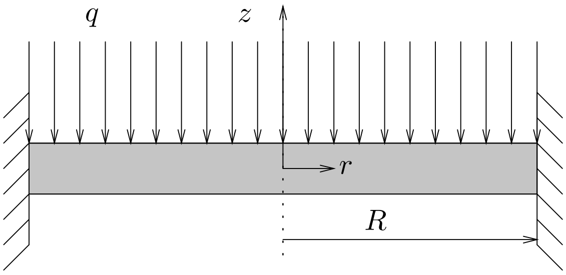


Fig. 1.17 : *Calculated values as a function of radial distance.*

Solid plate with distributed load

A solid circular plate has radius R and uniform thickness. It is clamped at its outer edge and loaded with a distributed force per unit of area q in negative z-direction. The equilibrium equation can be solved with proper boundary conditions, which are listed below.

Besides the obvious boundary conditions, there are also two symmetry conditions, one for the deformation and the other for the load.



boundary conditions

- $w(r = R) = 0$
- $\left(\dfrac{dw}{dr}\right)(r = R) = 0$
- $\left(\dfrac{dw}{dr}\right)(r = 0) = 0$
- $D(r = 0) = 0$

differential equation $\quad \dfrac{d^4w}{dr^4} + \dfrac{2}{r}\dfrac{d^3w}{dr^3} - \dfrac{1}{r^2}\dfrac{d^2w}{dr^2} + \dfrac{1}{r^3}\dfrac{dw}{dr} = -\dfrac{q}{B}$

general solution $\quad w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) + w_p$

particulate solution $\quad w_p = \alpha r^4 \quad \rightarrow \quad \text{substitution} \quad \rightarrow$

$$24\alpha + 48\alpha - 12\alpha + 4\alpha = -\frac{p}{B} \quad \rightarrow \quad \alpha = -\frac{q}{64B}$$

solution $\quad w = a_1 + a_2 r^2 + a_3 \ln(r) + a_4 r^2 \ln(r) - \dfrac{q}{64B}\, r^4$

The 4 constants in the general solution can be solved using the boundary and symmetry conditions. For this purpose these have to be elaborated to formulate the proper equations.

The two symmetry conditions result in two constants to be zero. The other two constants can be determined from the boundary conditions.

After solving the constants, the vertical displacement at $r = 0$ can be calculated.

$$\begin{aligned}
\frac{dw}{dr} &= 2a_2 r + \frac{a_3}{r} + a_4(2r\ln(r) + r) - \frac{q}{16B}\, r^3 \\
D &= \frac{dM_r}{dr} + \frac{1}{r}\,(M_r - M_t) \\
&= B\left(-\frac{4}{r}\, a_4 + \frac{q}{2B}\, r\right)
\end{aligned}$$

boundary conditions

$$\begin{aligned}
&\left(\frac{dw}{dr}\right)(r = 0) = 0 \quad \rightarrow \quad a_3 = 0 \\
&D(r = 0) = 0 \quad \rightarrow \quad a_4 = 0
\end{aligned}$$

$$\left(\frac{dw}{dr}\right)(r = R) = 2a_2 R - \frac{q}{16B} R^3 = 0 \quad \rightarrow \quad a_2 = \frac{q}{32B} R^2$$
$$w(r = R) = a_1 + a_2 R^2 - \frac{q}{64B} R^4 = 0 \quad \rightarrow \quad a_1 = -\frac{q}{64B} R^4$$

displacement in the center

$$w(r = 0) = -\frac{q}{64B} R^4$$

For the values listed in the table, calculated values are shown as a function of the radial distance r.

inner radius	R	0	m
outer radius	$2R$	0.2	m
thickness	h	0.01	m
Young's modulus	E	100	GPa
Poisson's ratio	ν	0.3	-
global load	q	-100	Pa

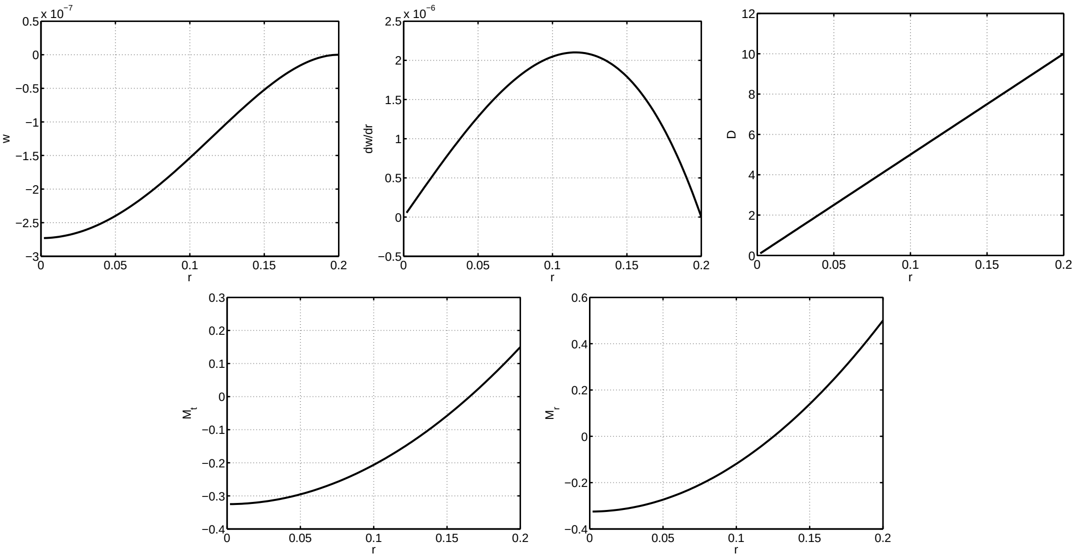
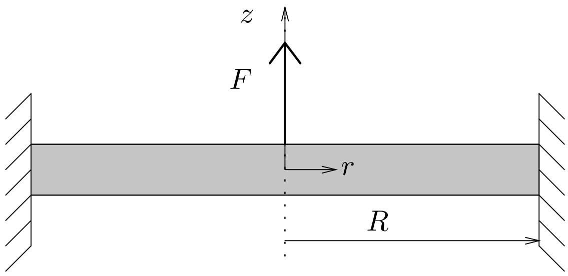


Fig. 1.18 : *Calculated values as a function of radial distance.*

Solid plate with vertical point load

A solid circular plate has radius R and uniform thickness d. The plate is clamped at its edge. It is loaded in its center ($r = 0$) by a point load F perpendicular to its plane in the positive z-direction. The equilibrium equation can be solved using appropriate boundary conditions, listed below.

Besides the obvious boundary conditions, there is also the symmetry condition that in the center of the plate the bending angle must be zero. In the same point the total cross-sectional force $K = 2\pi D$ must equilibrate the external force F.



boundary conditions

- $w(r = R) = 0$
- $\left(\dfrac{dw}{dr}\right)(r = R) = 0$
- $\left(\dfrac{dw}{dr}\right)(r = 0) = 0$
- $K(r = 0) = -F$

differential equation $\quad \dfrac{d^4w}{dr^4} + \dfrac{2}{r}\dfrac{d^3w}{dr^3} - \dfrac{1}{r^2}\dfrac{d^2w}{dr^2} + \dfrac{1}{r^3}\dfrac{dw}{dr} = 0$

general solution $\quad w = a_1 + a_2r^2 + a_3\ln(r) + a_4r^2\ln(r)$

The derivative of w and the cross-sectional load D can be written as a function of r.

The symmetry condition results in one integration constant to be zero. Because the derivative of w is not defined for $r = 0$, a limit has to be taken. The cross-sectional equilibrium results in a known value for another constant. The remaining constants can be determined from the boundary conditions at $r = R$.

After solving the constants, the vertical displacement at $r = 0$ can be calculated.

$$\frac{dw}{dr} = 2a_2r + \frac{a_3}{r} + a_4(2r\ln(r) + r)$$
$$D = \frac{dM_r}{dr} + \frac{1}{r}(M_r - M_t) = B\left(-\frac{4}{r}a_4\right)$$

boundary conditions

$$\lim_{r\longrightarrow 0}\left(\frac{dw}{dr}\right) = \lim_{r\longrightarrow 0}\left\{2a_2r + \frac{a_3}{r} + a_4(2r\ln(r) + r)\right\} = 0 \quad \rightarrow \quad a_3 = 0$$

$$K(r) = 2\pi r D(r) = -8\pi B a_4 = -F \quad \rightarrow \quad a_4 = \frac{F}{8\pi B}$$

$$\left.\begin{array}{l} w(r = R) = a_1 + a_2R^2 + \dfrac{F}{8\pi B}R^2\ln(R) = 0 \\[2ex] \left(\dfrac{dw}{dr}\right)(r = R) = 2Ra_2 + \dfrac{F}{8\pi B}(2R\ln(R) + R) = 0 \end{array}\right\} \rightarrow \left\{\begin{array}{l} a_1 = \dfrac{FR^2}{16\pi B} \\[2ex] a_2 = -\dfrac{F}{16\pi B}(2\ln R + 1) \end{array}\right.$$

displacement at the center

$$w(r=0) = \lim_{r \longrightarrow 0} \frac{FR^2}{16\pi B} \left\{ 1 - \left(\frac{r}{R} \right)^2 + 2 \left(\frac{r}{R} \right)^2 \ln \left(\frac{r}{R} \right) \right\} = \frac{FR^2}{16\pi B}$$

The stresses can also be calculated. They are a function of both coordinates r and z. We can conclude that at the center, where the force is applied, the stresses become infinite, due to the singularity, provoked by the point load.

$$\sigma_{rr} = -\frac{3Fz}{\pi d^3} \left\{ 1 + (1+\nu) \ln \left(\frac{r}{R} \right) \right\}$$

$$\sigma_{tt} = -\frac{3Fz}{\pi d^3} \left\{ \nu + (1+\nu) \ln \left(\frac{r}{R} \right) \right\}$$