NONLINEAR TRUSS

1 Nonlinear deformation

When deformation and/or rotation of the truss are large, various strains and stresses can be defined and related by material laws. The material behavior can be expected to be no longer linearly elastic.

1.1 Strains for large elongation

The deformation of the truss can be characterized uniquely by the two elongation factors λ and μ. However, it is common and useful to introduce deformation variables which are a function of the elongation factors : the strains. A wide variety of strain definitions is possible and used.

All strain definitions must obey some requirements, one of which is that they have to result in the same value for small elongations, being the value of the linear strain. When we plot the various strains as a function of the elongation factor, it is immediately clear that the strains, which are defined here, obey this requirement.

It is obvious that one and the same strain definition must be used throughout the same specimen and analysis. This implies that the contraction strain is defined analogously to the elongational strain. These strains are related by a material parameter, the Poisson's ratio ν. It is assumed, until stated otherwise, that this parameter is constant.

linear strain	$\varepsilon = \varepsilon_l = \lambda - 1$
logarithmic strain	$\varepsilon = \varepsilon_{ln} = \ln(\lambda)$
Green-Lagrange strain	$\varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

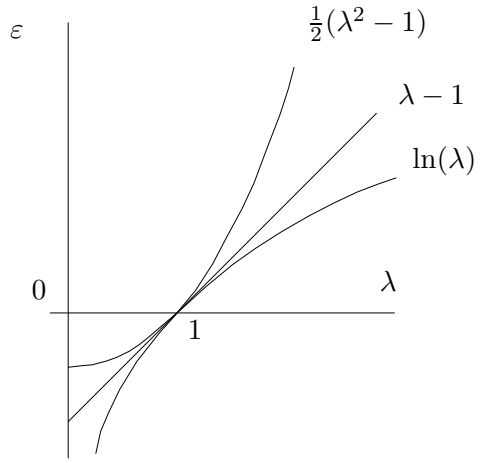

Fig. 1.1 : *Three strain definitions as function of the elongation factor*

Linear strain

The linear strain definition results in unrealistic contraction, when the elongation is too large. The cross-sectional area of the truss can become zero, which is of course not possible.

linear strain $\varepsilon = \varepsilon_l = \lambda - 1 = \dfrac{\Delta l}{l_0}$

contraction strain $\varepsilon_d = \mu - 1 = -\nu\varepsilon_l = -\nu(\lambda - 1)$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = 1 - \nu(\lambda - 1) \quad \rightarrow \quad A = A_0\{1 - \nu(\lambda - 1)\}^2$$

restriction of elongation

$$1 - \nu(\lambda - 1) > 0 \quad \rightarrow \quad \lambda - 1 < \frac{1}{\nu} \quad \rightarrow \quad \lambda < \frac{1 + \nu}{\nu}$$

Logarithmic strain

The logarithmic strain definition does not lead to unrealistic values for the contraction. Therefore it is very suitable to describe large deformations. [1]

logarithmic strain $\varepsilon = \varepsilon_{ln} = \ln(\lambda)$

contraction strain $\varepsilon_d = \ln(\mu) = -\nu\varepsilon_{ln} = -\nu \ln \lambda$

change of cross-sectional area

$$\mu = \sqrt{\frac{A}{A_0}} = e^{-\nu\varepsilon_{ln}} = e^{-\nu \ln(\lambda)} = \left[e^{\ln(\lambda)}\right]^{-\nu} = \lambda^{-\nu} \qquad \rightarrow \qquad A = A_0\lambda^{-2\nu}$$

A deformation process may be executed in a number of steps, as is often done in forming processes. The start of a new step can be taken to be the reference state to calculate current strains. In that case the logarithmic strain is favorably used, because the subsequent strains can be added to determine the total strain w.r.t. the initial state.

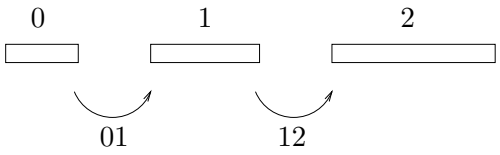

Fig. 1.2 : *Two-step deformation process*

[1] $\ln x = {}^{e}\log(x) = y \quad \rightarrow \quad x = e^y$

$l_0 \rightarrow l_1$ $\quad \varepsilon_l(01) = \frac{l_1 - l_0}{l_0}$
$\varepsilon_{ln}(01) = \ln(\frac{l_1}{l_0})$

$l_1 \rightarrow l_2$ $\quad \varepsilon_l(12) = \frac{l_2 - l_1}{l_1}$
$\varepsilon_{ln}(12) = \ln(\frac{l_2}{l_1})$

$l_0 \rightarrow l_2$ $\quad \varepsilon_l(02) = \frac{l_2 - l_0}{l_0} \neq \varepsilon_l(01) + \varepsilon_l(12)$
$\varepsilon_{ln}(02) = \ln(\frac{l_2}{l_0}) = \ln(\frac{l_2}{l_1}\frac{l_1}{l_0}) = \varepsilon_{ln}(01) + \varepsilon_{ln}(12)$

Green-Lagrange strain

Using the Green-Lagrange strain leads again to restrictions on the elongation to prevent the cross-sectional area to become zero.

Green-Lagrange strain $\quad \varepsilon = \varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$

contraction strain $\quad \varepsilon_d = \frac{1}{2}(\mu^2 - 1) = -\nu\varepsilon_{ln} = -\nu\frac{1}{2}(\lambda^2 - 1)$

change of cross-sectional area

$$1 - \nu(\lambda^2 - 1) > 0 \quad \rightarrow \quad \lambda < \sqrt{\frac{1+\nu}{\nu}}$$

1.2 Mechanical power for an axially loaded truss

The figure shows a tensile bar which is elongated due to the action of an axial force F. Its undeformed cross-sectional area and length are A_0 and l_0, respectively. In the deformed configuration the cross-sectional area and length are A and l.

At constant force F an infinitesimal small increase in length is associated with a change in mechanical energy per unit of time (power) : $P = F\dot{l}$. The elongation rate $\dot{l}$ can be expressed in various strain rates.

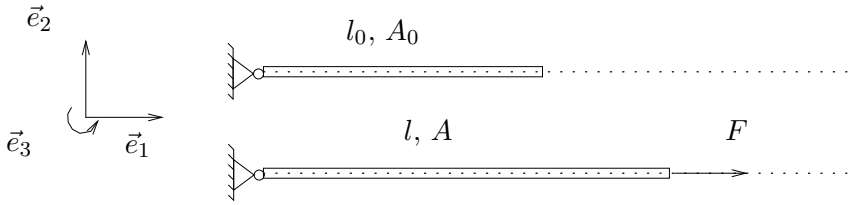

Fig. 1.3 : *Axial elongation of homogeneous truss*

linear strain	$\varepsilon_l = \lambda - 1$	$\rightarrow$	$\dot{\varepsilon}_l = \dot{\lambda} = \dfrac{\dot{l}}{l_0}$
logarithmic strain	$\varepsilon_{ln} = \ln(\lambda)$	$\rightarrow$	$\dot{\varepsilon}_{ln} = \dot{\lambda}\lambda^{-1} = \dfrac{\dot{l}}{l}$
Green-Lagrange strain	$\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$	$\rightarrow$	$\dot{\varepsilon}_{gl} = \dot{\lambda}\lambda = \lambda\,\dfrac{\dot{l}}{l_0} = \lambda^2\,\dfrac{\dot{l}}{l}$

$$
\begin{aligned}
P &= F\dot{\ell} = F\ell_0\dot{\varepsilon}_l = \frac{F}{A_0}A_0\ell_0\dot{\varepsilon}_l = \frac{F}{A_0}V_0\dot{\varepsilon}_l \\
P &= F\dot{\ell} = F\ell\dot{\varepsilon}_{ln} = \frac{F}{A}A\ell\dot{\varepsilon}_{ln} = \frac{F}{A}V\dot{\varepsilon}_{ln} \\
P &= F\dot{\ell} = F\ell_0\dot{\varepsilon}_l = \frac{F}{A}A\ell\frac{\ell_0}{\ell}\dot{\varepsilon}_l = \frac{F}{A}V\lambda^{-1}\dot{\varepsilon}_l \\
P &= F\dot{\ell} = F\ell\lambda^{-2}\dot{\varepsilon}_{gl} = \frac{F}{A}A\ell\lambda^{-2}\dot{\varepsilon}_{gl} = \frac{F}{A}V\lambda^{-2}\dot{\varepsilon}_{gl}
\end{aligned}
$$

Various stress definitions automatically emerge when the mechanical power is considered in the undeformed volume $V_0 = A_0 l_0$ or the current volume $V = Al$ of the tensile bar. The stresses are :

σ	:	Cauchy or true stress
σ_n	:	engineering or nominal stress
σ_{p1}	:	1st Piola-Kirchhoff stress $= \sigma_n$
σ_κ	:	Kirchhoff stress
σ_{p2}	:	2nd Piola-Kirchhoff stress

$$
\begin{aligned}
P &= = = V_0\sigma_n\dot{\varepsilon}_l \\
P &= V\sigma\dot{\varepsilon}_{ln} = V_0(J\sigma)\dot{\varepsilon}_{ln} = V_0\sigma_\kappa\dot{\varepsilon}_{ln} \\
P &= V(\sigma\lambda^{-1})\dot{\varepsilon}_l = V_0(J\sigma\lambda^{-1})\dot{\varepsilon}_l = V_0\sigma_{p1}\dot{\varepsilon}_l \\
P &= V(\sigma\lambda^{-2})\dot{\varepsilon}_{gl} = V_0(J\sigma\lambda^{-2})\dot{\varepsilon}_{gl} = V_0\sigma_{p2}\dot{\varepsilon}_{gl}
\end{aligned}
$$

specific mechanical power : $P = V_0\dot{W}_0 = V\dot{W}$

$$
\begin{aligned}
\dot{W}_0 &= \sigma_n\dot{\varepsilon}_l = \sigma_\kappa\dot{\varepsilon}_{ln} = \sigma_{p1}\dot{\varepsilon}_l = \sigma_{p2}\dot{\varepsilon}_{gl} \\
\dot{W} &= = \sigma\dot{\varepsilon}_{ln} = \sigma\lambda^{-1}\dot{\varepsilon}_l = \sigma\lambda^{-2}\dot{\varepsilon}_{gl}
\end{aligned}
$$

1.3 Equilibrium

Deformations may be so large that the geometry changes considerably. This and/or nonlinear boundary conditions render the deformation problem nonlinear. Proportionality and superposition do not hold in that case. The internal force f_i is a nonlinear function of the elongation u.

Nonlinear material behavior may also result in a nonlinear function $f_i(u)$. This nonlinearity is almost always observed when deformation is large.

Solving the elongation from the equilibrium equation is only possible with an iterative solution procedure.

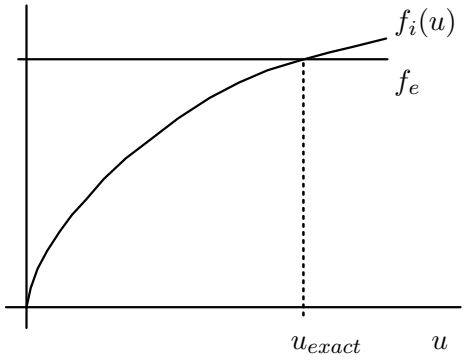

Fig. 1.4 : *Nonlinear internal load and constant external load*

external force	f_e
internal force	$f_i = \sigma A = f_i(u)$
equilibrium of point P	$f_i(u) = f_e$

1.4 Iterative solution procedure

It is assumed that an approximate solution u^* for the unknown exact solution u_{exact} exists. (Initially $u^* = 0$ is chosen.)

The *residual load* r^* is the difference between $f(u^*)$ and f_e. For the exact solution this residual is zero. What we want the iterative solution procedure to do, is generating better approximations for the exact solution so that the residual becomes very small (ideally zero).

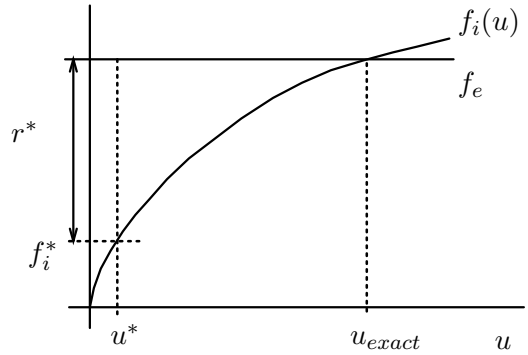

Fig. 1.5 : *Approximation of exact solution*

analytic solution	$f_i(u_{exact}) = f_e \quad \rightarrow \quad f_e - f_i(u_{exact}) = 0$
approximation u^*	$f_e - f_i(u^*) = r(u^*) \neq 0$
residual	$r^* = r(u^*)$

The unknown exact solution is written as the sum of the approximation and an unknown error δu. The internal force $f_i(u_{exact})$ is then written as a Taylor series expansion around u^* and linearized with respect to δu. The first derivative of f_i with respect to u is called the *tangential stiffness* K^*. Subsequently δu is solved from the linear iterative equation. The solution is called the *iterative displacement*.

$$\left.\begin{array}{l} f_i(u_{exact}) = f_e \\ u_{exact} = u^* + \delta u \end{array}\right\} \quad \rightarrow \quad f_i(u^* + \delta u) = f_e$$

$$f_i(u^*) + \left.\frac{df_i}{du}\right|_{u^*} \delta u = f_e \quad \rightarrow \quad f_i^* + K^* \delta u = f_e$$

$$K^*\, \delta u = f_e - f_i^* = r^* \quad \rightarrow \quad \delta u = \frac{1}{K^*}\, r^*$$

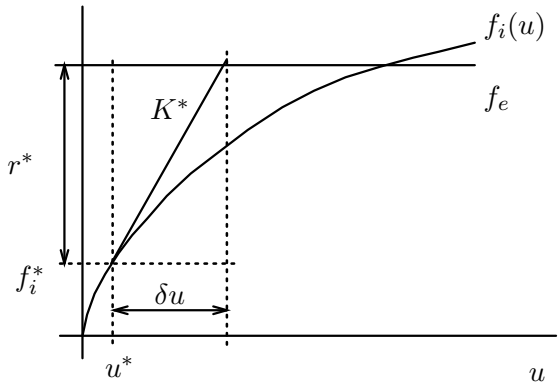

Fig. 1.6 : *Tangential stiffness and iterative solution*

With the iterative displacement δu a new approximate solution u^{**} can be determined by simply adding it to the known approximation.

When u^{**} is a better approximation than u^*, the iteration process is *converging*. As the exact solution is unknown, we cannot calculate the deviation of the approximation directly. There are several methods to quantify the *convergence*.

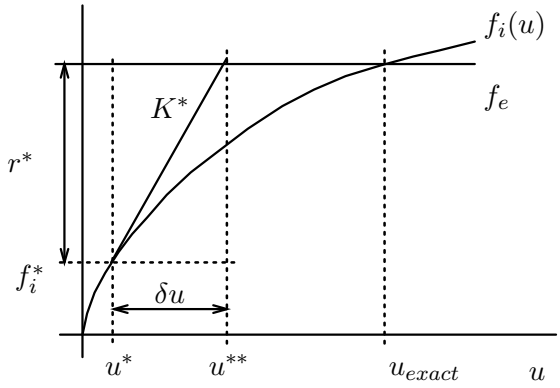

Fig. 1.7 : *New approximation of the exact solution*

new approximation	$u^{**} = u^* + \delta u$
error	$u_{exact} - u^{**}$
error smaller	$\rightarrow$ convergence

1.5 Convergence control

When the new approximation u^{**} is better than u^*, the residual r^{**} is smaller than r^*. If its value is not small enough, a new approximate solution is determined in a new iteration step. If its value is small enough, we are satisfied with the approximation u^{**} for the exact solution and the iteration process is terminated. To make this decision the residual is compared to a *convergence criterion* c_r. It is also possible to compare the iterative displacement δu with a convergence criterion c_u. If $\delta u < c_u$ it is assumed that the exact solution is determined close enough.

When the convergence criterion is satisfied, the displacement u will not satisfy the nodal equilibrium exactly, because the convergence limit is small but not zero. When incremental loading is applied, the difference between f_i and f_e is added to the load in the next increment, which is known as *residual load correction.*

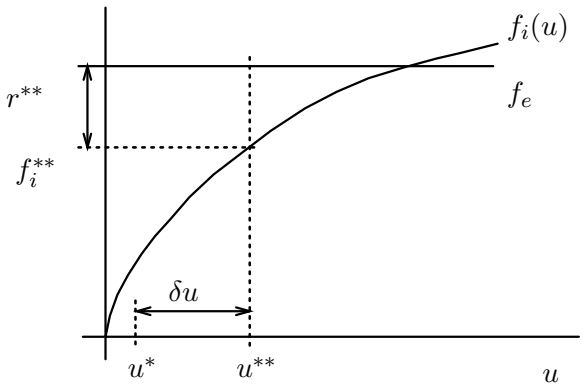

Fig. 1.8 : *New residual for approximate solution*

residual force	$\lvert r^{**}\rvert \leq c_r \quad \rightarrow$ stop iteration
iterative displacement	$\lvert \delta u\rvert \leq c_u \quad \rightarrow$ stop iteration

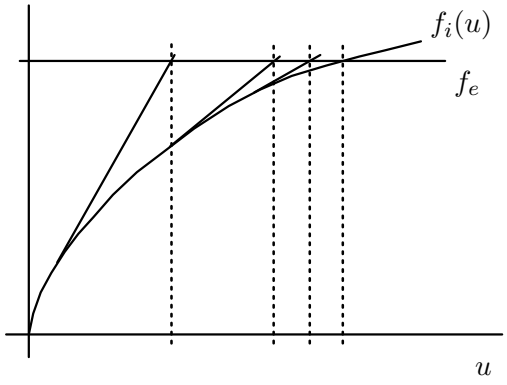

Fig. 1.9 : *Converging iteration process*

1.6 Residual and tangential stiffness

The residual and the tangential stiffness can be calculated from the material model, which describes the *material behavior*. It is assumed that this is a relation between the axial Cauchy stress σ and the elongation factor or stretch ratio $\lambda = \frac{l}{l_0}$: $\sigma = \sigma(\lambda)$. It is also necessary to now the relation between the cross-sectional area A and λ.

internal nodal force $\qquad f_i^* = N(\lambda^*) = A^*\sigma^*$

tangential stiffness $\qquad K^* = \left.\dfrac{\partial f_i}{\partial u}\right|_{u^*} = \left.\dfrac{\partial N(\lambda)}{\partial u}\right|_{u^*} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{d\lambda}{du}$

geometry $\qquad \lambda = 1 + \dfrac{\Delta l}{l_0} = 1 + \dfrac{1}{l_0}\,u \qquad \rightarrow \qquad \dfrac{d\lambda}{du} = \dfrac{1}{l_0}$

tangential stiffness $\qquad K^* = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{\partial\lambda}{\partial u} = \left.\dfrac{dN}{d\lambda}\right|_{\lambda^*} \dfrac{1}{l_0} = \left.\dfrac{dN}{d\lambda}\right|^* \dfrac{1}{l_0} = \dfrac{1}{l_0}\dfrac{d}{d\lambda}(\sigma A)\Big|^*$

$$K^* = \frac{1}{l_0}\left.\frac{d\sigma}{d\lambda}\right|^* A^* + \frac{1}{l_0}\,\sigma^*\left.\frac{dA}{d\lambda}\right|^*$$

1.7 Incremental loading

The external loading may be time-dependent. To determine the associated deformation, the time is discretized : the load is prescribed at subsequent, discrete moments in time and deformation is determined at these moments. A time interval between two discrete moments is called a *time increment* and the time dependent loading is referred to as *incremental loading*. This incremental loading is also applied for cases where the real time (seconds, hours) is not relevant, but when we want to prescribe the load gradually. One can than think of the "time" as a fictitious or virtual time.

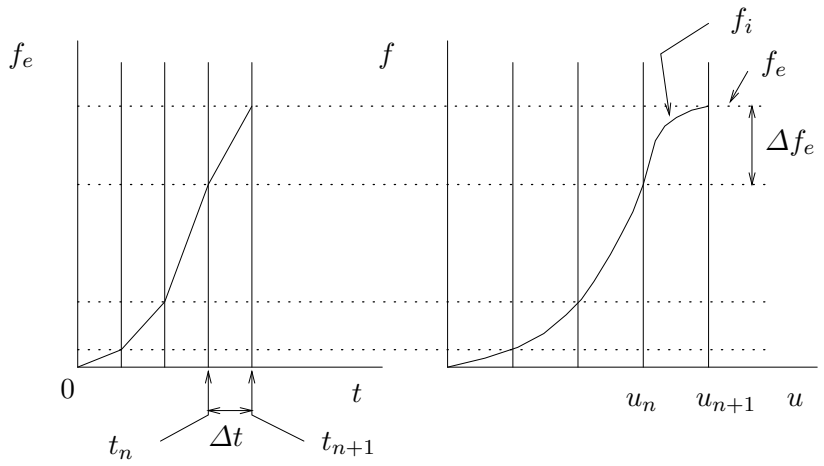

Fig. 1.10 : *Incremental loading*

Non-converging solution process

The iteration process is not always converging. Some illustrative examples are shown in the next figures.

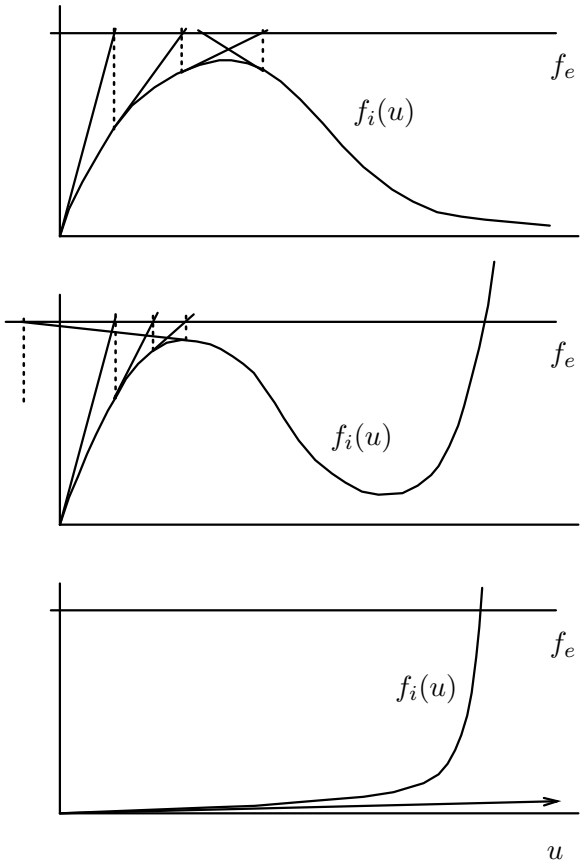

Fig. 1.11 : *Non-converging solution processes*

Modified Newton-Raphson procedure

Sometimes, it is possible to reach the exact solution by modifying the Newton-Raphson iteration process. The tangential stiffness is then not updated in every iteration step. Its initial value is used throughout the iterative procedure.

The figure shows a so-called "snap-through" problem, where no convergence can be reached due to a cycling *full* Newton-Raphson iteration process. With *modified* Newton-Raphson, iteration proceeds to the equilibrium $f_i = f_e$.

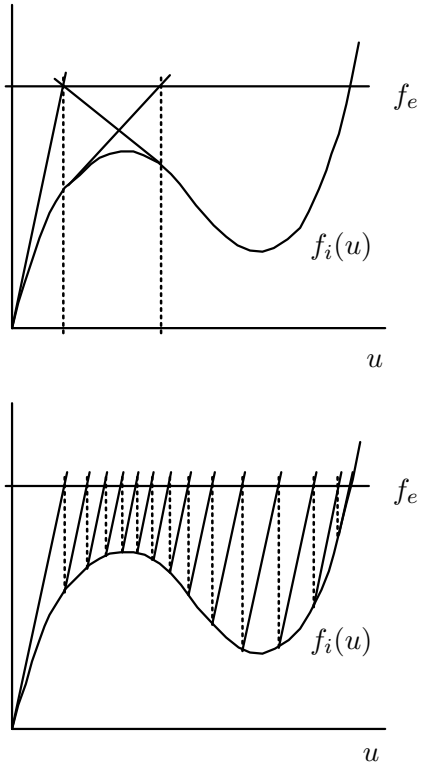

Fig. 1.12 : *Modified Newton-Raphson procedure*

2 Weighted residual formulation for nonlinear truss

In the initial configuration a truss has length ℓ_0. In the current configuration the truss is subjected to an axial load: concentrated forces N_0 and N_ℓ in begin and end point, and a volume load $q(s)$ per unit of length. It has length ℓ and is rotated with respect to the initial configuration. The coordinate along the truss axis is s and the direction of the axis is indicated by the unit vector $\vec{n}$.

In each point of the truss the equilibrium equation has to be satisfied. The equilibrium equation is derived under assumption of static loading conditions. It is a differential equation, for which analytical solutions do only exist for rather simple boundary conditions. For practical problems we have to be satisfied with an approximate solution.

The error represented by the approximation can be "smeared out" along the axis of the truss, by integrating the product of this error and a so-called *weighting function* over the length of the truss.

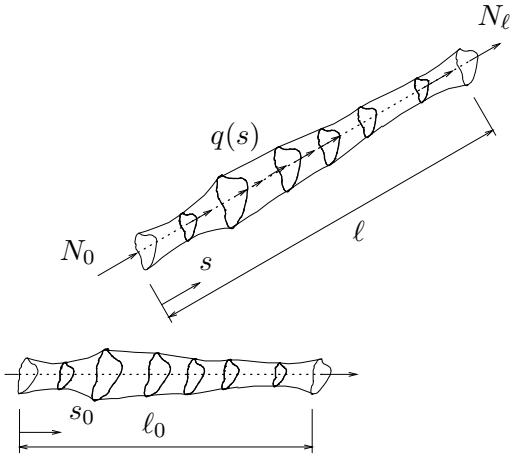

Fig. 2.13 : *Inhomogeneous truss*

equilibrium	$\dfrac{d\vec{N}}{ds} + \vec{q}(s) = \vec{0} \quad \rightarrow \quad \dfrac{d(\sigma A \vec{n})}{ds} + \vec{q}(s) = \vec{0} \quad \forall\ s \in [\,0, \ell\,]$
approximation	$\dfrac{d(\sigma^* A^* \vec{\bar{n}})}{ds} + \vec{q}(s) = \vec{\Delta}(s) \neq \vec{0} \quad \forall\ s \in [\,0, \ell\,]$
weighted error	$\vec{\Delta}(s)$ is "smeared out" over $[0, \ell] \quad \rightarrow \quad \displaystyle\int_{s=0}^{s=\ell} \vec{w}(s) \cdot \vec{\Delta}(s)\, ds$

2.1 Weighted residual formulation

The product of the left-hand side of the equilibrium equation and a weighting function $\vec{w}(s)$ can be integrated over the element length, resulting in the weighted residual integral. The *principle of weighted residuals* now states that :

> the requirement that the equilibrium equation is satisfied in each point of the truss, is equivalent to the requirement that the weighted residual integral is zero for every possible weighting function.

The first term in the integral is integrated by parts to reduce the continuity requirements of the axial stress. This results in the so-called *weak form* of the weighted residual formulation. The right hand part of the resulting integral equation represents the contribution of the external loads.

$$\int_{s=0}^{s=\ell} \vec{w} \cdot \left\{ \frac{d(\sigma A \vec{n})}{ds} + \vec{q} \right\} ds = 0 \qquad \forall \quad \vec{w}(s) \quad \rightarrow$$

$$\int_{s=0}^{s=\ell} \frac{d\vec{w}}{ds} \cdot (\sigma A \vec{n})\, ds = \int_{s=0}^{s=\ell} \vec{w} \cdot \vec{q}\, ds + \left[\vec{w}(\ell) \cdot \vec{N}(\ell) - \vec{w}(0) \cdot \vec{N}(0) \right] = f_e(\vec{w}) \quad \forall \quad \vec{w}(s)$$

2.2 State transformation

Because the current length of the truss is not known, the integration can not be carried out. Also the derivatives with respect to the coordinate s can not be evaluated. These problems can be circumvented by a transformation. In this case we transform everything to the initial configuration on time t_0 where the truss is undeformed. This procedure is generally referred to as the *Total Lagrange* approach. When transformation is carried out to the last known configuration, we would have followed the *Updated Lagrange* approach, which will not be considered here.

$$\frac{d(\ )}{ds} = \frac{ds_0}{ds} \frac{d(\ )}{ds_0} = \frac{1}{\lambda} \frac{d(\ )}{ds_0} \qquad ; \qquad ds = \lambda ds_0 \qquad \Rightarrow$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma A \vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

The current stress σ, cross-sectional area A and axis direction $\vec{n}$ have to be determined such that the integral is satisfied for every weighting function. Following a Newton-Raphson iteration procedure, the exact solutions are written as the sum of an approximation and a deviation. Subsequently linearisation is carried out.

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot (\sigma^* + \delta\sigma)(A^* + \delta A)(\vec{n}^* + \delta\vec{n})\, ds_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(s_0)$$

Linearisation with assumption $\delta A \approx 0$ leads to an iterative weighted residual integral.

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \delta\sigma A^* \vec{n}^* \, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \delta\vec{n} \, ds_0 = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^* \, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

2.3 Material model $\rightarrow$ iterative stress change

The material model relates the stress σ to the elongation λ. Using this relation the iterative change $\delta\sigma$ can be expressed in the iterative displacement $\delta\vec{u}$.

$$\sigma = \sigma(\lambda) \quad \rightarrow \quad \delta\sigma = \left.\frac{d\sigma}{d\lambda}\right|^* \delta\lambda = \left.\frac{d\sigma}{d\lambda}\right|^* \frac{d(\delta s)}{ds_0} = \left.\frac{d\sigma}{d\lambda}\right|^* \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0}$$

2.4 Rotation $\rightarrow$ iterative orientation change

Due to the rotation of the truss, the axis direction vector $\vec{n}$ is also a function of the iterative displacement. The vector $\vec{m}$ is a unit vector perpendicular to $\vec{n}$.

$$\begin{aligned}
\vec{n} &= \frac{d\vec{x}}{ds} = \frac{ds_0}{ds}\frac{d\vec{x}}{ds_0} = \frac{1}{\lambda}\frac{d\vec{x}}{ds_0} \\
\delta\vec{n} &= \left[-\frac{1}{\lambda^2}\frac{d\vec{x}}{ds_0}\right]^* \delta\lambda + \left[\frac{1}{\lambda}\right]^* \frac{d(\delta\vec{x})}{ds_0} = \left[-\frac{1}{\lambda}\vec{n}\right]^* \delta\lambda + \left[\frac{1}{\lambda}\right]^* \frac{d(\delta\vec{x})}{ds_0} \\
&= \left[-\frac{1}{\lambda}\vec{n}\vec{n}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0} + \left[\frac{1}{\lambda}\right]^* \frac{d(\delta\vec{u})}{ds_0} = \left[(\boldsymbol{I} - \vec{n}\vec{n})\frac{1}{\lambda}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0} \\
&= \left[\vec{m}\vec{m}\frac{1}{\lambda}\right]^* \cdot \frac{d(\delta\vec{u})}{ds_0}
\end{aligned}$$

2.5 Iterative weighted residual integral

The expressions for $\delta\sigma$ and $\delta\vec{n}$ are substituted into the iterative weighted residual integral.

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \left(\left.\frac{d\sigma}{d\lambda}\right|^* \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0} \right) A^* \vec{n}^* \, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \left(\vec{m}^* \vec{m}^* \cdot \frac{1}{\lambda^*} \frac{d(\delta\vec{u})}{ds_0} \right) ds_0 = f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^* \, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

$$\int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{n}^* \left(\frac{d\sigma}{d\lambda}\bigg|^* A^* \right) \vec{n}^* \cdot \frac{d(\delta\vec{u})}{ds_0}\, ds_0 + \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \vec{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*} \right) \vec{m}^* \cdot \frac{d(\delta\vec{u})}{ds_0}\, ds_0$$
$$= f_{e0}(\vec{w}) - \int_{s_0=0}^{s_0=\ell_0} \frac{d\vec{w}}{ds_0} \cdot \sigma^* A^* \vec{n}^*\, ds_0 \qquad \forall \quad \vec{w}(s_0)$$

3 Finite element method for nonlinear truss

The mechanical behavior of truss structures, which are build from nonlinear trusses, which may show large elongations and (thus) large rotations, can be analyzed with the finite element method. Individual truss elements are considered first, which means that the structure is discretized. Later the contributions of all trusses will be combined in an assembling procedure.

3.1 Element equation

We start with the weighted residual integral for one truss element, which length is ℓ_0^e in the initial state and ℓ^e in the current state. First, the global coordinate s_0 is replaced by a local element coordinate ξ.

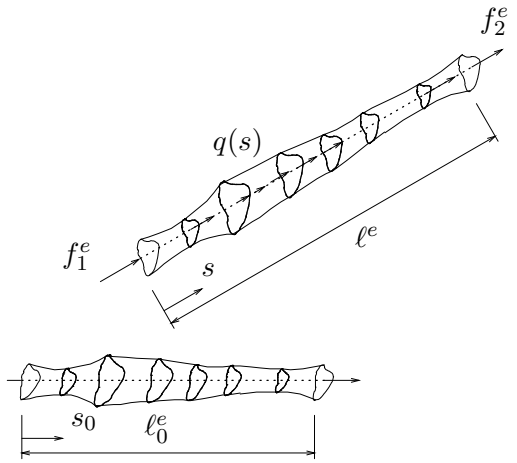

Fig. 3.14 : *Inhomogeneous truss element in undeformed state*

local coordinate : $\quad -1 \le \xi \le 1 \quad ; \quad ds_0 = \dfrac{l_0}{2}\, d\xi \quad ; \quad \dfrac{d(\ )}{ds_0} = \dfrac{2}{l_0}\, \dfrac{d(\ )}{d\xi}$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{n}^* \left(\left. \frac{d\sigma}{d\lambda} \right|^* A^* \frac{2}{l_0} \right) \vec{n}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi +$$

$$\int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \vec{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*} \frac{2}{l_0} \right) \vec{m}^* \cdot \frac{d(\delta\vec{u})}{d\xi}\, d\xi = f_{e0}^e(\vec{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\vec{w}}{d\xi} \cdot \sigma^* A^* \vec{n}^*\, d\xi$$

The vectors in the weighted residual integral are written in components with respect to a vector basis.

$$\int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{n}^* \left(\frac{d\sigma}{d\lambda}\bigg|^* A^* \frac{2}{l_0} \right) \underset{\sim}{n}^{*T} \frac{d(\delta\underset{\sim}{u})}{d\xi}\, d\xi + \int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \underset{\sim}{m}^* \left(\sigma^* A^* \frac{1}{\lambda^*} \frac{2}{l_0} \right) \underset{\sim}{m}^{*T} \frac{d(\delta\underset{\sim}{u})}{d\xi}\, d\xi$$
$$= f^e_{e0}(\underset{\sim}{w}) - \int_{\xi=-1}^{\xi=1} \frac{d\underset{\sim}{w}^T}{d\xi}\, \sigma^* A^* \underset{\sim}{n}^*\, d\xi$$

3.2 Interpolation

Both the iterative displacement and the weighting function components are interpolated between their values in the element nodes. Here we use a linear interpolation between two nodal values. The element nodes are located in the begin and end points of the element. Following the Galerkin procedure, the interpolation functions for $\delta\underset{\sim}{u}$ and $\underset{\sim}{w}$ are taken to be the same.

The derivatives of $\delta\underset{\sim}{u}$ and $\underset{\sim}{w}$ can also be interpolated directly.

$$\delta\underset{\sim}{u}^T = \left[\ \delta u_1 \quad \delta u_2\ \right] = \left[\ \delta u_{11}\psi^1 + \delta u_{21}\psi^2 \quad \delta u_{12}\psi^1 + \delta u_{22}\psi^2\ \right]$$
$$\underset{\sim}{w}^T = \left[\ w_{11}\psi^1 + w_{21}\psi^2 \quad w_{12}\psi^1 + w_{22}\psi^2\ \right]$$

$$\text{with} \qquad \psi^1(\xi) = \tfrac{1}{2}(1-\xi) \quad ; \quad \psi^2(\xi) = \tfrac{1}{2}(1+\xi)$$

$$\frac{d(\delta\underset{\sim}{u})}{d\xi} = \begin{bmatrix} \dfrac{d(\delta u_1)}{d\xi} \\ \dfrac{d(\delta u_2)}{d\xi} \end{bmatrix} = \begin{bmatrix} \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} & 0 & \frac{d\psi^2}{d\xi} \end{bmatrix} \begin{bmatrix} \delta u_{11} \\ \delta u_{12} \\ \delta u_{21} \\ \delta u_{22} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \delta\underset{\sim}{u}^e$$

$$\frac{d\underset{\sim}{w}^T}{d\xi} = \begin{bmatrix} \dfrac{dw_1}{d\xi} & \dfrac{dw_2}{d\xi} \end{bmatrix} = \begin{bmatrix}\ w_{11} & w_{12} & w_{21} & w_{22}\ \end{bmatrix} \begin{bmatrix} \frac{d\psi^1}{d\xi} & 0 \\ 0 & \frac{d\psi^1}{d\xi} \\ \frac{d\psi^2}{d\xi} & 0 \\ 0 & \frac{d\psi^2}{d\xi} \end{bmatrix} = \underset{\sim}{w}^{eT} \tfrac{1}{2} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Substitution of the interpolated variables leads to an element integral equation, where the internal nodal forces and the element tangential stiffness matrix can be recognized.

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c \\ s \end{bmatrix}^* \tfrac{1}{4} \left(\frac{d\sigma}{d\lambda}\bigg|^* A^* \frac{2}{l_0} \right) \begin{bmatrix}\ c & s\ \end{bmatrix}^* \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi\, \delta\underset{\sim}{u}^e +$$
$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -s \\ c \end{bmatrix}^* \tfrac{1}{4} \left(\sigma^* A^* \frac{1}{\lambda^*} \frac{2}{l_0} \right) \begin{bmatrix}\ -s & c\ \end{bmatrix}^* \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} d\xi\, \delta\underset{\sim}{u}^e$$

$$= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tfrac{1}{2} \begin{bmatrix} c \\ s \end{bmatrix}^* (\sigma^* A^*)\, d\xi$$

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left| \frac{d\sigma}{d\lambda} \right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* \begin{bmatrix} -c & -s & c & s \end{bmatrix}^* d\xi\, \delta\underset{\sim}{u}^e +$$

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) \begin{bmatrix} s \\ -c \\ -s \\ c \end{bmatrix}^* \begin{bmatrix} s & -c & -s & c \end{bmatrix}^* d\xi\, \delta\underset{\sim}{u}^e$$

$$= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2}\, (\sigma^* A^*) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* d\xi$$

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left| \frac{d\sigma}{d\lambda} \right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* d\xi\, \delta\underset{\sim}{u}^e +$$

$$\underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^* d\xi\, \delta\underset{\sim}{u}^e$$

$$= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2}\, (\sigma^* A^*) \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^* d\xi$$

With the introduction of some proper matrices and columns, the element equation can be written in short form.

$$\underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \left| \frac{d\sigma}{d\lambda} \right|^* A^* \frac{1}{l_0} \right) d\xi\, \underline{\underline{M}}^*_L \right] \delta\underset{\sim}{u}^e + \underset{\sim}{w}^{eT} \left[\int_{\xi=-1}^{\xi=1} \left(\tfrac{1}{2} \sigma^* A^* \frac{1}{\lambda^*} \frac{1}{l_0} \right) d\xi\, \underline{\underline{M}}^*_N \right] \delta\underset{\sim}{u}^e$$
$$= f^e_{e0}(\underset{\sim}{w}^e) - \underset{\sim}{w}^{eT} \int_{\xi=-1}^{\xi=1} \tfrac{1}{2}\, (\sigma^* A^*)\, \underset{\sim}{V}^*\, d\xi$$

$$\underset{\sim}{w}^{eT} \underline{K}^{e^*} \delta\underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}^e_{e0} - \underset{\sim}{w}^{eT} \underset{\sim}{f}^{e^*}_i = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^*}$$

3.3 Integration

Integation over the element length is needed to determine the element stiffness matrix $\underline{K}^{e^*}$ and the internal force column $\underset{\sim}{f}_i^{e^*}$.

For a homogeneous element, e.g. an element with uniform cross-sectional area and material properties, this leads to the following expressions.

$$\underline{K}^{e^*} = \left(\left. \frac{d\sigma}{d\lambda} \right|^* A^* \frac{1}{l_0} \right) \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}^* + \left(\sigma^* A^* \frac{1}{l^*} \right) \begin{bmatrix} s^2 & -cs & -s^2 & cs \\ -cs & c^2 & cs & -c^2 \\ -s^2 & cs & s^2 & -cs \\ cs & -c^2 & -cs & c^2 \end{bmatrix}^*$$

$$\underset{\sim}{f}_i^{e^*} = \sigma^* A^* \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix}^*$$

3.4 Assembling

The contributions of the individual elements are added in the assembling procedure. The result is an integral equation for the total system, which, according to the principle of weighted residuals, has to be satisfied for every column with nodal weighting function values. This requirement leads to a system of algebraic equations from which the iterative nodal displacement components must be solved.

element contribution	$\underset{\sim}{w}^{eT} \underline{K}^{e^*} \delta \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}_{e0}^e - \underset{\sim}{w}^{eT} \underset{\sim}{f}_i^{e^*} = \underset{\sim}{w}^{eT} \underset{\sim}{r}^{e^*}$
assembled equation	$\underset{\sim}{w}^T \underline{K}^* \delta \underset{\sim}{u} = \underset{\sim}{w}^T \underset{\sim}{f}_{e0} - \underset{\sim}{w}^T \underset{\sim}{f}_i^* = \underset{\sim}{w}^T \underset{\sim}{r}^* \qquad \forall\ \underset{\sim}{w}$
iterative equation system	$\underline{K}^* \delta \underset{\sim}{u} = \underset{\sim}{r}^*$

3.5 Boundary conditions

Boundary conditions are only applied at the beginning of an incremental step. Links – relations between degrees of freedom – can be incorporated as usual, but now of course for the iterative displacements.

3.6 Program structure

A finite element program starts with reading data from an input file and initialization of variables and databases.

The loading is prescribed as a function of the (fictitious) time in an incremental loop. In each increment the system of nonlinear equilibrium equations is solved iteratively.

In each iteration loop the system of equations is build. In a loop over all elements, the stresses are calculated and the material stiffness is updated. The element internal nodal force column and the element stiffness matrix are assembled into the global column and matrix.

After taking tyings and boundary conditions into account, the unknown nodal displacements and reaction forces are calculated.

When the convergence criterion is not reached, a new iteration step is performed. After convergence output data are stored and the next incremental step is carried out.

```
read input data from input file
calculate additional variables from input data
initialize values and arrays

while load increments to be done

   for all elements
      calculate initial element stiffness matrix
      assemble global stiffness matrix
   end element loop

   determine external incremental load from input

   while non-converged iteration step

      take tyings into account
      take boundary conditions into account

      calculate iterative nodal displacements
      calculate total deformation

      for all elements
         calculate stresses from material behavior
         calculate material stiffness from material behavior
         calculate element internal nodal forces
         calculate element stiffness matrix
         assemble global stiffness matrix
         assemble global internal load column
      end element loop

      calculate residual load column
```

```
      calculate convergence norm

   end iteration step

   store data for post-processing
end load increment
```

3.7 FE program `tr2d`

The Matlab program `tr2d` is used to model and analyze two-dimensional truss structures, where large deformations and nonlinear material behavior may occur.

In this section, examples of two-dimensional truss structures are shown. The material behavior is always elastic and described by a linear relation between the Cauchy stress and the linear strain. Other material models have also been implemented in the program.

Large deformation of a truss structure

A structure is made of five trusses. The vertical truss is 0.5 m and the horizontal truss is 1 m in length. Cross-sectional areas are 100 mm^2. The modulus is 2.5 GPa. Contraction is not considered ($\nu = 0$). The vertical displacement of node 4 is prescribed to increase from 0 to -0.25 m. The reaction force, the horizontal displacement of node 4 and the vertical displacement of node 2 is plotted against the fictitious time t.

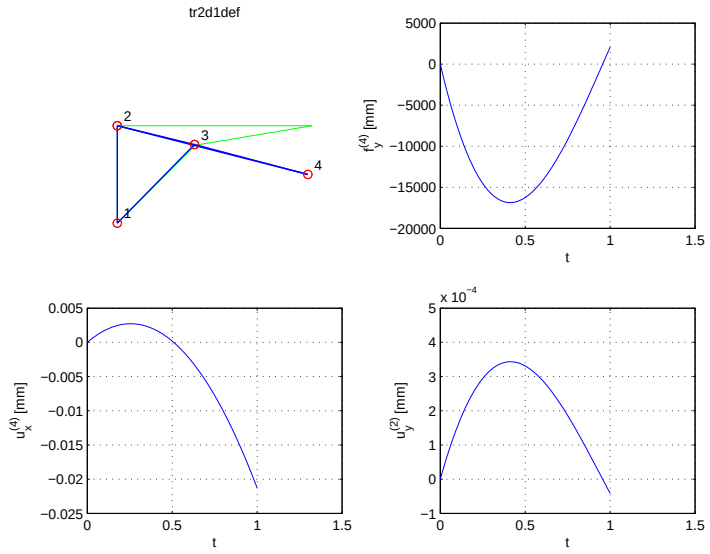

Fig. 3.15 : *Large deformation of a truss structure.*

Buckling

Large rotations occur when buckling leads to a sudden increase in deformation. The theoretical buckling load can be calculated analytically for a simple systems as shown here.

The numerical calculation starts with a very small imperfection being an initial vertical displacement of the inner node(s) of $\pm$ 0.0001 m. This allows us to reach not only the first and smallest buckled state, the symmetric shape, but also the second mode, the anti-symmetric shape. Also a larger imperfection is analized for both buckling modes.

The horizontal trusses have a high stiffness of $k_t = (EA)/l = (100e9)(100e-6)/1$ N/m, while the springs have a very low stiffness of $k = 1$ N/m. The displacement in node 4 is prescribed to increase from 0 to -0.02 m.

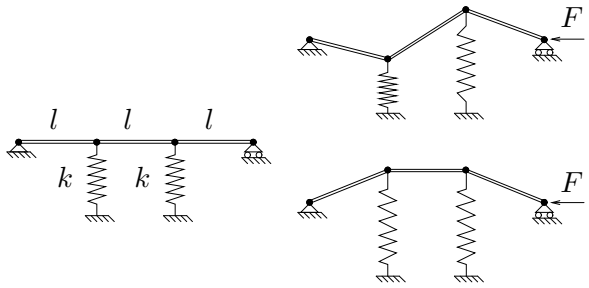

Fig. 3.16 : *Symmetric and anti-symmetric buckling.*

$$\text{symm} \;:\; F_c = \frac{kl}{3} \quad ; \quad \text{anti-symm} \;:\; F_c = kl$$

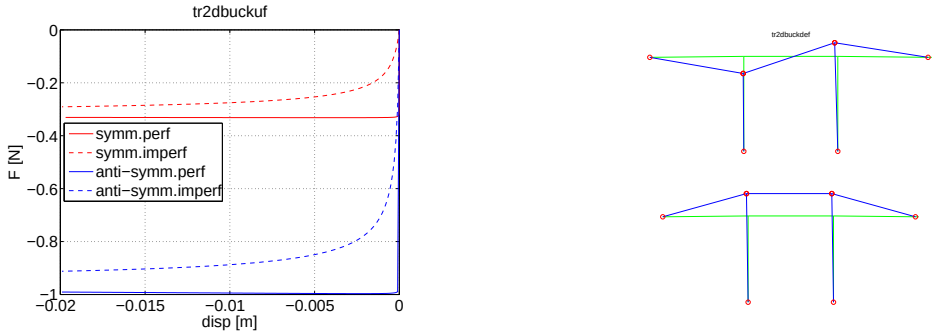

Fig. 3.17 : *Buckling forces versus displacement (left). Symmetric and anti-symmetric buckling shapes (right).*