

ONE-DIMENSIONAL MATERIAL BEHAVIOR

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1 One-dimensional material models

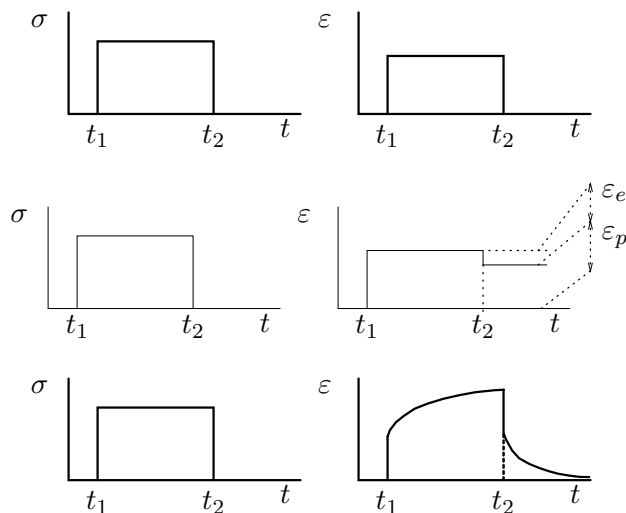
In the following sections material behavior is described in a one-dimensional context. and used to analyze the axial deformation of trusses. The material behavior is modeled, using a discrete mechanical model of springs, dashpots and friction sliders. The axial stress is related to the axial strain by one or more (differential) equation(s) from which the stress response must be calculated when the strain excitation is prescribed. This stress update procedure is implemented in Matlab files.

The various material models are incorporated in a finite element program, which is used to model and analyze the mechanical behavior of truss structures, subjected to prescribed displacements and/or forces. In the iterative solution procedure, the material stiffness plays an essential role and must be derived from the material law.

2 Material behavior

Characterization of the mechanical behavior of an unknown material almost always begins with performing a tensile experiment. A stepwise change in the axial stress σ may be prescribed and the strain ε of the tensile bar can be measured and plotted as a function of time. From these plots important conclusions can be drawn concerning the material behavior.

For *elastic* material behavior the strain follows the stress immediately and becomes zero after stress release. For *elastoplastic* material behavior the strain also follows the stress immediately, but there is permanent deformation after stress release. When the material is *viscoelastic* the strain shows time delayed response on a stress step, which indicates a time dependent behavior. When time dependent behavior is accompanied by permanent deformation, the behavior is referred to as *viscoplastic*.



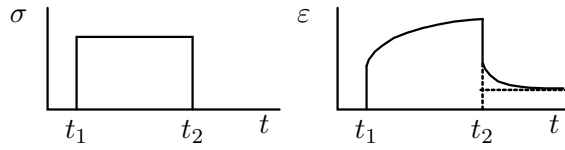


Fig. 2.1 : Strain response for a stress-step for a) elastic b) elastoplastic, c) viscoelastic and d) viscoplastic material behavior

Another way of representing the measurement data of the tensile experiment is by plotting the stress against the strain, resulting in the stress-strain curve. The relation between stress and strain may be *linear* or *nonlinear*. Also, the relation may be *history dependent*, due to changes in the material structure. Different behavior in tensile and compression may be observed.

2.1 Tensile curve : elastic behavior

When elastic behavior is well described by a linear relation between stress and strain, the elastic behavior is referred to as linear.

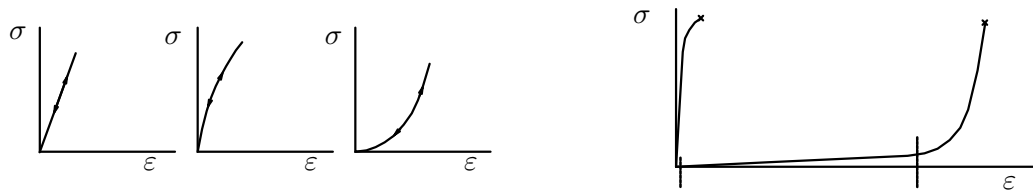


Fig. 2.2 : Tensile curves for elastic material behavior

2.2 Tensile curve : viscoelastic behavior

Viscoelastic behavior is time-dependent. The stress is a function of the strain rate. There is a phase difference between stress and strain, which results in a hysteresis loop when the loading is periodic.

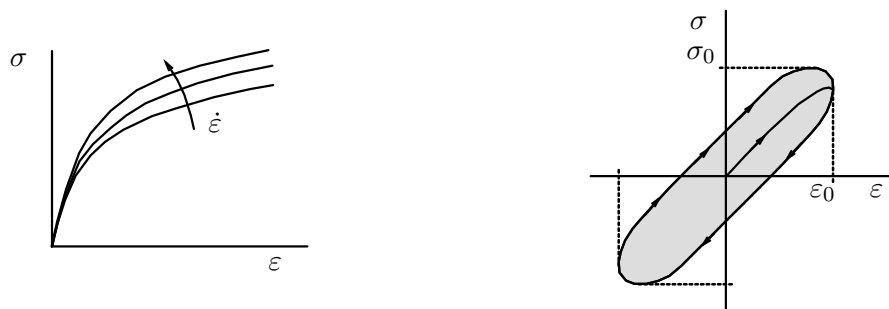


Fig. 2.3 : Tensile curve and hysteresis loop for viscoelastic material behavior

2.3 Tensile curve : elastoplastic behavior

When a material is loaded or deformed above a certain threshold, the resulting deformation will be permanent or plastic. When time (strain rate) is of no importance, the behavior is referred to as elastoplastic. Stress-strain curves may indicate different characteristics, especially when the loading is reversed from tensile to compressive.

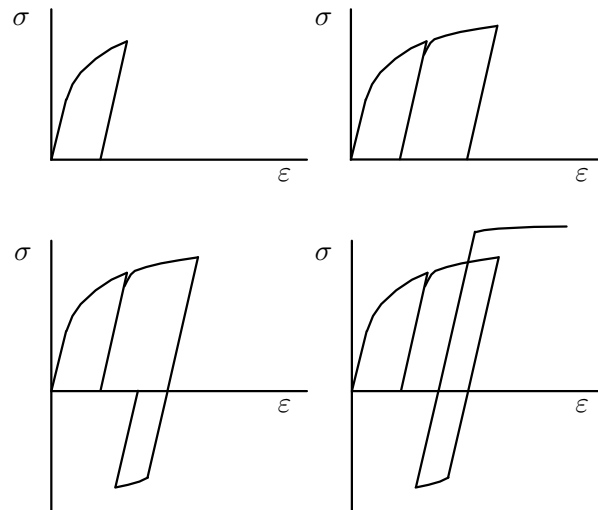


Fig. 2.4 : Tensile curves for elastoplastic material behavior

2.4 Tensile curve : viscoplastic behavior

A combination of plasticity and time-dependency is called viscoplastic behavior. This behavior is often observed for polymeric materials. For some polymers the stress reaches a maximum and subsequently drops with increasing strain. This phenomenon is referred to as *intrinsic softening*. In a tensile experiment it will provoke necking of the tensile bar.

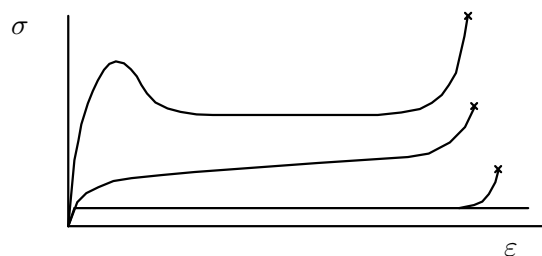


Fig. 2.5 : Tensile curves for viscoplastic material behavior

2.5 Tensile curve : damage

Structural damage influences the material properties. The onset and evolution of damage can be described with a damage model. For materials like concrete and ceramics, the onset and propagation of damage causes softening. Because damage is often associated with the initiation and growth of voids, the stress-strain curve is different for tensile and compressive loading.

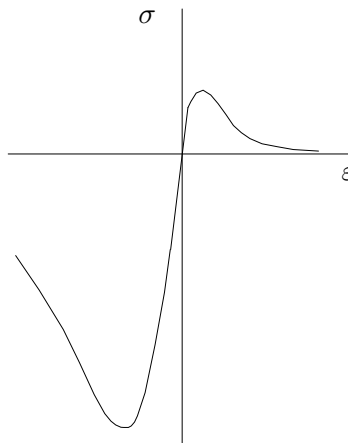
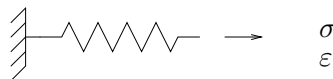


Fig. 2.6 : *Tensile curve for damaging material with different behavior in tension and compression*

2.6 Discrete material models

Material models relate stresses to deformation and possibly deformation rate. For three-dimensional continua the material model is often represented by a (large) number of coupled (differential) equations. As a simplified introduction, we will present material models first in a one-dimensional setting. The material behavior is represented by the behavior of a one-dimensional, discrete, mechanical system of springs, dashpots and friction sliders. For such a system the relation between the axial stress σ and the axial strain ε can be derived.

In the following sections models for elastic, elastoplastic, linear viscoelastic, creep, viscoplastic and nonlinear viscoelastic behavior will be presented.



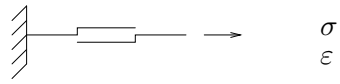


Fig. 2.7 : *Discrete elements : spring, dashpot and friction slider*

3 Elastic material behavior

When a material behaves elastically, the current stress can be calculated directly from the current strain, because there is no path and/or time dependency. When the stress is released, the strain will become zero, so there is no permanent deformation at zero stress. All stored strain energy is released and there is no energy dissipation. For the one-dimensional case of an axially loaded truss the elastic behavior is described by a relation between the stress σ and the elongation factor λ or the strain ε .

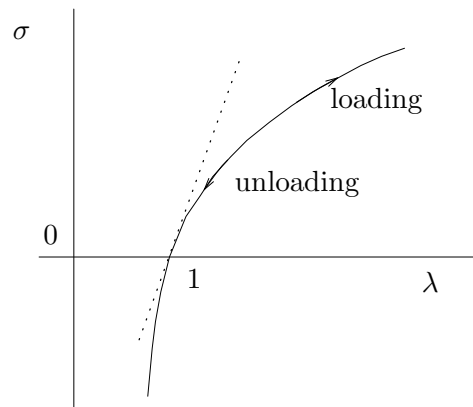


Fig. 3.8 : *Non-linear elastic material behavior*

3.1 Small strain elastic behavior

For small elongations, all strain definitions are the same, as are all stress definitions. The relation between stress and strain is linear and the constant material parameter is the Young's modulus.

strain $\varepsilon = \varepsilon_{gl} = \varepsilon_{ln} = \varepsilon_l = \lambda - 1$

stress $\sigma = \frac{F}{A} = \frac{F}{A_0} = \sigma_n$

linear elastic behavior $\sigma = E\varepsilon = E(\lambda - 1)$

modulus $E = \lim_{\lambda \rightarrow 1} \frac{d\sigma}{d\lambda} = \lim_{\varepsilon \rightarrow 0} \frac{d\sigma}{d\varepsilon}$

3.2 Large strain elastic behavior

For large deformations, nonlinear elastic behavior can be observed in polymers, elastomeric materials (rubbers) and, on a small scale, in atomic bonds, when a tensile/compression test is carried out and the axial force F is plotted as a function of λ . In a material model we

want to describe such behavior with a mathematical relation between a stress and a strain. Consideration of the stored elastic energy per unit of volume learns that each stress definition is associated with a certain strain definition, so these should be combined in a material model. However, when the observed material behavior is described accurately by another stress/strain combination, it can be used as well.

For three-dimensional models more considerations have to be taken into account. Care has to be taken that the material model does not generate stresses for large rigid body rotations of the material, which is known as the requirement of *objectivity*.

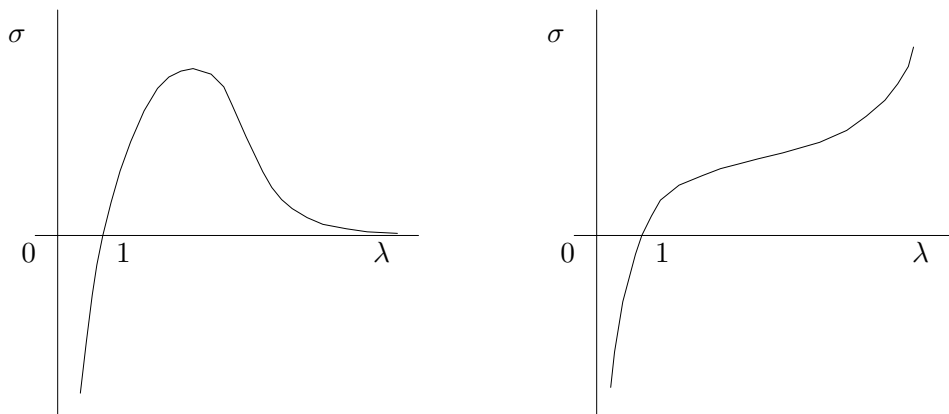


Fig. 3.9 : Non-linear stress-strain relations for an atomic bond and for an elastomeric material

3.3 Elasticity models

The discrete one-dimensional model for elastic material behavior is a spring. The behavior is modeled with a relation between the stress σ and the elongation factor λ or a strain ε . The material stiffness C_λ is the derivative of σ w.r.t. the stretch ratio λ . The derivative w.r.t. the strain ε results in the stiffness C_ε .

Consideration of the stored elastic energy per unit of material volume (see ??) learns that, in a material model, true stress σ should be combined with logarithmic strain ε_{ln} , engineering stress σ_n with linear strain ε_l or 2nd-Piola-Kirchhoff stress σ_{p2} with Green-Lagrange strain ε_{gl} . Experimentally observed tensile behavior can often be described with a linear relation between a certain stress and its associated strain.

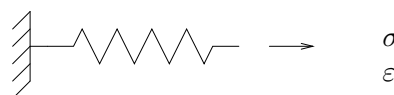


Fig. 3.10 : Spring

constitutive equation $\sigma = \sigma(\lambda)$

stiffness $C_\lambda = \frac{d\sigma}{d\lambda} = \frac{d\sigma}{d\varepsilon} \frac{d\varepsilon}{d\lambda} = C_\varepsilon \frac{d\varepsilon}{d\lambda}$

4 Hyper-elastic models

Elastomeric materials (rubbers) show very large elastic deformations (elongation up to $\lambda = 5$). The material models for these materials are therefore referred to as *hyper-elastic*. They are derived from an elastic energy function, which has to be determined experimentally. The three-dimensional versions of these so-called Rivlin or Mooney models are expressed in the principal elongation factors $\lambda_i, i = 1, 2, 3$. Experimental observations indicate that elastomeric materials are incompressible, so that we have $\lambda_1 \lambda_2 \lambda_3 = 1$.

$$W = \sum_i^n \sum_j^m C_{ij} (I_1 - 3)^i (I_2 - 3)^j \quad \text{with } C_{00} = 0$$

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

$$I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 = \frac{1}{\lambda_3^2} + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}$$

The incremental change of the elastically stored energy per unit of deformed volume, can be expressed in the principal stresses and the principal logarithmic strains.

$$dW = \sigma_1 d\varepsilon_{ln1} + \sigma_2 d\varepsilon_{ln2} + \sigma_3 d\varepsilon_{ln3}$$

4.1 Mooney models

For incompressible materials like elastomer's (rubber) the stored elastic energy per unit of deformed volume is specified and fitted onto experimental data. Several specific energy functions are used.

Neo-Hookean $W = C_{10} (I_1 - 3)$

Mooney-Rivlin $W = C_{10} (I_1 - 3) + C_{01} (I_2 - 3)$

Signiorini $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2$

Yeoh $W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$

Klosner-Segal $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{03}(I_2 - 3)^3$

2-order invariant $W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2$

Third-order model of James, Green and Simpson

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + \\ C_{20}(I_1 - 3)^2 + C_{02}(I_2 - 3)^2 + C_{21}(I_1 - 3)^2(I_2 - 3) + \\ C_{30}(I_1 - 3)^3 + C_{03}(I_2 - 3)^3 + C_{12}(I_1 - 3)(I_2 - 3)^2$$

4.2 Ogden models

For 'slightly' compressible materials the Ogden specific energy functions are used. Because the volume change is not zero, these functions depend on the volume change factor $J = \lambda_1 \lambda_2 \lambda_3$. The second part of the energy function accounts for the volumetric deformation. Because the volumetric behavior is characterized by a constant bulk modulus K , the model is confined to slightly compressible deformation.

For 'highly' compressible materials like foams, specific energy functions also exist. The first part of the energy function also describes volume change.

slightly compressible $W = \sum_{i=1}^N \frac{a_i}{b_i} \left[J^{-\frac{b_i}{3}} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} \right) - 3 \right] + 4.5K \left(J^{\frac{1}{3}} - 1 \right)^2$

highly compressible $W = \sum_{i=1}^N \frac{a_i}{b_i} \left(\lambda_1^{b_i} + \lambda_2^{b_i} + \lambda_3^{b_i} - 3 \right) + \sum_{i=1}^N \frac{a_i}{c_i} (1 - J^{c_i})$

4.3 One-dimensional models

For tensile (or compressive) loading of a homogeneous and isotropic truss, where the axial direction is taken to be the 1-direction, we have : $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda}$. In this case there is only an axial stress $\sigma_1 = \sigma$, so that we have

$$dW = \sigma d\varepsilon_{ln} \quad \rightarrow \quad \sigma = \frac{dW}{d\varepsilon_{ln}} = \frac{dW}{d\lambda} \frac{d\lambda}{d\varepsilon_{ln}} = \frac{dW}{d\lambda} \lambda$$

The Neo-Hookean model is the simplest model as it contains only one material parameter. Axial stress σ and axial force F can be calculated easily. From statistical mechanics it is known that for an ideal rubber material the stress is :

$$\sigma = \frac{\rho RT}{M} \left(\lambda^2 - \frac{1}{\lambda} \right) \quad \text{with} \quad \begin{array}{ll} \rho & : \text{ density} \\ R & : \text{ gas constant} = 8.314 \text{ JK}^{-1}\text{mol}^{-1} \\ T & : \text{ absolute temperature} \\ M & : \text{ average molecular weight} \end{array}$$

Most rubber materials cannot be characterized well with the Neo-Hookean model. The more complex Mooney-Rivlin model yields better results. The stiffness C_λ is a function of the elongation factor λ . The initial stiffness E is often referred to as the modulus.

Neo – Hookean

$$\begin{aligned}W &= C_{10} \left(\lambda^2 + \frac{2}{\lambda} - 3 \right) \\ \sigma &= C_{10} \left(2\lambda - \frac{2}{\lambda^2} \right) \lambda = 2C_{10} \left(\lambda^2 - \frac{1}{\lambda} \right) \\ C_\lambda &= \frac{\partial \sigma}{\partial \lambda} = 2C_{10} \left(2\lambda + \frac{1}{\lambda^2} \right) \quad ; \quad E = \lim_{\lambda \rightarrow 1} \frac{\partial \sigma}{\partial \lambda} = 6C_{10} \\ F &= \sigma A = \sigma \frac{1}{\lambda} A_0 = 2C_{10} A_0 \left(\lambda - \frac{1}{\lambda^2} \right)\end{aligned}$$

Mooney – Rivlin

$$\begin{aligned}W &= C_{10} \left(\lambda^2 + \frac{2}{\lambda} - 3 \right) + C_{01} \left(\frac{1}{\lambda^2} + 2\lambda - 3 \right) \\ \sigma &= 2C_{10} \left(\lambda^2 - \frac{1}{\lambda} \right) + 2C_{01} \left(\lambda^2 - \frac{1}{\lambda} \right) \frac{1}{\lambda} \\ C_\lambda &= \frac{\partial \sigma}{\partial \lambda} = 2C_{10} \left(2\lambda + \frac{1}{\lambda^2} \right) + 2C_{01} \left(1 + \frac{2}{\lambda^3} \right) \quad ; \quad E = \lim_{\lambda \rightarrow 1} \frac{\partial \sigma}{\partial \lambda} = 6(C_{10} + C_{01}) \\ F &= \sigma A = \sigma \frac{1}{\lambda} A_0 = A_0 \frac{1}{\lambda} \left[2C_{10} \left(\lambda^2 - \frac{1}{\lambda} \right) + 2C_{01} \left(\lambda^2 - \frac{1}{\lambda} \right) \frac{1}{\lambda} \right]\end{aligned}$$

4.4 Examples

Some examples of analytical solutions

4.5 Limitations of axial stretch

A cylindrical tensile bar with initial length l_0 and initial cross-sectional area A_0 is loaded with an axial force F . The elongation is described by the stretch ratio $\lambda = \frac{l}{l_0}$. The contraction is described by the stretch ratio $\mu = \sqrt{\frac{A}{A_0}}$.

The material is homogeneous and the elastic behavior is described by a linear relation between the Cauchy stress σ and the Green-Lagrange strain $\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$.

$$\sigma = C(\lambda^2 - 1) \quad \text{with} \quad C = \text{constant} > 0$$

In each direction the same strain definition must be used, so the contraction strain is

$$\varepsilon_d = \frac{1}{2}(\mu^2 - 1)$$

When the contraction strain is related to the axial strain with Poisson's ratio ν , we have

$$\varepsilon_d = \frac{1}{2}(\mu^2 - 1) = -\nu \varepsilon_{gl} = -\nu \frac{1}{2}(\lambda^2 - 1) \quad \rightarrow \quad \mu^2 = 1 - \nu(\lambda^2 - 1)$$

When Poisson's ration is assumed to be constant, the axial elongation is limited because the cross-sectional area obviously cannot become zero.

$$1 - \nu(\lambda^2 - 1) = 0 \rightarrow \nu(\lambda^2 - 1) = 1 \rightarrow \lambda^2 - 1 = \frac{1}{\nu} \rightarrow \lambda^2 = \frac{1 + \nu}{\nu} \rightarrow \lambda = \sqrt{\frac{1 + \nu}{\nu}}$$

When the cross-sectional area is plotted as a function of λ with the value $\nu = 0.25$, we clearly see the limit value for λ where $A = 0$.

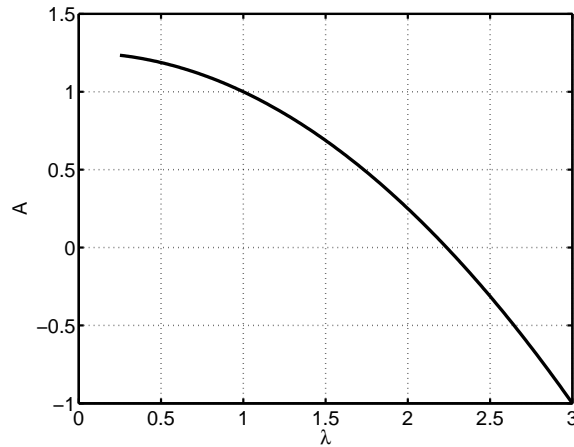


Fig. 4.11 : *Cross-sectional area versus stretch ratio.*

The axial force F can be calculated and expressed as function of λ .

$$\begin{aligned} F &= \sigma A = \sigma \mu^2 A_0 = \sigma \{1 - \nu(\lambda^2 - 1)\} A_0 \\ &= C A_0 \{1 - \nu(\lambda^2 - 1)\} (\lambda^2 - 1) \end{aligned}$$

When we plot this relation for values $\nu = 0.25$ and $A_0 = 1$, it becomes clear that the proposed material law has some physical inconsistencies.

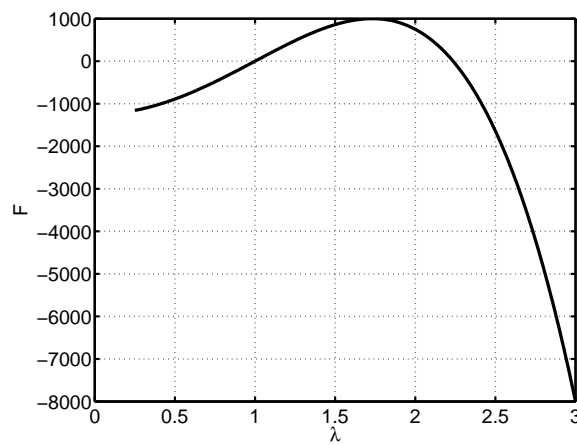


Fig. 4.12 : *Axial force versus stretch ratio.*

The volume change ratio is :

$$J = \lambda \mu^2 = \lambda \{1 - \nu(\lambda^2 - 1)\}$$

When the material is assumed to be incompressible, $J = 1$ and Poisson's ratio ν cannot be constant any more but is a function of λ :

$$\begin{aligned} \mu^2 &= \frac{1}{\lambda} = 1 - \nu(\lambda^2 - 1) \quad \rightarrow \\ \nu(\lambda^2 - 1) &= 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda} \quad \rightarrow \quad \nu = \frac{\lambda - 1}{\lambda(\lambda^2 - 1)} = \frac{1}{\lambda(\lambda + 1)} \end{aligned}$$

For very small elongations, the value of ν becomes $\frac{1}{2}$, which is already known from three-dimensional Hooke's law for linear elasticity.

4.6 Instability and localisation

A tensile test has revealed that the elastic behavior of a material is described very well by a linear relation between the engineering stress σ_n and a nonlinear function of the elongation factor λ .

$$\sigma_n = C \frac{1}{\lambda} \ln(\lambda) \quad \text{with} \quad \begin{array}{l} \sigma_n = \text{engineering stress} \\ C = \text{elasticity constant} > 0 \\ \lambda = \text{axial elongation factor} \end{array}$$

The undeformed cross-sectional area of the truss is A_0 . Poisson's ratio is ν and is assumed to be constant.

The relation between the axial force F and the axial stretch factor λ is

$$F = C A_0 \lambda^{-1} \ln(\lambda)$$

For values $C = 1000$ and $A_0 = 1$, this relation is shown in the figure below.

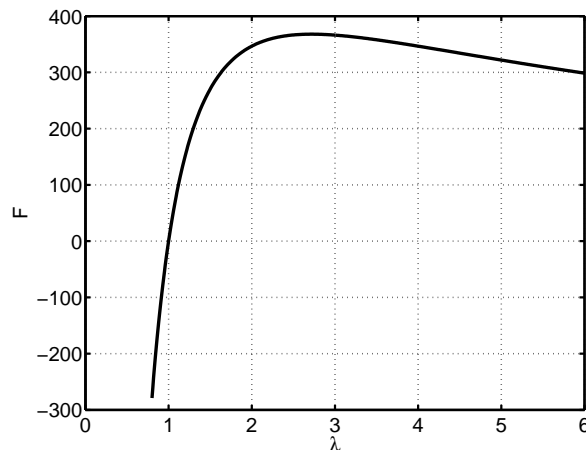


Fig. 4.13 : Axial force versus stretch ratio.

The value for λ for which the axial force reaches the maximum value can be determined by differentiation.

$$\frac{dF}{d\lambda} = CA_0 \left\{ -\frac{1}{\lambda^2} \ln(\lambda) + \frac{1}{\lambda^2} \right\} = CA_0 \frac{1}{\lambda^2} (1 - \ln(\lambda))$$

$$\frac{dF}{d\lambda} = 0 \quad \rightarrow \quad \ln(\lambda) = 1 \quad \rightarrow \quad \lambda = e = \pm 2.7$$

The maximum value of the force is

$$F_{max} = F(\lambda = e) = CA_0 \frac{1}{e} \ln(e) = CA_0 \frac{1}{e}$$

When this maximum is reached, the deformation increases while the load diminishes. Because in a real material there will always be some inhomogeneity, e.g. a cross-section with a slightly smaller area, or with a slightly lower value of C , The maximum will be reached there first. All the elongation will then be concentrated in this weakest cross-section, a phenomenon which is referred to as localisation.

4.7 Inflating a spherical balloon

A spherical balloon has an inner diameter D_0 and a uniform wall thickness $w_0 \ll D_0$ in the undeformed state. The balloon is loaded with an internal pressure p whereupon it deforms homogeneously to have an inner diameter D and wall thickness w .

To describe the mechanics of the balloon, we use three coordinate axes : two perpendicular tangential directions t and the radial direction r , as is shown in the figure below for the deformed state.

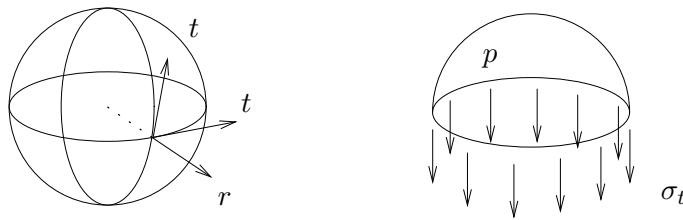


Fig. 4.14 : *Balloon in deformed state.*

The wall of the balloon is made from elastomeric material, the behavior of which is characterised by the Neo-Hookean elastic energy function, expressed in the principal stretch ratios λ_1 , λ_2 and λ_3 :

$$W = C_{10} \{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 \}$$

where C_{10} is a positive valued material constant. The material is incompressible.

In the case of the pressurized balloon, the principal directions of deformation are the perpendicular tangential directions and the radial direction in each point of the balloon wall, so we have :

$$\lambda_1 = \lambda_2 = \lambda_t = \frac{D}{D_0} = \lambda \quad \text{and} \quad \lambda_3 = \lambda_r = \frac{w}{w_0}$$

With the knowledge that the material is incompressible, the elastic energy function can be expressed in the tangential elongation factor λ .

$$J = \lambda_1 \lambda_2 \lambda_3 = \lambda_t^2 \lambda_r = 1 \quad \rightarrow \quad \lambda_r = \frac{1}{\lambda_t^2} \quad \rightarrow$$

$$W = C_{10} (\lambda_t^2 + \lambda_t^2 + \lambda_r^2 - 3) = C_{10} \left(2\lambda^2 + \frac{1}{\lambda^4} - 3 \right)$$

The principal stress directions coincide with the principal strain directions and are :

$$\sigma_1 = \sigma_2 = \sigma_t \quad \text{and} \quad \sigma_3 = \sigma_r \approx 0$$

For the isotropic hyperelastic model, the incremental specific elastic energy can be written as

$$dW = \sigma_1 d\varepsilon_{ln_1} + \sigma_2 d\varepsilon_{ln_2} + \sigma_3 d\varepsilon_{ln_3} = 2\sigma_t d\varepsilon_{ln_t} \quad \rightarrow \quad \sigma_t = \frac{1}{2} \frac{dW}{d\varepsilon_{ln_t}}$$

The tangential stress σ_t is then expressed in λ .

$$\sigma_t = \frac{1}{2} \frac{dW}{d\lambda} \lambda = 2C_{10} \left(\lambda^2 - \frac{1}{\lambda^4} \right)$$

From equilibrium in the deformed state, the relation between the internal pressure p and the tangential stress σ_t is derived.

$$p = 4\sigma_t \frac{w}{D}$$

The internal pressure can then be expressed in λ and the initial dimensions D_0 and w_0 .

$$p = 4\sigma_t \frac{w}{D} = 8C_{10} \frac{w_0}{D_0} \left(\lambda^2 - \frac{1}{\lambda^4} \right) \frac{1}{\lambda^3} = 8C_{10} \frac{w_0}{D_0} \left(\frac{1}{\lambda} - \frac{1}{\lambda^7} \right)$$

The plot of *pagainst* λ for $1 \leq \lambda \leq 6$ shows a clearly nonlinear relation. This nonlinearity comes from the large reduction of the load-carrying wall thickness and also from the nonlinear material behavior.

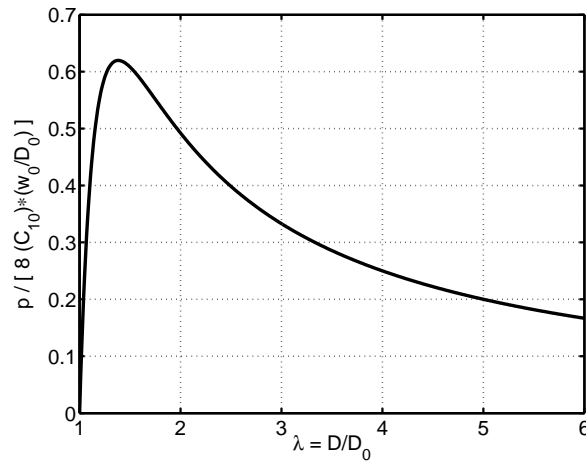


Fig. 4.15 : Pressure versus diameter change.

4.8 Stress update

The relation between σ and λ can be used to update the stress directly when the strain is known.

$$\sigma = \sigma(\lambda)$$

4.9 Stiffness

The material stiffness is determined by taking the derivative of the stress with respect to the elongation ratio or the strain.

$$C_\lambda = \frac{\partial \sigma}{\partial \lambda}$$

4.10 Implementation

See the files `tr2delas.m` and `tr2delam.m` for the implementation of the elastic and elastomeric material models.

4.11 Examples

A truss is loaded axially with a prescribed elongation/force. The initial length l_0 of the truss is 100 mm and the initial cross-sectional area A_0 is 10 mm². The axial force/elongation is calculated. The cross-sectional area will change as a function of the elongation.

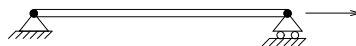


Fig. 4.16 : Tensile loading of truss element

For all elastic models the elastic constant is taken $C = 100000$ MPa and Poisson's ratio ν is 0.3. The stress-elongation results are shown in the next figures. The models with a linear relation between stress (σ or P) and Green-Lagrange strain, clearly lack a physically realistic description of the material behavior during compression.

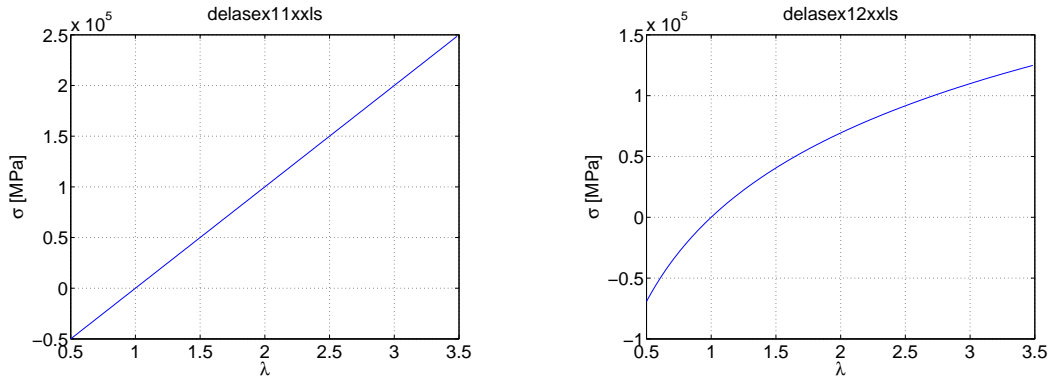


Fig. 4.17 : Stress versus λ for $\sigma \sim \varepsilon_l$ and $\sigma \sim \varepsilon_{ln}$ models

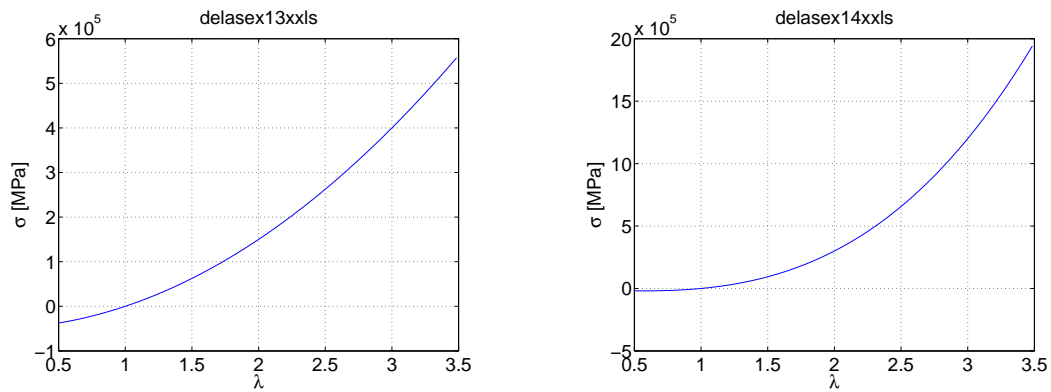


Fig. 4.18 : Stress versus λ for $\sigma \sim \varepsilon_{gl}$ and $P \sim \varepsilon_{gl}$ models

The axial force and the cross-sectional area are calculated and shown in the next figures as a function of the elongation. The cross-sectional areas of some models become zero and even negative, which clearly shows the limited use of these models.

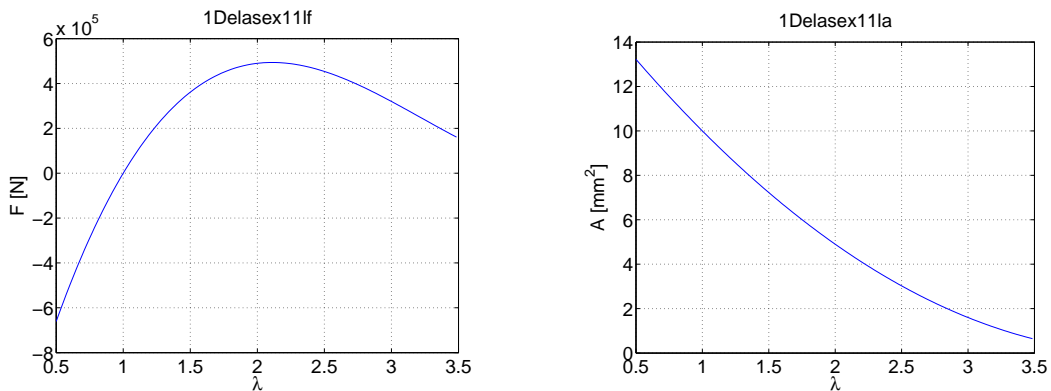


Fig. 4.19 : Axial force and cross-sectional area versus the elongation for $\sigma \sim \varepsilon_l$ model

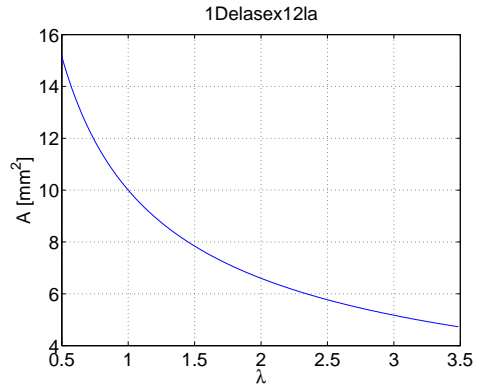
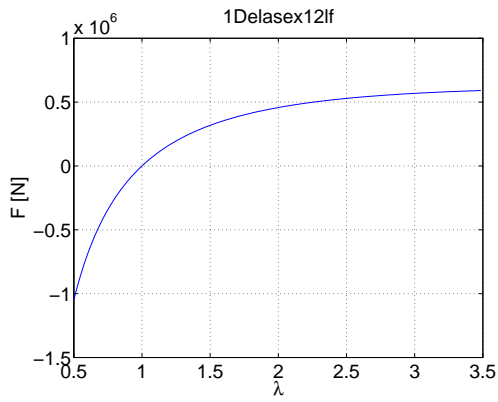


Fig. 4.20 : Axial force and cross-sectional area versus the elongation for $\sigma \sim \varepsilon_{ln}$ model

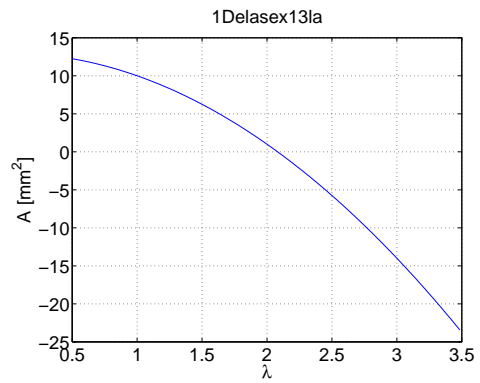
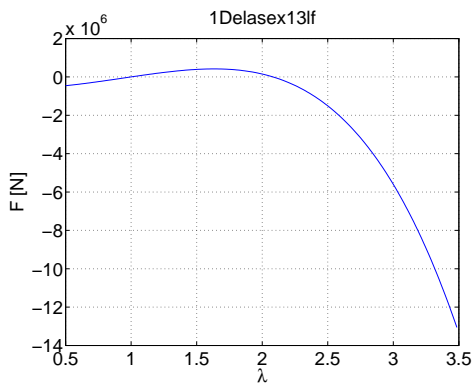


Fig. 4.21 : Axial force and cross-sectional area versus the elongation for $\sigma \sim \varepsilon_{gl}$ model

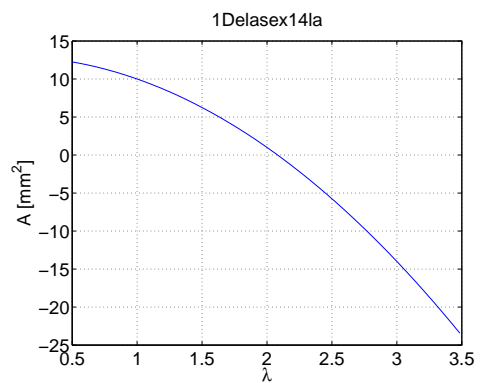
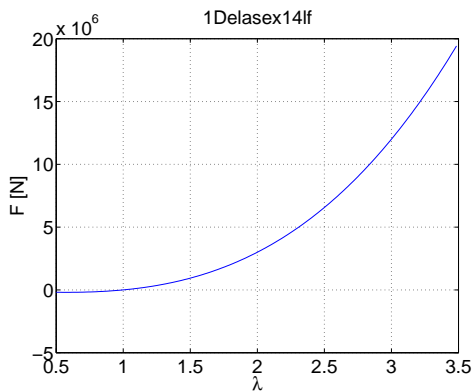


Fig. 4.22 : Axial force and cross-sectional area versus the elongation for $P \sim \varepsilon_{gl}$ model

For the elastomeric Neo-Hookean and Mooney-Rivlin models the material constants are : $C_{10} = 20000$ MPa and $C_{01} = 20000$ MPa. Stress versus elongation is shown.

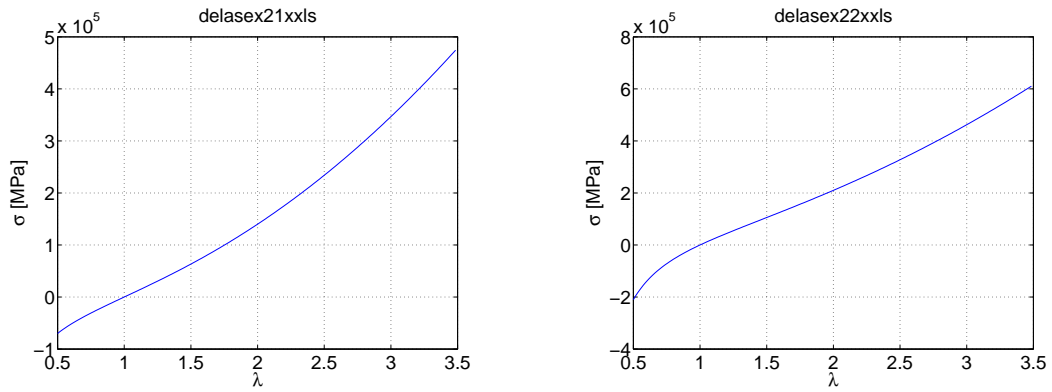


Fig. 4.23 : Stress versus λ for Neo-Hookean and Mooney-Rivlin models

The axial force is calculated for a prescribed axial elongation.

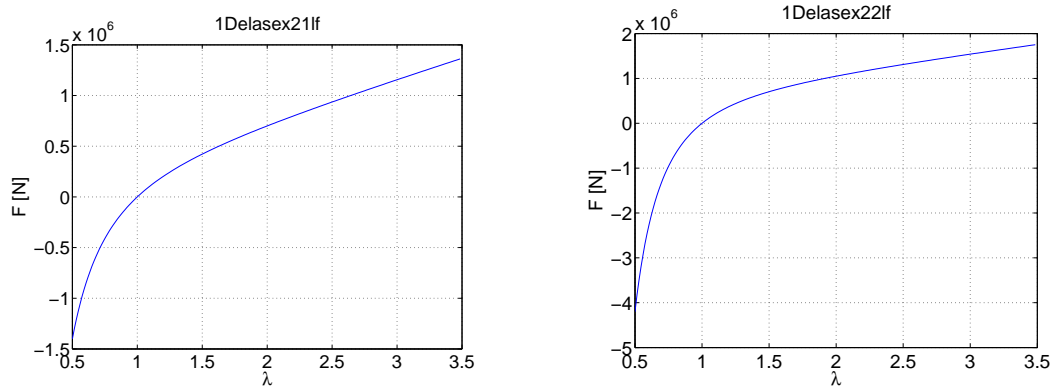


Fig. 4.24 : Axial force and cross-sectional area versus the elongation for Neo-Hookean and Mooney-Rivlin model

5 Elastoplastic material behavior

Below a certain load (stress) value, the deformation of all materials will be elastic. When the stress exceeds a limit value, plastic deformation occurs, which means that permanent elongation is observed after release of the load. At increased loading above the limit value, the stress generally increases with increasing elongation, a phenomenon referred to as *hardening*.

Reversed loading will first result in elastic deformation, but after reaching a limit value of the stress, plastic deformation will be observed again. Looking at the stress-strain curve after a few loading reversals, it can be seen that elastoplastic material behavior is history dependent: the stress is not uniquely related to the strain; its value depends on the deformation history. The total stress-strain history must be taken into account to determine the current stress.

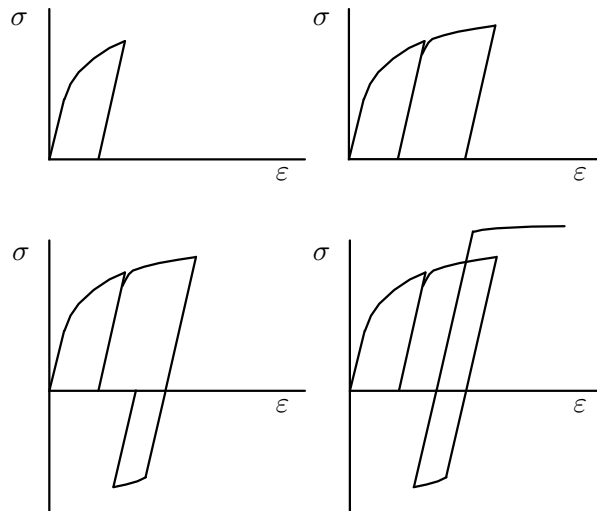


Fig. 5.25 : Stress-strain curves for elastoplastic material behavior

5.1 Tensile test

When a tensile bar, with undeformed length l_0 and cross-sectional area A_0 , is subjected to a tensile test, the axial force F and the length l can be measured. The axial strain ε can be calculated from the elongation factor λ . To calculate the true stress $\sigma = \frac{F}{A}$, the actual cross-sectional area of the tensile bar must be measured during the experiment. The nominal stress $\sigma_n = \frac{F}{A_0}$ can be calculated straightforwardly. The nominal stress σ_n can be plotted against the linear strain $\varepsilon_l = \lambda - 1 = \frac{l-l_0}{l_0} = \frac{\Delta l}{l_0}$ resulting in the $\sigma_n - \varepsilon_l$ stress-strain curve.

Until the proportionality limit $\sigma_n = \sigma_P$ is reached, the material behavior is assumed to be linear elastic : $\sigma_n = E\varepsilon$, where E is Young's modulus. When the stress exceeds the initial yield stress $\sigma_{y0} > \sigma_P$, unloading will reveal permanent (= plastic) deformation of the bar. The exact value of σ_{y0} cannot be determined so in practice σ_{y0} is taken to be the stress where a plastic strain of 0.2 % remains. In the following however, we will assume that σ_{y0} is exactly known and that $\sigma_{y0} = \sigma_P$.

The axial force and therefore the nominal stress will reach a maximum value. At that

point necking of the tensile bar will be observed. The maximum nominal stress is the tensile strength σ_T . In forming processes strains can be much higher than in a tensile test, because of the compression in certain directions.

After reaching the tensile strength the nominal stress will decrease while the strain is still increasing. Fracture occurs at the fracture stress $\sigma_n = \sigma_F$. The fracture strain ε_F is for metals and metal alloys about $10\% = 0.1$. This is a rather small elongation which means that for these materials we can assume $\sigma = \sigma_n$ and also that all strain definitions are approximately equivalent, so $\varepsilon = \varepsilon_l$.

Experiments have shown that during plastic deformation the volume of metals and metal alloys remains constant : plastic deformation is taken to be incompressible.

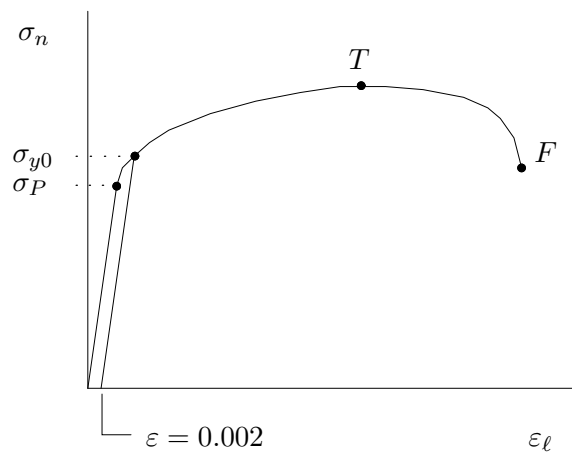


Fig. 5.26 : *Stress-strain curve during tensile test*

5.2 Compression test

For metal alloys a compression test instead of a tensile test will reveal that first yield will occur at $\sigma = \sigma_n = -\sigma_{y0}$. The initial material behavior is the same in tension and compression. In general terms the transition from purely elastic behavior to elastoplastic behavior is determined by a yield criterion. For the one-dimensional case this criterion says that first yielding will occur when :

$$f = \sigma^2 - \sigma_{y0}^2 = 0$$

The function f is the yield function.

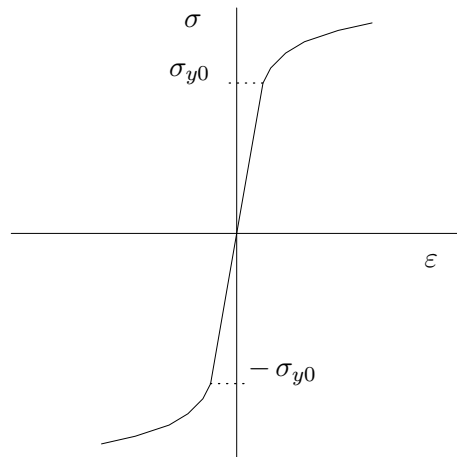


Fig. 5.27 : Stress-strain curve during tensile or compression test

5.3 Interrupted tensile test

When the axial load is released at σ_A (see figure below) with $\sigma_{y0} < \sigma_A < \sigma_T$, the unloading stress-strain path is elastic and characterized by the initial Young's modulus E . The permanent or plastic elongation is represented by the plastic strain ε_p . The difference between the total strain in point A and the plastic strain is the elastic strain $\varepsilon_e = \varepsilon_A - \varepsilon_p = \frac{\sigma_A}{E}$.

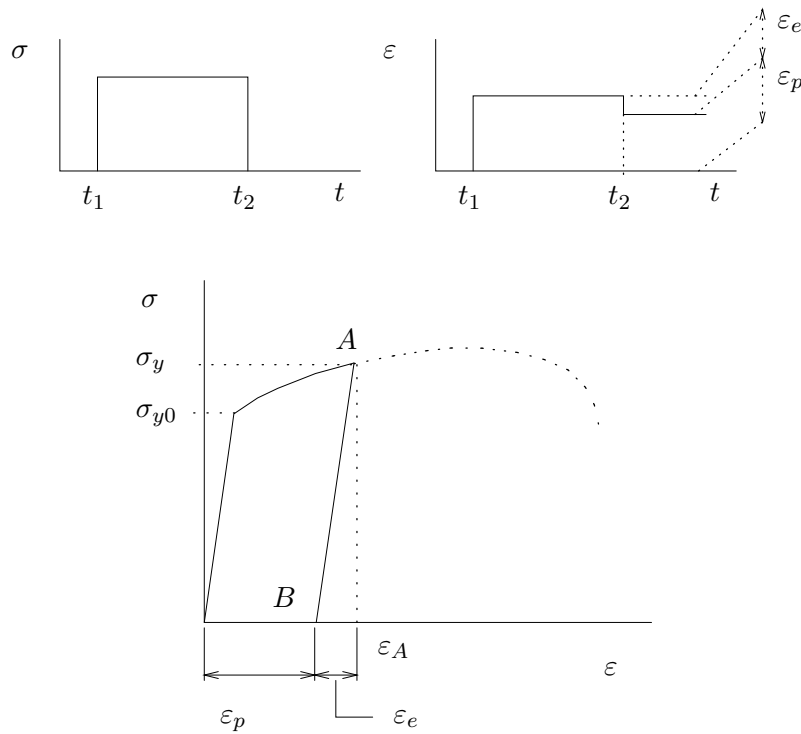


Fig. 5.28 : Stress-strain curve after interrupted tensile test

5.4 Resumed tensile test

When after unloading, the bar is again loaded with a tensile force, the elastic line BA will be followed where $\Delta\sigma = E\Delta\varepsilon = E\Delta\varepsilon_e$ holds. For $\sigma \geq \sigma_A$ ($\varepsilon \geq \varepsilon_A$) further elastoplastic deformation takes place and the stress-strain curve will be followed as if unloading were not occurred.

The stress σ_A is the *current yield stress* σ_y , which is generally larger than the initial yield stress σ_{y0} . The increase, referred to as *hardening*, is related to the plastic strain by a hardening law.

5.5 Hardening

To study the hardening phenomenon, the tensile bar is not reloaded in tension but in compression. Two extreme observations may be made, illustrated in the figure below.

In the first case the elastic trajectory increases in length due to plastic deformation : $AA' > Y_0Y'_0$. The elastic trajectory is symmetric about $\sigma = 0$ ($BA = BA'$). What we observe is *isotropic hardening*.

In the second case the elastic trajectory remains of constant length : $AA' = Y_0Y'_0$. It is symmetric about the line OC ($CA = CA'$). After unloading the yield stress under compression is different than the yield stress under tension. This is called *kinematic hardening*. The stress in point C , the center of the elastic trajectory, is the *shift stress* $\sigma = q$. This phenomenon is also referred to as the *Bauschinger effect*.

Real materials will show a combination of isotropic and kinematic hardening.

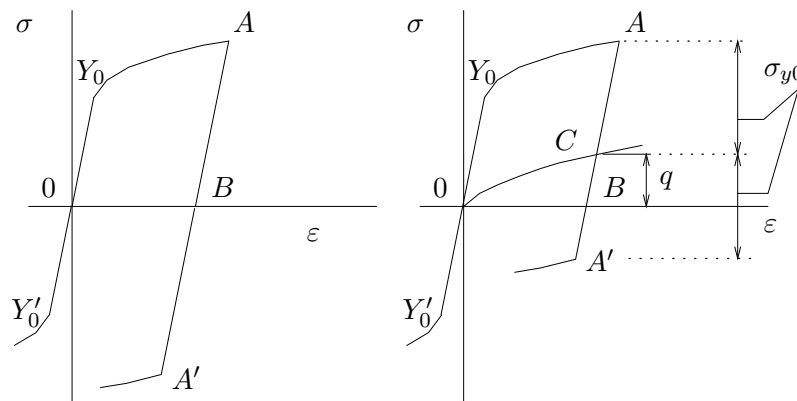


Fig. 5.29 : *Isotropic and kinematic hardening*

isotropic hardening : elastic area larger & symmetric w.r.t. $\sigma = 0$

$$\left. \begin{array}{l} \text{tensile} \\ \text{compression} \end{array} \right\} \begin{array}{l} : \\ : \end{array} \left. \begin{array}{l} \sigma = \sigma_y \\ \sigma = -\sigma_y \end{array} \right\} \rightarrow f = \sigma^2 - \sigma_y^2 = 0$$

kinematic hardening : elastic area constant & symmetric w.r.t. $\sigma = q$

$$\left. \begin{array}{l} \text{tensile} \\ \text{compression} \end{array} \right\} \begin{array}{l} : \\ : \end{array} \left. \begin{array}{l} \sigma = q + \sigma_{y0} \\ \sigma = q - \sigma_{y0} \end{array} \right\} \rightarrow f = (\sigma - q)^2 - \sigma_{y0}^2 = 0$$

combined isotropic/kinematic hardening

$$\left. \begin{array}{l} \text{tensile} \quad : \quad \sigma = q + \sigma_y \\ \text{compression} \quad : \quad \sigma = q - \sigma_y \end{array} \right\} \rightarrow f = (\sigma - q)^2 - \sigma_y^2 = 0$$

5.6 Effective plastic strain

Isotropic hardening could be described by relating the yield stress σ_y to the plastic strain ε_p . However, as the figure below shows, this would lead to the unrealistic conclusion that the yield stress increases while the plastic strain decreases. To prevent this problem, the *effective plastic strain* $\bar{\varepsilon}_p$ is taken as the history parameter. It is a measure of the total plastic strain, be its change positive or negative, and as such cannot decrease.

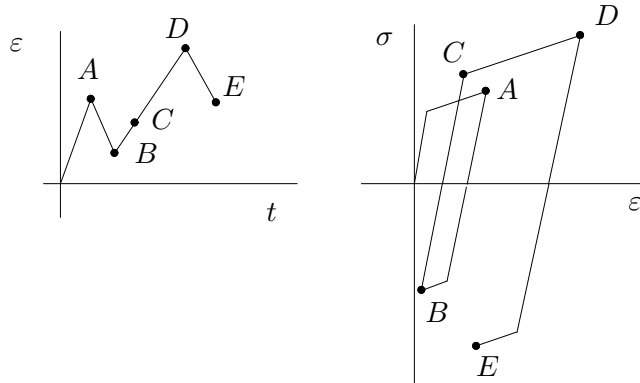


Fig. 5.30 : Increasing yield stress at decreasing plastic strain

$$\bar{\varepsilon}_p = \sum_{\varepsilon} |\Delta\varepsilon_p| = \sum_{\tau=0}^{\tau=t} \frac{|\Delta\varepsilon_p|}{\Delta t} \Delta t = \int_{\tau=0}^t |\dot{\varepsilon}_p| d\tau = \int_{\tau=0}^t \dot{\bar{\varepsilon}}_p d\tau$$

5.7 Hardening laws

For one-dimensional stress states encountered in the axial loading of a truss, several hardening laws are formulated, based on experimental observations. They can be generalized to three-dimensional stress-strain states. For isotropic hardening the current yield stress is related to the effective plastic strain and the initial yield stress. The isotropic hardening parameter is $H = \frac{d\sigma_y}{d\bar{\varepsilon}_p}$. For kinematic hardening the shift stress q is related to the plastic strain ε_p . The kinematic hardening parameter is $K = \frac{dq}{d\varepsilon_p}$.

$$\text{linear isotropic hardening} \quad \left. \begin{array}{l} \sigma_y = \sigma_{y0} + H\bar{\varepsilon}_p \\ q = 0 \end{array} \right\}$$

linear kinematic hardening $\sigma_y = \sigma_{y0}$
 $q = K\varepsilon_p$

exponential hardening $\sigma_y = \sigma_{y0} + C(\bar{\varepsilon}_p)^n$
 $q = q(\varepsilon_p)$

no hardening : ideal plastic $\sigma = \sigma_{y0}$
 $q = 0$

5.8 Cyclic load

A truss can be loaded with a prescribed strain $-\varepsilon_m \leq \varepsilon \leq \varepsilon_m$. It is assumed that the stress will reach values above the initial yield stress σ_{y0} and that linear hardening occurs.

For purely isotropic hardening the stress will increase after each load reversal and finally no further plastic deformation will take place.

For purely kinematic hardening the stress-strain path will be one single hysteresis loop, where the stress cycles, as does the strain, between two constant values $-\sigma_m \leq \sigma \leq \sigma_m$.

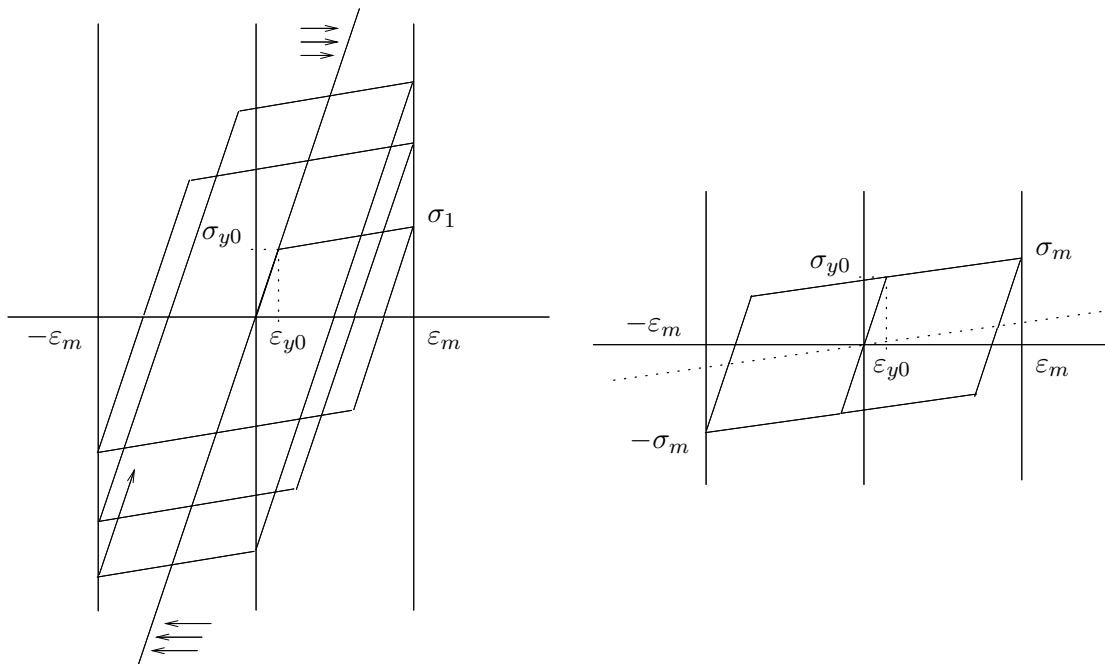


Fig. 5.31 : Stress-strain curve during cyclic loading for isotropic and for kinematic hardening

5.9 Examples

5.10 Reversed plasticity in a tensile bar

A cylindrical tensile bar is loaded with an axial stress σ , which is applied as a function of the (pseudo)time t as indicated in the figure below.

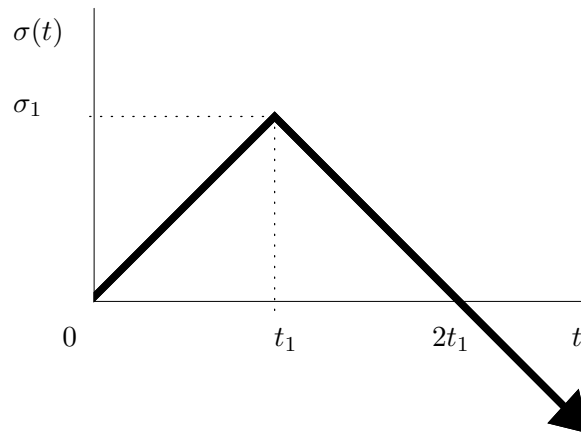


Fig. 5.32 : Prescribed stress as a function of (pseudo)time.

When the load is increased from $t = 0$, the material behaves linearly elastic – Young’s modulus is E – until the initial yield stress σ_{y0} is reached. The maximum stress σ_1 is such that $\sigma_1 > \sigma_{y0}$. After reaching this maximum value the stress is reduced to zero and compressive loading is applied. After yielding, the material shows linear, isotropic hardening with hardening constant H .

First the stress-strain diagram is plotted and relevant stress-strain points are indicated.

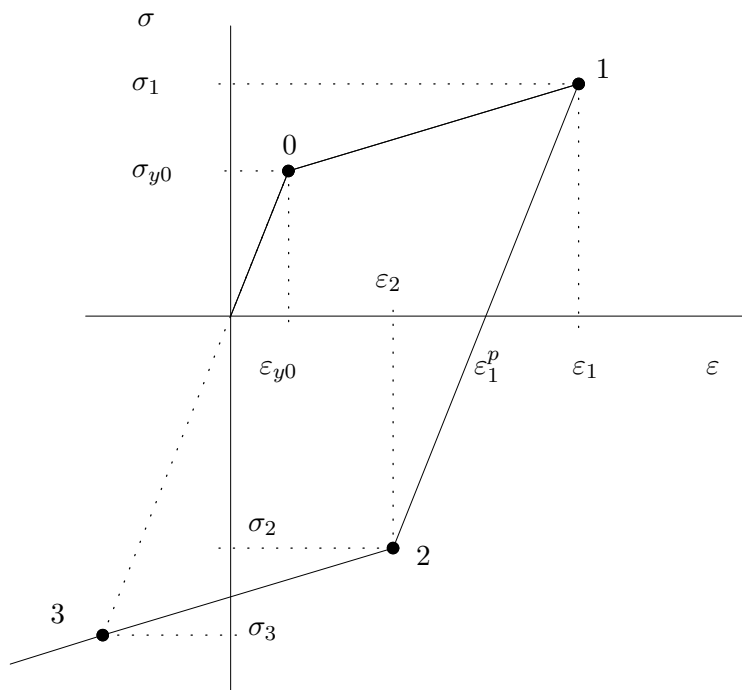


Fig. 5.33 : Stress-strain diagram.

When the stress reaches the maximum value σ_1 at $t = t_1$ the strain ε_1 can be calculated and expressed in σ_1 , E , σ_{y0} and H .

$$\begin{aligned}\varepsilon_1 &= \varepsilon_{y0} + \frac{E+H}{EH} (\sigma_1 - \sigma_{y0}) = \frac{\sigma_{y0}}{E} + \frac{E+H}{EH} (\sigma_1 - \sigma_{y0}) \\ &= \frac{\sigma_{y0}}{E} \left(1 - \frac{E+H}{H}\right) + \frac{E+H}{EH} \sigma_1 \\ &= -\frac{\sigma_{y0}}{H} + \frac{E+H}{EH} \sigma_1\end{aligned}$$

The plastic strain ε_1^p at maximum stress σ_1 is

$$\begin{aligned}\varepsilon_1^p &= \varepsilon_1 - \frac{\sigma_1}{E} = -\frac{\sigma_{y0}}{H} + \left(\frac{E+H}{H} - 1\right) \frac{\sigma_1}{E} \\ &= \frac{\sigma_1 - \sigma_{y0}}{H}\end{aligned}$$

From $t = t_1$ the stress is reduced to zero after which compression occurs. The strain ε_2 at yield under compressive loading is :

$$\varepsilon_2 = \varepsilon_1 - 2\frac{\sigma_1}{E}$$

The stress σ_3 for which the plastic strain is zero, is :

$$\varepsilon_3^p = 0 \quad \rightarrow \quad -(\sigma_3 - \sigma_2) = \sigma_1 - \sigma_{y0} = H\varepsilon_1^p \quad \rightarrow \quad \sigma_3 = -2\sigma_1 + \sigma_{y0}$$

5.11 Parallel truss structure

The figure shows two parallel bars a and b , which are connected to a rigid wall and to a rigid block, which can only translate in horizontal direction. The bars have the same initial length L . The cross-sectional area of a and b is A and $2A$, respectively. Both a and b have the same Young's modulus E and initial yield stress σ_{y0} . Upon yielding bar a is ideal plastic, while bar b shows isotropic linear hardening with hardening constant H .

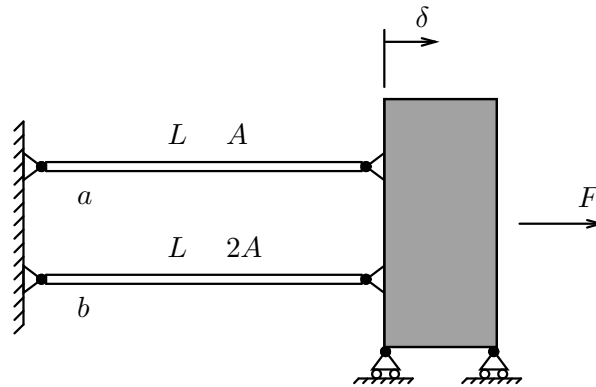


Fig. 5.34 : Parallel truss structure.

The axial deformation is provoked by the force F , which increases to a maximum value after which it will be reduced to zero. The maximum is such that both trusses will undergo elastoplastic deformation. The displacement of the rigid block is δ .

For all deformation states the strain in both trusses will be equal : $\varepsilon^a = \varepsilon^b$, due to the fact that their initial length and their elongation is the same. With this in mind we draw the stress-strain diagram of both bars below each other.

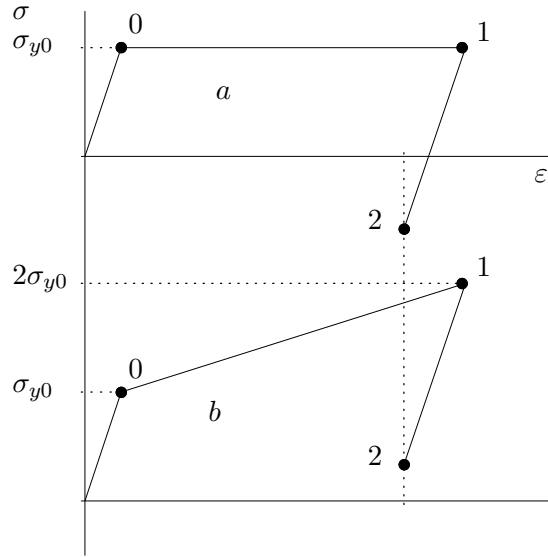


Fig. 5.35 : *Stress-strain diagrams of both trusses.*

The force F_0 at which yielding occurs for the first time, is

$$F_0 = \sigma_{y0}A + 2\sigma_{y0}A = 3\sigma_{y0}A$$

The strain ε_0 and the displacement δ_0 is

$$\varepsilon_0 = \varepsilon_{y0} = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \delta_0 = L \frac{\sigma_{y0}}{E}$$

The force F is increased to $F = F_1 = \frac{5}{3}F_0$. Because truss a doesn't show any hardening, the axial stress does not change : $\sigma_1^a = \sigma_0^a = \sigma_{y0}$, and thus $F_1^a = \sigma_{y0}A$. The axial force and the stress in truss b is then :

$$F_1^b = F_1 - F_1^a = 5\sigma_{y0}A - \sigma_{y0}A = 4\sigma_{y0}A \quad \rightarrow \quad \sigma_1^b = 2\sigma_{y0}$$

The strain ε_1^b in truss b and the displacement δ_1 can now be calculated.

$$\varepsilon_1^b = \frac{\sigma_{y0}}{E} + \frac{E+H}{EH} \sigma_{y0} = \frac{\sigma_{y0}}{E} \left(1 + \frac{E+H}{H} \right) = \frac{\sigma_{y0}}{E} \left(\frac{E+2H}{H} \right)$$

$$\delta_1 = L\varepsilon_1^b = L \frac{\sigma_{y0}}{E} \left(\frac{E+2H}{H} \right)$$

The plastic strain of the trusses is

$$\begin{aligned}\varepsilon_{p1}^a &= \varepsilon_1^a - \frac{\sigma_{y0}}{E} = \frac{E+H}{EH} \sigma_{y0} \\ \varepsilon_{p1}^b &= \varepsilon_1^b - \frac{2\sigma_{y0}}{E} = -\frac{\sigma_{y0}}{E} + \frac{E+H}{EH} \sigma_{y0} = \frac{\sigma_{y0}}{E} \left(-1 + \frac{E+H}{H} \right) = \frac{\sigma_{y0}}{H}\end{aligned}$$

The force F is now reduced to zero. The residual stresses in both trusses can be calculated.

$$\left. \begin{aligned}\varepsilon_2^a - \varepsilon_1^a &= \frac{1}{E} (\sigma_2^a - \sigma_1^a) \\ \varepsilon_2^b - \varepsilon_1^b &= \frac{1}{E} (\sigma_2^b - \sigma_1^b) \\ \varepsilon_2^a &= \varepsilon_2^b \quad ; \quad \varepsilon_1^a = \varepsilon_1^b\end{aligned}\right\} \rightarrow \begin{aligned}\sigma_2^a - \sigma_1^a &= \sigma_2^b - \sigma_1^b \rightarrow \\ \sigma_1^a &= \sigma_{y0} \quad ; \quad \sigma_1^b = 2\sigma_{y0}\end{aligned}$$

$$\left. \begin{aligned}\sigma_2^a - \sigma_{y0} &= \sigma_2^b - 2\sigma_{y0} \rightarrow & \sigma_2^b - \sigma_2^a &= \sigma_{y0} \\ \text{equilibrium} &\rightarrow \sigma_2^a A + \sigma_2^b 2A = 0 \rightarrow & \sigma_2^a &= -2\sigma_2^b\end{aligned}\right\} \rightarrow$$

$$3\sigma_2^b = \sigma_{y0} \rightarrow \sigma_2^b = \frac{1}{3}\sigma_{y0} \quad ; \quad \sigma_2^a = -\frac{2}{3}\sigma_{y0}$$

5.12 Serial truss structure

A cylindrical truss is divided in two parts a and b as is shown in the figure below. In the undeformed state, part a has length L and cross-sectional area $2A$, part b has length $2L$ and cross-sectional area A . In point R an axial force F is applied. Point P is fixed and point R will show a displacement δ .

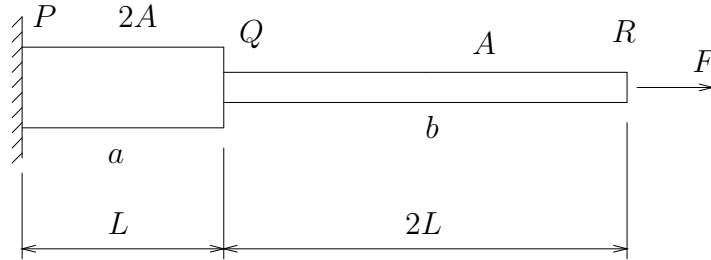


Fig. 5.36 : Series configuration of two trusses.

For each state in the deformation process, we will have :

$$N^a = N^b \quad \rightarrow \quad \sigma^a = \frac{1}{2} \sigma^b$$

When the load is increased from $F = 0$ at $t = 0$ the material is linearly elastic – Young's modulus E – as long as the axial stress is below the initial yield stress σ_{y0} . After yielding the material will show linear isotropic hardening with hardening constant H .

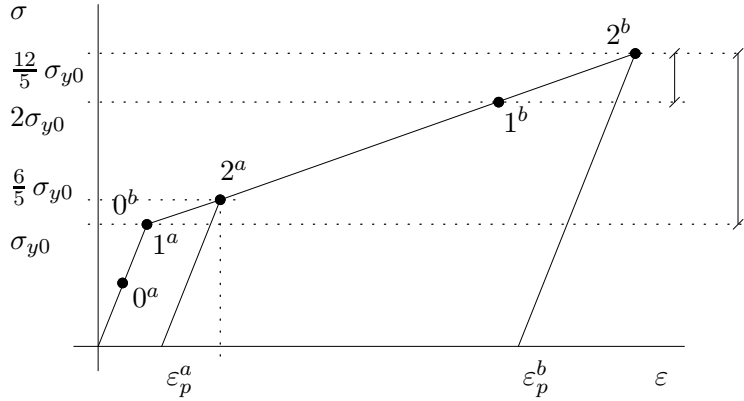


Fig. 5.37 : Stress-strain diagram of the two trusses.

When yielding occurs for the first time the force F_0 is part b will yield first at a force

$$F_0 = \sigma_{y0}A$$

The displacement δ_0 at that moment is

$$\left. \begin{array}{l} \sigma_0^b = \sigma_{y0} \quad \rightarrow \quad \varepsilon_0^b = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \Delta L_0^b = 2L \frac{\sigma_{y0}}{E} \\ \sigma_0^a = \frac{1}{2}\sigma_{y0} \quad \rightarrow \quad \varepsilon_0^a = \frac{\sigma_{y0}}{2E} \quad \rightarrow \quad \Delta L_0^a = L \frac{\sigma_{y0}}{2E} \end{array} \right\} \rightarrow \delta_0 = \Delta L_0^a + \Delta L_0^b = \frac{5}{2} L \frac{\sigma_{y0}}{E}$$

The load is increased to $F = F_1 > F_0$ when part a will yield for the first time.

$$\sigma_1^a = \sigma_{y0} \quad \rightarrow \quad \sigma_1^b = 2\sigma_{y0} \quad \rightarrow \quad F_1 = 2\sigma_{y0}A$$

The displacement δ_1 can then be calculated :

$$\left. \begin{array}{l} \sigma_1^a = \sigma_{y0} \quad \rightarrow \quad \varepsilon_1^a = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \Delta L_1^a = L \frac{\sigma_{y0}}{E} \\ \varepsilon_1^b = \varepsilon_{y0}^b + \frac{E+H}{EH} \sigma_{y0} = \frac{\sigma_{y0}}{E} + \frac{E+H}{EH} \sigma_{y0} = \frac{\sigma_{y0}}{E} \left(\frac{E+2H}{H} \right) \quad \rightarrow \quad \Delta L_1^b = 2L \frac{\sigma_{y0}}{E} \left(\frac{E+2H}{H} \right) \end{array} \right\} \rightarrow$$

$$\delta_1 = \Delta L_1^a + \Delta L_1^b = L \frac{\sigma_{y0}}{E} + 2L \frac{\sigma_{y0}}{E} \left(\frac{E+2H}{H} \right) = L \frac{\sigma_{y0}}{E} \left(\frac{2E+5H}{H} \right)$$

The force F is increased further to $F = F_2 = \frac{6}{5}F_1 = \frac{12}{5}\sigma_{y0}A$ and is subsequently reduced to zero : $F = F_3 = 0$. The residual plastic strain in each part can be determined.

$$\sigma_2^b = \frac{F_2}{A} = \frac{12}{5}\sigma_{y0} \quad \rightarrow \quad \varepsilon_2^b - \varepsilon_1^b = \frac{E+H}{EH}(\sigma_2^b - \sigma_1^b) \quad \rightarrow \quad \varepsilon_2^b = \frac{1}{5} \frac{\sigma_{y0}}{EH} (7E + 12H)$$

$$\sigma_2^a = \frac{F_2}{2A} = \frac{6}{5}\sigma_{y0} \quad \rightarrow \quad \varepsilon_2^a - \varepsilon_1^a = \frac{E+H}{EH}(\sigma_2^a - \sigma_1^a) \quad \rightarrow \quad \varepsilon_2^a = \frac{1}{5} \frac{\sigma_{y0}}{EH} (E + 6H)$$

$$\varepsilon_{p2}^b = \varepsilon_3^b = \varepsilon_2^b - \frac{12}{5} \frac{\sigma_{y0}}{E} = \frac{7}{5} \frac{\sigma_{y0}}{H} \quad ; \quad \varepsilon_{2p}^a = \varepsilon_3^a = \varepsilon_2^a - \frac{6}{5} \frac{\sigma_{y0}}{E} = \frac{1}{5} \frac{\sigma_{y0}}{H}$$

The total displacement after unloading is

$$\delta_3 = \Delta L_3^a + \Delta L_3^b = L \frac{1}{5} \frac{\sigma_{y0}}{H} + 2L \frac{7}{5} \frac{\sigma_{y0}}{H} = 3L \frac{\sigma_{y0}}{H}$$

5.13 Elastoplastic model

The elastoplastic deformation characteristics can be represented by a discrete mechanical model. A friction element represents the yield limit and a hardening spring – stiffness H ($H > 0$) – provides the stiffness reduction after reaching the yield limit. The elastoplastic model describes rate-independent plasticity – there is no dashpot in the discrete model –, so the time is fictitious and "rate" is just referring to momentary change.

The yield criterion is used to decide at which stress level a purely elastic deformation will be followed by elastoplastic deformation. During elastoplastic deformation the total strain rate ($\dot{\varepsilon}$) is additively decomposed in an elastic ($\dot{\varepsilon}_e$) and a plastic ($\dot{\varepsilon}_p$) part. The plastic strain rate $\dot{\varepsilon}_p$ is related to $\frac{\partial f}{\partial \sigma}$ by the rate of the *plastic multiplier* λ , the so-called *consistency parameter* $\dot{\lambda}$. During ongoing plastic deformation the consistency equation $\dot{f} = 0$ must be satisfied, because f must remain zero.

The hardening law relates the current yield stress σ_y to the initial yield stress σ_{y0} and the effective plastic strain $\bar{\varepsilon}_p$. The shift stress q is related to the plastic strain ε_p .

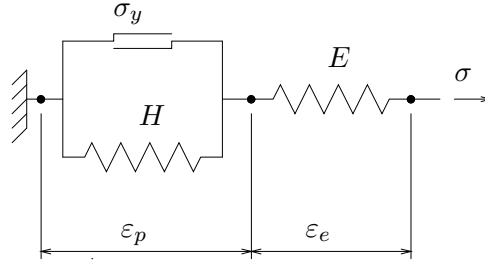


Fig. 5.38 : Discrete mechanical model for elastoplastic material behavior

- $f = (\sigma - q)^2 - \sigma_y^2$ with $f < 0 \mid f = 0 \wedge \dot{f} < 0 \rightarrow$ elastic
 $f = 0 \wedge \dot{f} = 0 \rightarrow$ elastoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p)$; $q = q(\varepsilon_p)$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_p$
- $\sigma = E\varepsilon_e \rightarrow \dot{\varepsilon}_e = \frac{1}{E} \dot{\sigma}$
- $\dot{\varepsilon}_p = \dot{\lambda} \frac{\partial f}{\partial \sigma} = 2\dot{\lambda}(\sigma - q)$; $\dot{\bar{\varepsilon}}_p = |\dot{\varepsilon}_p| = 2\dot{\lambda}|\sigma - q|$
- $\bar{\varepsilon}_p = \int_{\tau=0}^t \dot{\bar{\varepsilon}}_p d\tau = \sum_t |\Delta\varepsilon_p|$

5.14 Constitutive equations

From the constitutive relations a set of constitutive equations can be derived.

$$\left. \begin{array}{l} \dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_p) = E\{\dot{\varepsilon} - 2\dot{\lambda}(\sigma - q)\} \\ f = 0 \end{array} \right\} \rightarrow$$

$$\left. \begin{aligned} \dot{\sigma} + 2E(\sigma - q)\dot{\lambda} - E\dot{\varepsilon} &= 0 \\ f &= 0 \end{aligned} \right\}$$

The current stress has to be determined from these constitutive equations. The first one is a differential equation in pseudo-time. To solve it we use the incremental approach, where the total time is discretized and where we assume to have reached a solution for the begin-increment time t_n , i.e. values at the beginning of the current increment are known.

Although very general solution procedures can be used, we first consider a special case. It is assumed that the stress state at both the begin-increment time and the end-increment time is on the yield trajectory. Also linear hardening is considered, first isotropic then kinematic.

5.15 Linear isotropic hardening

For linear isotropic hardening with constant hardening parameter H , the stress increment can be derived straightforwardly. In a point of the elastic trajectory we know that $\Delta\sigma = E\Delta\varepsilon$ holds. In a point of the elastoplastic trajectory we can write $\Delta\sigma = S\Delta\varepsilon$, where the material stiffness $S = C_\varepsilon$ will depend on E and H .

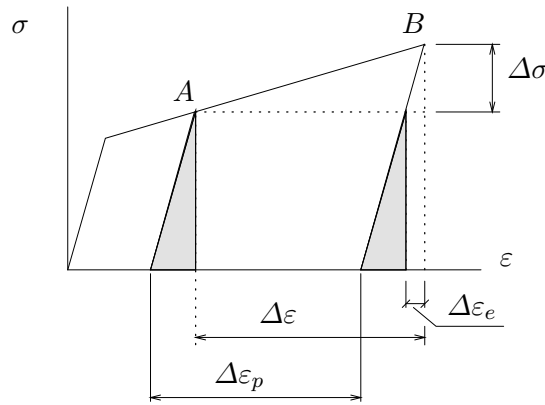


Fig. 5.39 : *Stress-strain curve for monotonic tensile loading*

$$\Delta\sigma = E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta\sigma_y}{H}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{H}\right) \rightarrow$$

$$\Delta\sigma = \frac{EH}{E+H} \Delta\varepsilon = S\Delta\varepsilon \quad ; \quad \Delta\varepsilon_p = \frac{\Delta\sigma}{H} = \frac{E}{E+H} \Delta\varepsilon$$

5.16 Kinematic hardening

For linear kinematic hardening, the result is similarly derived.

$$\Delta\sigma = \frac{EK}{E+K} \Delta\varepsilon \quad ; \quad \Delta\varepsilon_p = \frac{1}{K} \Delta\sigma = \frac{E}{E+K} \Delta\varepsilon$$

Again these relations can be derived straightforwardly from the figure.

$$\begin{aligned} \Delta\sigma &= E\Delta\varepsilon_e = E(\Delta\varepsilon - \Delta\varepsilon_p) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right) = E\left(\Delta\varepsilon - \frac{\Delta\sigma}{K}\right) \quad \rightarrow \\ \Delta\sigma &= \frac{EK}{E+K} \Delta\varepsilon = S\Delta\varepsilon \quad ; \quad \Delta\varepsilon_p = \frac{\Delta\sigma}{K} = \frac{E}{E+K} \Delta\varepsilon \end{aligned}$$

Note that the stiffness equals Young's modulus when H (or K) approaches infinity.

5.17 Stress update

In a general case of elastoplastic deformation, the begin-increment state, indicated with index n , may reside on the elastic trajectory or on the elastoplastic trajectory. The end-increment state is indicated with an index $n+1$ but this is skipped furtheron. Depending of $\Delta\varepsilon = \varepsilon - \varepsilon_n$ (further) elastoplastic deformation or elastic unloading can occur. Several possibilities are indicated in the figure below.

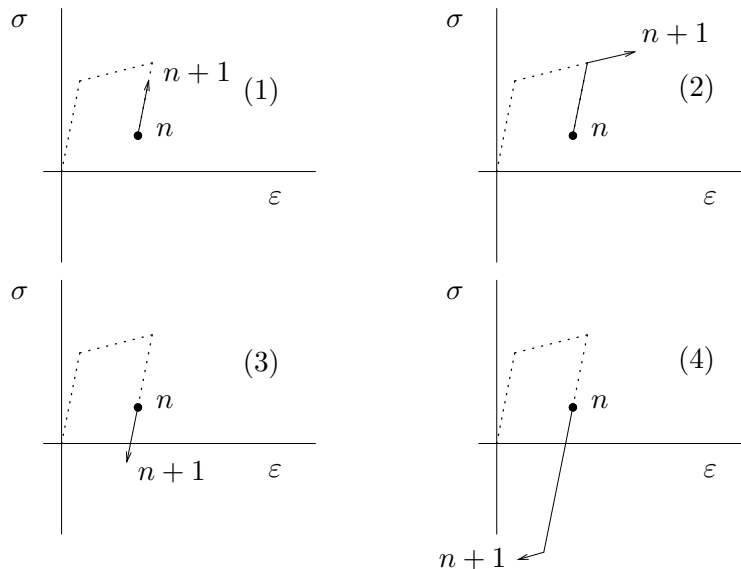


Fig. 5.40 : Various incremental stress-strain changes

5.18 Elastic stress predictor

Because it is not known a priori whether (ongoing) elastoplastic deformation or elastic unloading will have taken place in the current increment $t_n \rightarrow t_{n+1}$, the stress calculation starts from the assumption that the strain increment is completely elastic. The *elastic stress predictor* σ_e is calculated and subsequently the yield criterion is evaluated with the yield function f .

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 \leq 0 \quad \rightarrow \quad$ elastic increment
- $f = (\sigma_e - q_n)^2 - \sigma_{y_n}^2 > 0 \quad \rightarrow \quad$ elastoplastic increment

5.19 Elastic increment

When the increment is fully elastic, the end-increment stress equals the calculated elastic stress. As no plastic deformation has occurred during the increment, the effective plastic strain and the yield stress remain unchanged.

$$\begin{aligned} \sigma(t_{n+1}) &= \sigma_e & ; \quad \bar{\varepsilon}_p(t_{n+1}) &= \bar{\varepsilon}_p(t_n) = \bar{\varepsilon}_{p_n} \\ \sigma_y(t_{n+1}) &= \sigma_y(t_n) = \sigma_{y_n} & ; \quad q(t_{n+1}) &= q(t_n) = q_n \end{aligned}$$

5.20 Elastoplastic increment

If the elastic stress predictor indicates that the yield criterion is violated, the increment is elastoplastic. The end-increment stress has to be determined by integration of the constitutive equations, such that at the end of the increment the stress satisfies the yield criterion as will be discussed.

There are many procedures which can be followed to solve the differential equation for the stress. They can be classified as *implicit* or *explicit*. The implicit methods are more accurate and more stable than the explicit methods.

We assume that the begin-increment state resides on the yield trajectory, so $f_n = 0$. In reality this is of course not always the case: the begin-increment case may be elastic ($f_n < 0$) and plastic deformation will develop during the increment. The implicit procedures can well cope with this phenomenon. Explicit procedures will need some correction.

5.21 Implicit solution procedure

In an implicit procedure we want to satisfy the constitutive equations at the current time, which is the end-increment time. Because various variables are unknown, the equations are non-linear and have to be solved iteratively. Remember that at the begin-increment time, the equations are satisfied – in the former increment – so all values are known and $f_n \leq 0$.

In an iterative approach, an unknown value is written as the sum of an approximate value

and an iterative change, which is assumed to be very small, allowing linearisation. Concerning the yield function, it has to be recalled that it depends on the stress σ , the shift stress q and the yield stress σ_y and that both the yield stress and the shift stress depend on the plastic strain – through the hardening law – and thus on λ .

$$\left. \begin{aligned} \sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) &= E(\varepsilon - \varepsilon_n) \\ f - f_n &= f = 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} \sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) &= E(\varepsilon - \varepsilon_n) \\ f^* + \delta f = 0 \rightarrow f^* + \frac{\partial f}{\partial \sigma} \delta\sigma + \frac{\partial f}{\partial \lambda} \delta\lambda &= 0 \end{aligned} \right\}$$

with

$$\begin{aligned} \frac{\partial f}{\partial \sigma} &= 2(\sigma - q) \\ \frac{\partial f}{\partial \lambda} &= \frac{\partial f}{\partial q} \frac{\partial q}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial \lambda} + \frac{\partial f}{\partial \sigma_y} \frac{\partial \sigma_y}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial \lambda} \\ &= [-2(\sigma - q)][K][2(\sigma - q)] + [-2\sigma_y][H][2|\sigma - q|] \\ &= -4K(\sigma - q)^2 - 4H\sigma_y|\sigma - q| \end{aligned}$$

this becomes

$$\left. \begin{aligned} \sigma^* + \delta\sigma - \sigma_n + 2E(\sigma^* + \delta\sigma - q^* - \delta q)(\lambda^* + \delta\lambda - \lambda_n) &= E(\varepsilon - \varepsilon_n) \\ f^* + 2(\sigma^* - q^*)\delta\sigma - [4K^*(\sigma^* - q^*)^2 + 4H^*\sigma_y^*|\sigma^* - q^*|]\delta\lambda &= 0 \end{aligned} \right\}$$

After linearization and solving the unknown $\delta\sigma$ and $\delta\lambda$ until convergence is reached, the result is a new approximate value for σ , ε_p , q and σ_y at the end of the increment.

$$\begin{aligned} \sigma^* &= \sigma^* + \delta\sigma \\ \lambda^* &= \lambda^* + \delta\lambda \\ \Delta\varepsilon_p &= 2(\lambda^* - \lambda_n)(\sigma^* - q_n) \rightarrow \varepsilon_p \rightarrow q^*, K^* \\ \Delta\bar{\varepsilon}_p &= |\Delta\varepsilon_p| \rightarrow \bar{\varepsilon}_p \rightarrow \sigma_y^*, H^* \end{aligned}$$

5.22 Stiffness

From the solution procedure, also a relation for the material stiffness $C_\varepsilon = \frac{\partial \sigma}{\partial \varepsilon}$ can be derived.

$$\begin{cases}
\sigma - \sigma_n + 2E(\sigma - q)(\lambda - \lambda_n) - E(\varepsilon - \varepsilon_n) = 0 \\
f = 0
\end{cases}$$

$$\begin{cases}
\delta\sigma + 2E\delta\sigma(\lambda - \lambda_n) + 2E(\sigma - q)\delta\lambda - E\delta\varepsilon = 0 \\
(\sigma - q)\delta\sigma - 2K(\sigma - q)^2\delta\lambda - 2H\sigma_y|\sigma - q|\delta\lambda = 0
\end{cases}$$

$$\left[1 + 2E(\lambda - \lambda_n) + \frac{2E(\sigma - q)^2}{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|} \right] \delta\sigma = E\delta\varepsilon$$

$$C_\varepsilon = \frac{E\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\}}{\{1 + 2E(\lambda - \lambda_n)\}\{2K(\sigma - q)^2 + 2H\sigma_y|\sigma - q|\} + 2E(\sigma - q)^2}$$

yield at $\tau = t = t_{n+1} \rightarrow (\sigma - q)^2 = \sigma_y^2$ and $|\sigma - q| = \sigma_y \rightarrow$

$$C_\varepsilon = \frac{E(K + H)}{E + K + H + 2E(K + H)(\lambda - \lambda_n)}$$

5.23 Explicit solution procedure

An explicit procedure starts from the known state at the beginning of the increment and calculates incremental changes directly, assuming that values of some variables remain the same during the increment. Obviously, this is not through, so these procedures are not very accurate. The final solution may not satisfy the yield criterion $f = 0$ exactly, which calls for a correction, where the final state is projected onto the yield trajectory.

$$\left. \begin{aligned}
&\Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon \\
&\Delta f = 0 \rightarrow \left. \begin{aligned}
&\frac{\partial f}{\partial\sigma}\Big|_n \Delta\sigma + \frac{\partial f}{\partial\lambda}\Big|_n \Delta\lambda = 0
\end{aligned} \right\}
\end{aligned}$$

$$\left. \begin{aligned}
&\Delta\sigma + 2E(\sigma_n - q_n)\Delta\lambda = E\Delta\varepsilon \\
&2(\sigma_n - q_n)\Delta\sigma - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{yn}|\sigma_n - q_n|\Delta\lambda = 0 \rightarrow \\
&\Delta\lambda = \frac{(\sigma_n - q_n)}{2K_n(\sigma_n - q_n)^2 + 2H_n\sigma_{yn}|\sigma_n - q_n|} \Delta\sigma = \frac{1}{2K_n(\sigma_n - q_n) + 2H_n(\sigma_n - q_n)} \Delta\sigma
\end{aligned} \right\} \rightarrow$$

$$\Delta\sigma = \frac{E[K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|]}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n| + E(\sigma_n - q_n)^2} \Delta\varepsilon$$

$$\Delta\varepsilon_p = 2(\sigma_n - q_n)\Delta\lambda = \frac{(\sigma_n - q_n)^2}{K_n(\sigma_n - q_n)^2 + H_n\sigma_{yn}|\sigma_n - q_n|}$$

When we calculate the stress from the equations above, it is found that the result is not correct due to the bad transition from the elastic to the elastoplastic regime. To solve this problem, the elastoplastic increment is split in an elastic and an elastoplastic part.

A scaling factor β is calculated from the requirement that $\beta(\varepsilon - \varepsilon_n)$ brings us to the yield trajectory where $\varepsilon = \varepsilon^f$. This strain to yield ε^f is determined, and the current stress is

calculated, using the current stiffness. Calculation of β is generalized for tension and pressure. Notice that $\text{sign}(\alpha)$ is the *sign* of α .

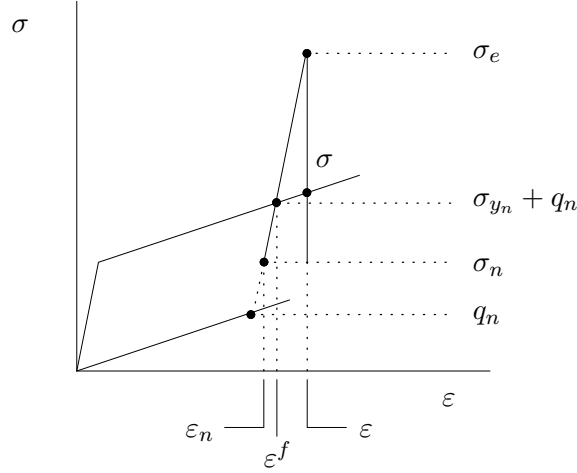


Fig. 5.41 : *Explicit stress update by increment splitting*

$$\begin{aligned} \sigma_e &= \sigma_n + E(\varepsilon - \varepsilon_n) & \rightarrow & \quad \Delta\sigma_e = \sigma_e - \sigma_n = E(\varepsilon - \varepsilon_n) \\ \beta &= \frac{|\text{sign}(\varepsilon - \varepsilon_n)\sigma_{y_n} - (\sigma_n - q_n)|}{|\sigma_e - \sigma_n|} \\ \varepsilon^f &= \varepsilon_n + \beta(\varepsilon - \varepsilon_n) & \rightarrow & \quad \Delta\varepsilon^f = \varepsilon - \varepsilon^f = (1 - \beta)(\varepsilon - \varepsilon_n) \end{aligned}$$

The elastoplastic stress increment $\Delta\sigma^f$ and the increment of λ is now calculated from the constitutive equations. The total stress increment and other state variables can then be calculated.

$$\begin{cases} \Delta\sigma^f + 2E(\sigma_n - q)\Delta\lambda = E\Delta\varepsilon^f \\ 2(\sigma_n - q)\Delta\sigma^f - 4K_n(\sigma_n - q_n)^2\Delta\lambda - 4H_n\sigma_{ym}|\sigma_n - q_n|\Delta\lambda = 0 \end{cases}$$

The incremental changes are then :

$$\begin{aligned} \Delta\sigma &= \beta\Delta\sigma_e + \Delta\sigma^f & \rightarrow & \quad \sigma = \sigma_n + \Delta\sigma \\ \lambda &= \lambda_n + \Delta\lambda \\ \Delta\varepsilon_p &= 2(\lambda - \lambda_n)(\sigma - q_n) & \rightarrow & \quad \varepsilon_p \rightarrow q, K \\ \Delta\bar{\varepsilon}_p &= |\Delta\varepsilon_p| & \rightarrow & \quad \bar{\varepsilon}_p \rightarrow \sigma_y, H \end{aligned}$$

5.24 Implementation

See `tr2delpl.m` for the implementation.

5.25 Examples

5.26 Cyclic loading

The stress-strain behavior of a truss is calculated for a prescribed cyclic strain, for linear isotropic and linear kinematic hardening. The truss has initial length $l_0 = 100$ mm and cross-sectional area $A_0 = 10$ mm² and is loaded with a prescribed cyclic axial strain. The axial stress is calculated. Material parameters are :

Young's modulus	E	100000	MPa
Poisson's ratio	ν	0.3	-
initial yield stress	σ_{y0}	250	MPa
hardening coefficient	H	5000	MPa
hardening coefficient	K	5000	MPa

The isotropic hardening leads to an increasingly larger elastic trajectory. After many load reversals, the behavior will become purely elastic. The kinematic hardening results in a steady state hysteresis loop.

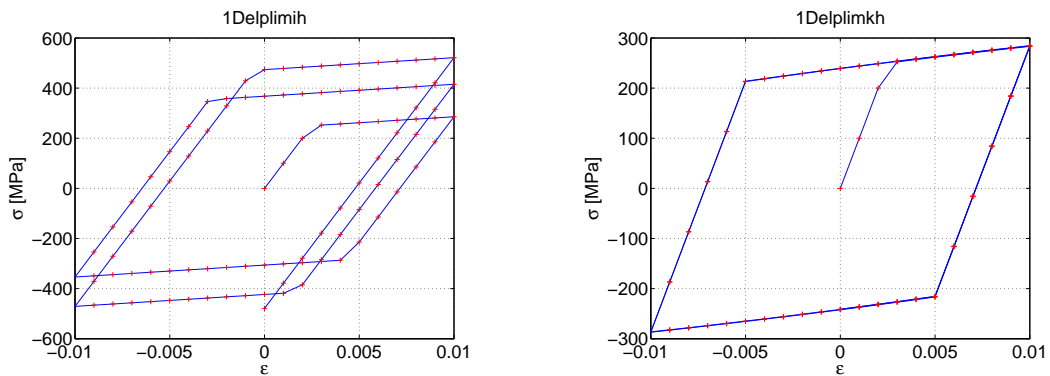


Fig. 5.42 : *Cyclic stress-strain behavior for linear isotropic and kinematic hardening*

5.27 Clamped truss

A prismatic truss is clamped between two rigid walls, as is shown in the figure. The length L of the truss is 2×1000 [mm], its cross-sectional area A is 100 mm². The material is elastic with Young's modulus $E = 200000$ N/mm² as long as the axial stress is below the initial yield stress of 200 N/mm². Above this value the material shows linear isotropic hardening with hardening coefficient $H = 1000$ M/mm².

In the middle of the truss, in point Q , a point load F is applied, which first increases and then is decreased to zero. The displacement of point Q is calculated with `tr2d`. The force F as function of the displacement is shown in the figure below.

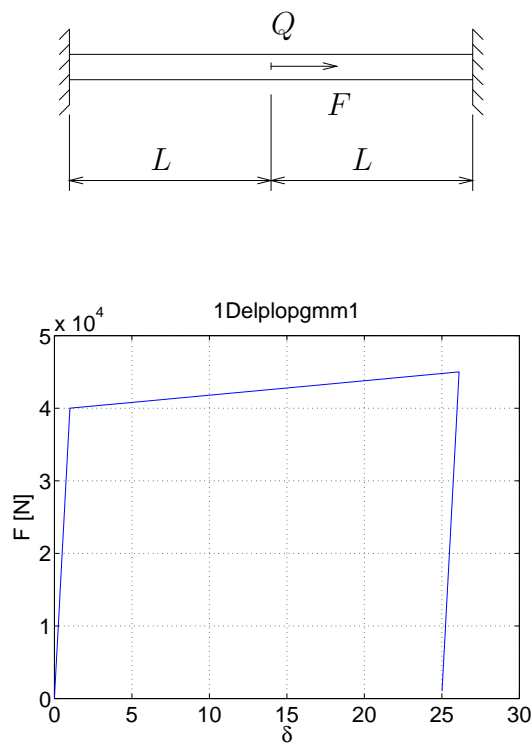


Fig. 5.43 : Force F versus displacement of point Q .

5.28 Truss structure

A structure of three trusses is loaded by a vertical displacement. When the axial stress exceeds a certain limit value, a trusses will deform plastically.

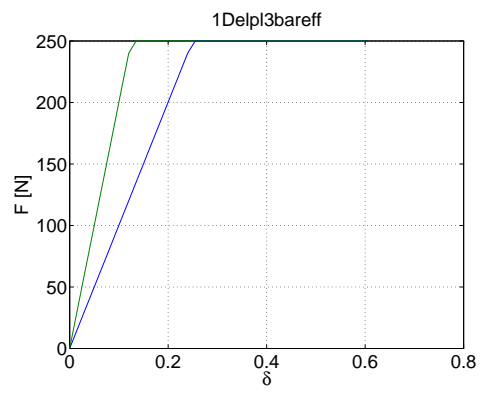
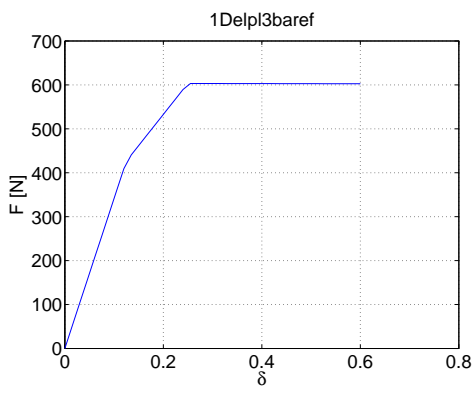
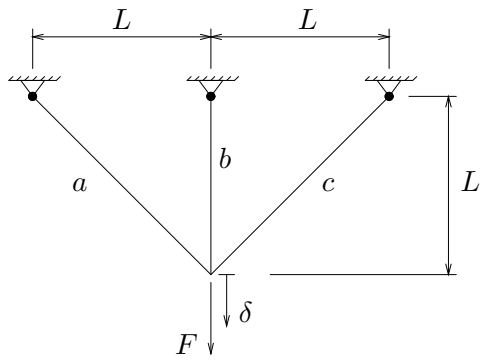


Fig. 5.44 : Force F versus displacement δ .

6 Linear viscoelastic material behavior

Viscoelastic materials show time dependent behavior. When during a tensile test the stress/strain is prescribed stepwise, the strain/stress will not react immediately, but show a delayed response, which is called creep/relaxation. Viscoelastic material behavior is a combination of elastic and viscous behavior. Both cases will be illustrated first.

Here, we only consider linear viscoelastic behavior and also assume that strains are small.

6.1 Linear elastic material behavior

For a linear elastic material the stress is uniquely related to the strain by the Young's modulus E [Pa]. The linear elastic truss behaves like a spring with constant stiffness k [N/m].

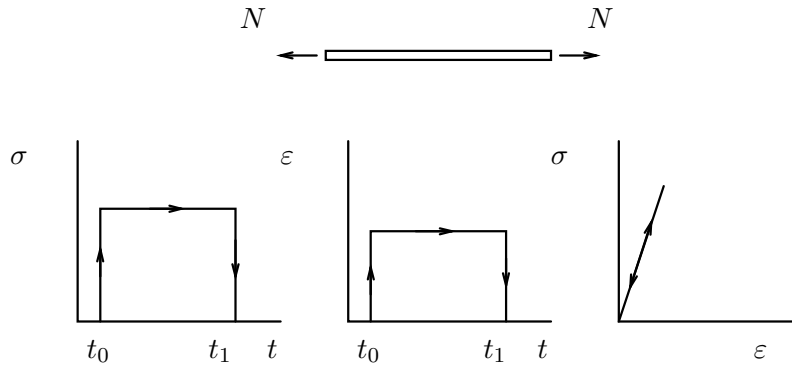


Fig. 6.45 : Tensile experiment for linear elastic truss

$$\varepsilon = \frac{1}{E} \sigma \quad \rightarrow \quad \sigma = E\varepsilon \quad \rightarrow \quad N = \sigma A = EA\varepsilon = \frac{EA}{l} \Delta l = k \Delta l$$

A linear elastic material can be subjected to a loading stress cycle. The work per unit of volume during the cycle appears to be zero indicating that there has been no dissipation. This is also obvious when looking at the stress-strain curve associated with the load cycle : the area below the curve is zero.

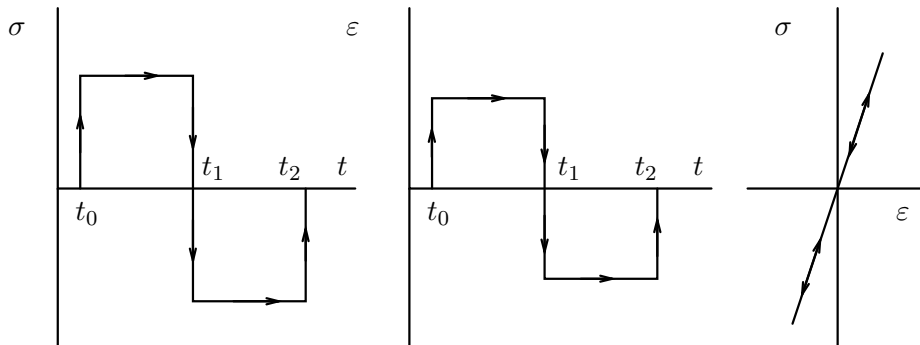


Fig. 6.46 : Loading cycle applied to linear elastic truss

$$\begin{aligned}
 U_d &= \int_{t_0}^{t_1} \sigma d\varepsilon + \int_{t_1}^{t_2} \sigma d\varepsilon = \int_{t_0}^{t_1} E\varepsilon d\varepsilon + \int_{t_1}^{t_2} E\varepsilon d\varepsilon \\
 &= \frac{1}{2}E[\varepsilon_1^2 - \varepsilon_0^2 + \varepsilon_2^2 - \varepsilon_1^2] = 0
 \end{aligned}$$

6.2 Linear viscous material behavior

For a linear viscous material the stress is uniquely related to the strain rate by the viscosity η [Pa.s]. The linear viscous "truss" behaves like a dashpot with constant damping value b [Ns/m].

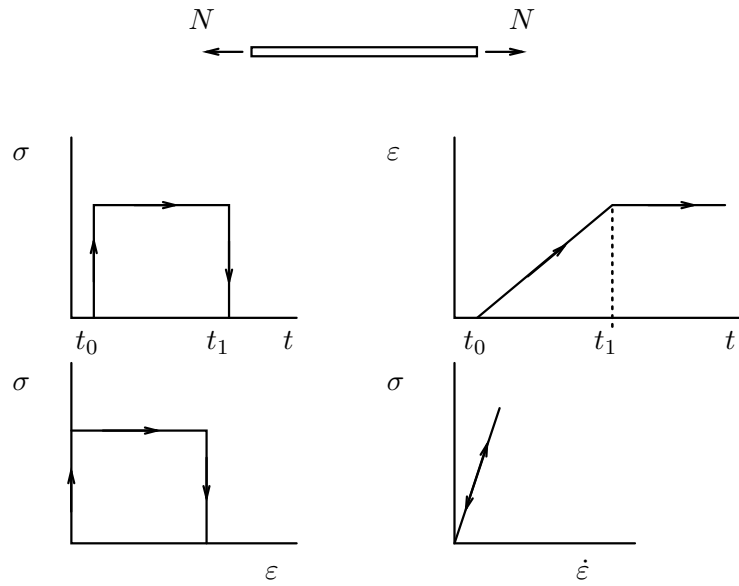


Fig. 6.47 : Tensile experiment for linear viscous truss

$$\dot{\varepsilon} = \frac{1}{\eta} \sigma \quad \rightarrow \quad \sigma = \eta \dot{\varepsilon} \quad \rightarrow \quad N = \sigma A = \eta A \dot{\varepsilon} = \frac{\eta A}{l} \dot{\Delta}l = b \dot{\Delta}l$$

The linear viscous material is subjected to a loading stress cycle. The work per unit of volume can be calculated and appears to be non-zero. All the work is dissipated as can be seen from the stress-strain curve : the area included by the stress-strain trajectory represents the specific dissipated energy.

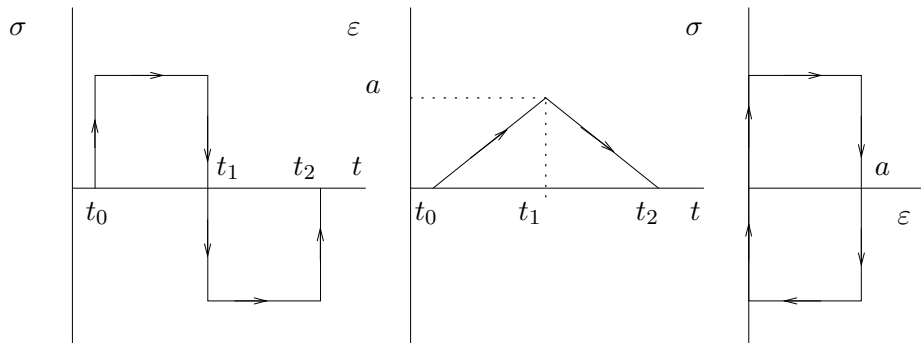


Fig. 6.48 : Loading cycle applied to linear viscous truss

$$\begin{aligned}
 U_d &= \int_{t_0}^{t_1} \sigma d\varepsilon + \int_{t_1}^{t_2} \sigma d\varepsilon = \int_{t_0}^{t_1} \eta \dot{\varepsilon} d\varepsilon + \int_{t_1}^{t_2} \eta \dot{\varepsilon} d\varepsilon = \int_{t_0}^{t_1} \eta c d\varepsilon - \int_{t_1}^{t_2} \eta c d\varepsilon \\
 &= \eta c [\varepsilon_1 - \varepsilon_0 - \varepsilon_2 + \varepsilon_1] = 2\eta c a
 \end{aligned}$$

6.3 Viscoelastic material behavior

Viscoelastic material behavior is a combination of elastic and viscous behavior. Part of the deformation energy will be dissipated, while the rest is stored as reversible elastic energy. Viscoelastic behavior can be characterized by mechanical models build from springs and dashpots.

We will assume the deformation to be small, so that the choice of stress and strain definitions is irrelevant. First the characteristics of the viscoelastic material behavior will be described, based on experimental observations. To predict the behavior, viscoelastic models are needed, which will be based on the behavior of springs and dashpots.

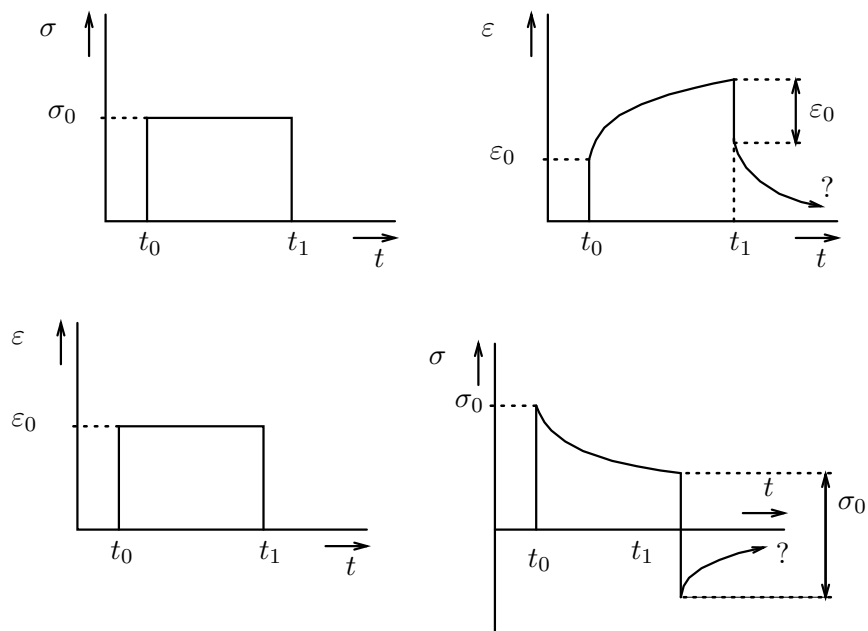


Fig. 6.49 : *Stress-strain curves for viscoelastic material behavior*

To investigate the characteristics of viscoelastic material behavior, a tensile test is carried out, where a tensile bar is loaded with stress excitations, which are prescribed as a step-function in time.

6.4 Proportionality

In a tensile test, a stress-step is applied and the strain is measured as a function of time. The test is repeated for increased stress amplitudes. From the measurement data, the strain values at the same time after loading are plotted against the stress amplitudes. The resulting plots are *isochrones*, because they represent the strain at the same time after loading.

For linear viscoelastic material behavior the isochrones are straight lines. This means that the strain as a function of time is *proportional* to the stress. The strain response can be written as the product of the stress amplitude $\Delta\sigma$ and a function of the time $D(t - t_0)$, whose value is zero for $t < t_0$.

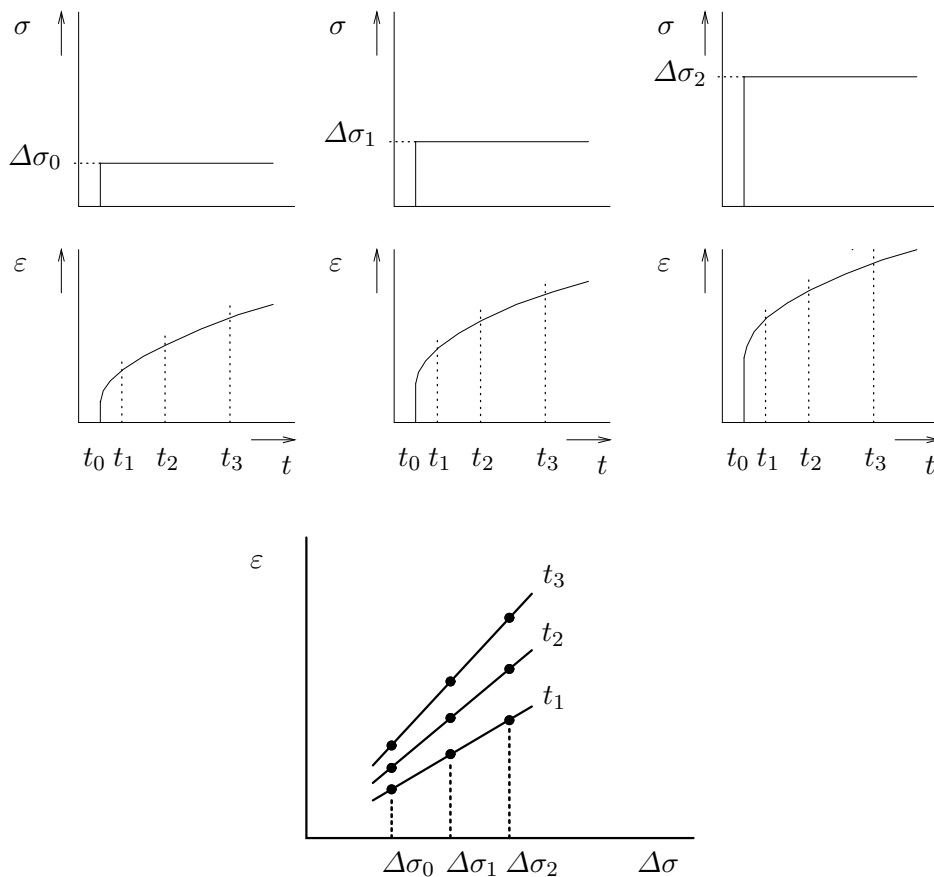


Fig. 6.50 : Proportionality of strain response and stress excitation

$$\varepsilon(t) = \Delta\sigma D(t - t_0) \quad \text{for} \quad \forall t \geq t_0$$

6.5 Superposition

A tensile test is carried out three times. In the first two tests, a stress step with different amplitude is applied and the strain response is measured. Then, in the third experiment, the

two stress steps are applied subsequently and again the strain response is measured.

For linear viscoelastic material behavior, the strain response in the third experiment is the sum of the separate responses in the first two experiments. This means that strain responses can be determined by *superposition*.

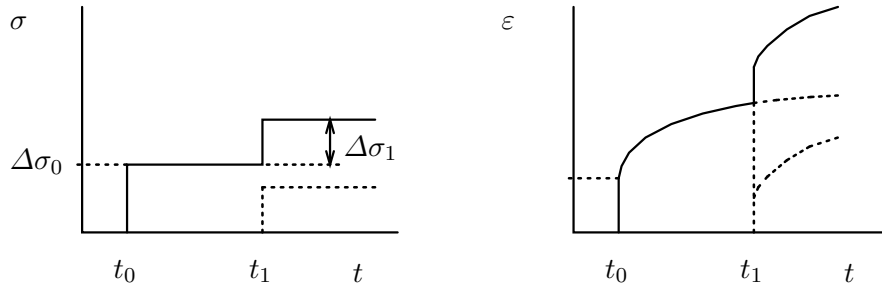


Fig. 6.51 : *Superposition of strain responses to two stress excitations*

separate excitations

$$\begin{aligned} \Delta\sigma = \Delta\sigma_0 &\quad \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) && \text{for } t > t_0 \\ \Delta\sigma = \Delta\sigma_1 &\quad \rightarrow \quad \varepsilon(t) = \Delta\sigma_1 D(t - t_1) && \text{for } t > t_1 \end{aligned}$$

subsequent excitations

$$\begin{aligned} \Delta\sigma = \Delta\sigma_0 &\quad \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) && \text{for } t_0 < t < t_1 \\ \Delta\sigma = \Delta\sigma_0 + \Delta\sigma_1 &\quad \rightarrow \quad \varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1) && \text{for } t > t_1 \end{aligned}$$

6.6 Boltzmann integral

Linear viscoelasticity is characterized by the two properties described in the previous sections.

1. proportionality : At every time the strain response is proportional to the amplitude of a constant stress step which is applied at $t = t_0$: $\varepsilon_i(t) = \Delta\sigma_i D(t - t_0)$ for $t > t_0$
2. superposition : The strain response to two subsequently (at time $t = t_0$ and $t = t_1$) applied constant amplitude ($\Delta\sigma_0$ and $\Delta\sigma_1$) stress steps equals the sum of the separate responses for $t > t_1$: $\varepsilon(t) = \Delta\sigma_0 D(t - t_0) + \Delta\sigma_1 D(t - t_1)$ for $t > t_1$

Every stress excitation can be seen as an infinite sequence of infinitesimal small stress steps. The superposition property then leads to the Boltzmann integral expressing the strain response. This integral is also called Duhamel or memory integral.

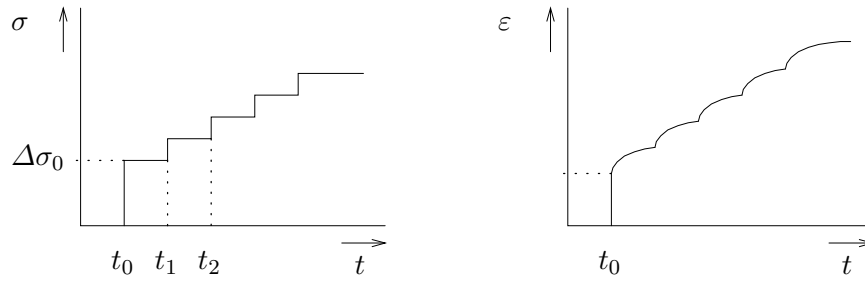


Fig. 6.52 : Superposition of strain responses to subsequent stress excitations

$$\begin{aligned}
 \varepsilon(t) &= \Delta\sigma_0 D(t-t_0) + \Delta\sigma_1 D(t-t_1) + \Delta\sigma_2 D(t-t_2) + \dots \\
 &= \sum_{i=1}^n \Delta\sigma_i D(t-t_i) \quad \rightarrow \quad \text{limit } n \rightarrow \infty \quad (t \rightarrow \tau) \\
 &= \int_{\tau=t_0^-}^t D(t-\tau) d\sigma(\tau) = \int_{\tau=t_0^-}^t D(t-\tau) \frac{d\sigma(\tau)}{d\tau} d\tau \\
 \varepsilon(t) &= \int_{\tau=t_0^-}^t D(t-\tau) \dot{\sigma}(\tau) d\tau
 \end{aligned}$$

For strain excitation and stress response the same observations can be made and we arrive at the Boltzmann integral for the stress response.

$$\sigma(t) = \int_{\tau=t_0^-}^t E(t-\tau) \dot{\varepsilon}(\tau) d\tau$$

6.7 Step excitations

Step excitations are important for the (experimental) characterization of viscoelastic materials. A unit step (amplitude = 1) can be described with the Heaviside function. The derivative of the unit step function is the Dirac function or unit pulse. It has the important property that integration of the product of a function $f(\tau)$ and $\delta(\tau, t^*)$ over an interval which contains $\tau = t^*$, the "location" of the Dirac pulse, results in the value $f(t^*)$.

$$\text{Heaviside function} \quad H(t, t^*) \quad \left\{ \begin{array}{ll} t < t^* & : \quad H(t, t^*) = 0 \\ t > t^* & : \quad H(t, t^*) = 1 \end{array} \right\}$$

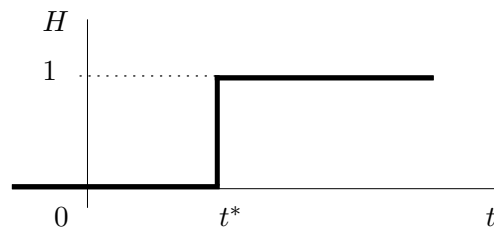


Fig. 6.53 : Unit step or Heaviside function

Dirac function $\delta(t, t^*) = \frac{d}{dt} \{H(t, t^*)\}$

$$\int_{\tau=0}^{t > t^*} \delta(\tau, t^*) d\tau = 1 \quad ; \quad \int_{\tau=0}^{t > t^*} f(\tau) \delta(\tau, t^*) d\tau = f(t^*)$$

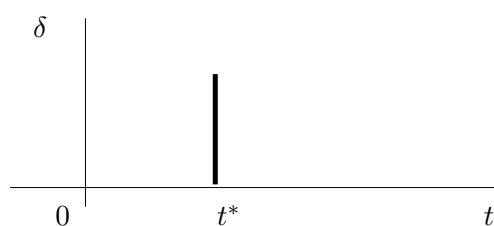


Fig. 6.54 : Unit pulse or Dirac function

6.8 Creep (retardation)

The strain response to a stress step excitation at $t = 0$ having an amplitude σ_0 equals $\sigma_0 D(t)$. The measured strain response can be used to fit a proposed model for $D(t)$. Experiments have revealed some characteristic properties.

$$\sigma(t) = \sigma_0 H(t, 0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t, 0)$$

$$\varepsilon(t) = \int_{\tau=0^-}^t D(t - \tau) \dot{\sigma}(\tau) d\tau = \int_{\tau=0^-}^t D(t - \tau) \sigma_0 \delta(\tau, 0) d\tau = \sigma_0 D(t)$$

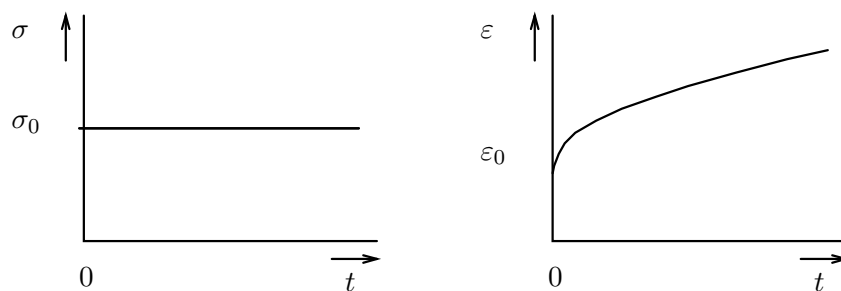


Fig. 6.55 : Creep strain response to unit stress step

- $\dot{D}(t) \geq 0 \quad \forall \quad t \geq 0$
- $\ddot{D}(t) < 0 \quad \forall \quad t \geq 0$

6.9 Relaxation

The stress response to a strain step excitation at $t = 0$ having an amplitude ε_0 equals $\varepsilon_0 E(t)$. The measured stress response can be used to fit a proposed model for $E(t)$. Experiments have revealed some characteristic properties.

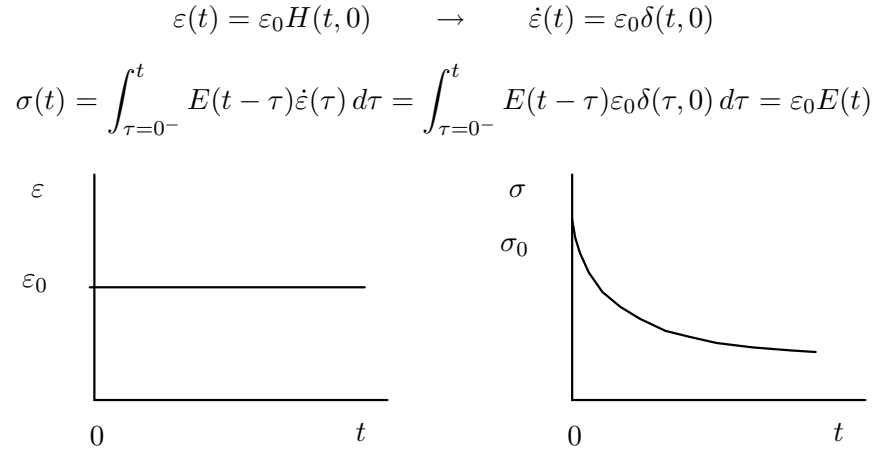


Fig. 6.56 : *Stress relaxation response to unit strain step*

- $\dot{E}(t) \leq 0 \quad \forall \quad t \geq 0$
- $\ddot{E}(t) > 0 \quad \forall \quad t \geq 0$
- $\int_{t=0}^{\infty} \dot{E}(t) dt \geq 0 \quad \rightarrow \quad \lim_{t \rightarrow \infty} \dot{E}(t) = 0$

6.10 Harmonic strain excitation

For the experimental characterization of viscoelastic materials, harmonic excitation is of great importance. We consider first a tensile test where the strain is prescribed harmonically with an angular frequency ω [rad s⁻¹] and amplitude ε_0 .

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

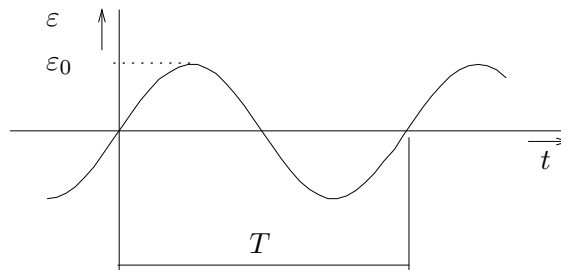


Fig. 6.57 : *Harmonic strain excitation*

6.11 Stress response

The stress response to the harmonic strain excitation can be calculated using the Boltzmann integral. Evaluating the integral involves transformation to another integration variable.

The result reveals two important viscoelastic material parameters : the *storage modulus* E' and the *loss modulus* E'' , which both are a function of the angular frequency ω . They can be measured with a Dynamic Mechanical Analysis (DMA) experiment and transformed into a relaxation function $E(t)$. This experiment is more easy to perform and more accurate than the direct measurement of $E(t)$ in a relaxation experiment.

$$\begin{aligned}
 \sigma(t) &= \int_{\tau=-\infty}^t E(t-\tau) \varepsilon_0 \omega \cos(\omega\tau) d\tau = \varepsilon_0 \omega \int_{\xi=-\infty}^t E(t-\tau) \cos(\omega\tau) d\tau \\
 &\quad t - \tau = s \quad \rightarrow \quad \tau = t - s \quad \rightarrow \quad d\tau = -ds \\
 &= \varepsilon_0 \omega \int_{s=0}^{\infty} E(s) \cos\{\omega(t-s)\} ds \\
 &\quad \cos(\omega t - \omega s) = \cos(\omega t) \cos(\omega s) + \sin(\omega t) \sin(\omega s) \\
 &= \varepsilon_0 \left[\omega \int_{s=0}^{\infty} E(s) \sin(\omega s) ds \right] \sin(\omega t) + \varepsilon_0 \left[\omega \int_{s=0}^{\infty} E(s) \cos(\omega s) ds \right] \cos(\omega t) \\
 &= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)
 \end{aligned}$$

$$\begin{aligned}
 E'(\omega) &= \omega \int_{s=0}^{\infty} E(s) \sin(\omega s) ds && : \text{ storage modulus} \\
 E''(\omega) &= \omega \int_{s=0}^{\infty} E(s) \cos(\omega s) ds && : \text{ loss modulus}
 \end{aligned}$$

6.12 Energy dissipation

The dissipated energy per unit of volume during one period of the harmonic strain excitation can be calculated. This dissipated energy must always be positive. As shown below, it must be concluded that the loss modulus E'' is also positive.

Referring to the calculated stress response, we can conclude that the stress at time $t = 0$ where the strain was taken to be $\varepsilon = 0$, has a positive value. We thus have proved something which we already knew from experiments : there is a phase difference between strain and stress and the stress shows a gain w.r.t. the strain.

$$\begin{aligned}
 U_d &= \int_{\varepsilon(0)}^{\varepsilon(T)} \sigma d\varepsilon = \int_{t=0}^T \sigma \dot{\varepsilon} dt \\
 &= \int_{t=0}^T \{ \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t) \} \{ \varepsilon_0 \omega \cos(\omega t) \} dt \\
 &= \int_{t=0}^T \varepsilon_0^2 \omega \{ E' \sin(\omega t) \cos(\omega t) + E'' \cos^2(\omega t) \} dt \\
 &= \int_{t=0}^T \varepsilon_0^2 \omega \left\{ \frac{1}{2} E' \sin(2\omega t) + \frac{1}{2} E'' + \frac{1}{2} E'' \cos(2\omega t) \right\} dt
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\varepsilon_0^2\omega \left[-E' \frac{1}{2\omega} \cos(2\omega t) + E''t + E'' \frac{1}{2\omega} \sin(2\omega t) \right]_{t=0}^{t=\frac{2\pi}{\omega}} \\
&= \frac{1}{2}\varepsilon_0^2\omega \left[-E' \frac{1}{2\omega} + E' \frac{1}{2\omega} + E'' \frac{2\pi}{\omega} \right] \\
&= \pi\varepsilon_0^2 E'' > 0 \quad \Rightarrow \quad E'' > 0 \quad \rightarrow
\end{aligned}$$

$$\sigma(t=0) = \varepsilon_0 E'' > 0$$

The phase difference between stress and strain results in a so-called hysteresis loop, when a stress-strain diagram is drawn. The area enclosed by the hysteresis loop is a measure for the dissipated energy per unit of volume during one cycle.

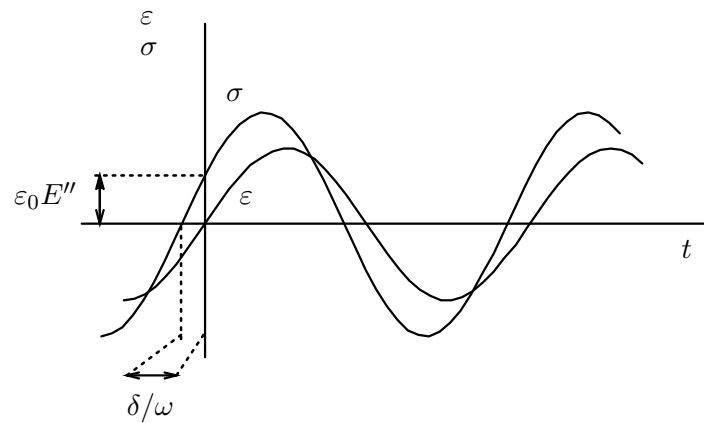


Fig. 6.58 : *Harmonic strain excitation and stress response*

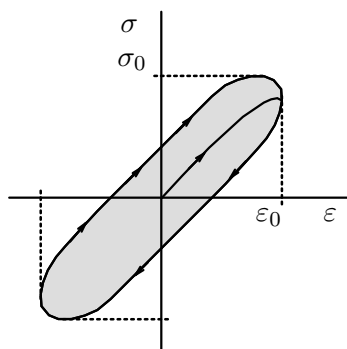


Fig. 6.59 : *Hysteresis of stress and strain*

6.13 Relation between E' , E'' and δ

Writing the stress response with two different relations, results in relations between E' , E'' and δ . The amplitude σ_0 of the stress response can also be calculated.

$$\begin{aligned}\sigma(t) &= \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\delta) \sin(\omega t) + \sigma_0 \sin(\delta) \cos(\omega t) \\ \sigma(t) &= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t)\end{aligned}$$

storage and loss modulus

$$\left. \begin{aligned} E' &= \frac{\sigma_0}{\varepsilon_0} \cos(\delta) \\ E'' &= \frac{\sigma_0}{\varepsilon_0} \sin(\delta) \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \frac{E''}{E'} &= \tan(\delta) \rightarrow \\ \delta &= \arctan\left(\frac{E''}{E'}\right) \end{aligned} \right.$$

amplitude $\sigma_0 = \varepsilon_0 \sqrt{(E')^2 + (E'')^2}$

6.14 Measured E' , E'' and $\tan(\delta)$

Typical measured values for $E'(\omega)$, $E''(\omega)$ and $\tan(\delta)$ are shown in the plots below. For low and high frequencies, the loss modulus is zero, indicating that there is no dissipation and the material behaves elastically. For high frequencies, the "stiffness" E' is much higher than for low frequencies.

Storage and loss moduli can be measured accurately using DMA test equipment. From $E'(\omega)$ and $E''(\omega)$, the relaxation function $E(t)$ can be calculated using dedicated software.

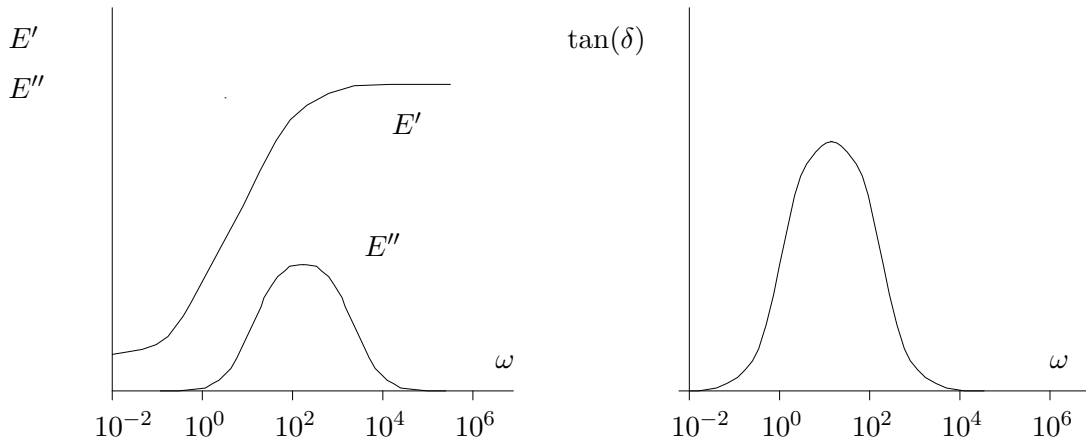


Fig. 6.60 : Characteristic values of E' , E'' and $\tan(\delta)$

6.15 Harmonic stress excitation

The axial stress can be prescribed harmonically with an angular frequency ω [rad s⁻¹]. The strain response can be calculated with the Boltzmann integral and appears to be characterized by the *storage compliance* $D'(\omega)$ and the *loss compliance* $D''(\omega)$. Both compliances are positive for all ω . Because $\varepsilon(t = 0) < 0$, the definition of D' includes a minus sign.

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

$$\begin{aligned} \varepsilon(t) &= \int_{\tau=-\infty}^t D(t-\tau) \dot{\sigma}(\tau) d\tau = \int_{\tau=-\infty}^t D(t-\tau) \sigma_0 \omega \cos(\omega \tau) d\tau \\ &= \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s) \sin(\omega s) ds \right] \sin(\omega t) + \sigma_0 \left[\omega \int_{s=0}^{\infty} D(s) \cos(\omega s) ds \right] \cos(\omega t) \\ &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t) \end{aligned}$$

$$\begin{aligned} D'(\omega) &= \omega \int_{s=0}^{\infty} D(s) \sin(\omega s) ds && : \text{ storage compliance} \\ D''(\omega) &= -\omega \int_{s=0}^{\infty} D(s) \cos(\omega s) ds && : \text{ loss compliance} \end{aligned}$$

6.16 Relation between D' , D'' and δ

Writing the strain response with two different relations, results in relations between D' , D'' and δ . The amplitude ε_0 of the strain response can also be calculated.

$$\begin{aligned} \varepsilon(t) &= \varepsilon_0 \sin(\omega t - \delta) = \varepsilon_0 \cos(\delta) \sin(\omega t) - \varepsilon_0 \sin(\delta) \cos(\omega t) \\ \varepsilon(t) &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t) \end{aligned}$$

storage and loss compliance

$$\left. \begin{aligned} D' &= \frac{\varepsilon_0}{\sigma_0} \cos(\delta) \\ D'' &= \frac{\varepsilon_0}{\sigma_0} \sin(\delta) \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \frac{D''}{D'} &= \tan(\delta) \rightarrow \\ \delta &= \arctan\left(\frac{D''}{D'}\right) \end{aligned} \right.$$

amplitude

$$\varepsilon_0 = \sigma_0 \sqrt{(D')^2 + (D'')^2}$$

6.17 Measured D' and D''

Typical measured values for $D'(\omega)$, $D''(\omega)$ are shown in the plots below. Again it is obvious that the loss compliance is zero for both very low and very high frequencies. The storage compliance is reversely proportional to the frequency.

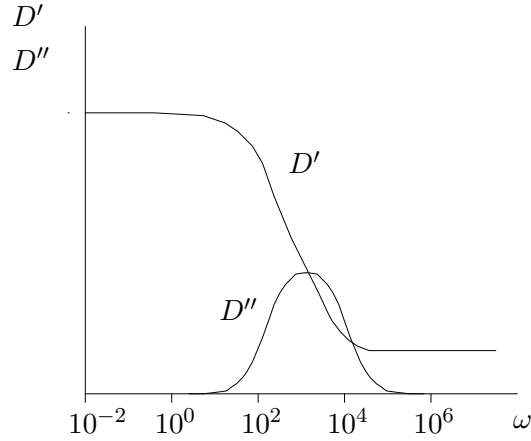


Fig. 6.61 : Characteristic values of D' and D''

6.18 Relation between (D', D'') and (E', E'')

There is a relation between storage and loss modulus on the one hand and storage and loss compliance on the other. Remember that there is **not** such a relation between the relaxation function and the creep function.

$$\left. \begin{aligned} \sigma_0 &= \varepsilon_0 \sqrt{(E')^2 + (E'')^2} \\ \varepsilon_0 &= \sigma_0 \sqrt{(D')^2 + (D'')^2} \end{aligned} \right\} \rightarrow [(E')^2 + (E'')^2][(D')^2 + (D'')^2] = 1 \quad (1)$$

$$\frac{D''}{D'} = \frac{E''}{E'} \rightarrow D'' = D' \frac{E''}{E'} \quad (2)$$

$$(1) \ \& \ (2) \quad \rightarrow \quad D' = \frac{E'}{(E')^2 + (E'')^2} \quad ; \quad D'' = \frac{E''}{(E')^2 + (E'')^2}$$

$$\text{idem} \quad \quad \quad E' = \frac{D'}{(D')^2 + (D'')^2} \quad ; \quad E'' = \frac{D''}{(D')^2 + (D'')^2}$$

6.19 Complex variables

In literature on viscoelastic behavior and modeling, complex variables are often used. They can be derived easily by writing the strain excitation and the stress response as the real part of a complex number, where Euler's formula $e^{i\alpha x} = \cos(\alpha x) + i \sin(\alpha x)$ is used.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) = \varepsilon_0 \cos(\omega t - \frac{\pi}{2}) = \text{Re} \left[\varepsilon_0 e^{-i\frac{\pi}{2}} e^{i\omega t} \right] = \text{Re} \left[\varepsilon^* e^{i\omega t} \right]$$

$$\sigma(t) = \sigma_0 \sin(\omega t + \delta) = \sigma_0 \cos(\omega t - \frac{\pi}{2} + \delta) = \text{Re} \left[\sigma_0 e^{i(\delta - \frac{\pi}{2})} e^{i\omega t} \right] = \text{Re} \left[\sigma^* e^{i\omega t} \right]$$

complex modulus and compliance

$$E^* = \frac{\sigma^*}{\varepsilon^*} = \frac{\sigma_0}{\varepsilon_0} e^{i\delta} = \frac{\sigma_0}{\varepsilon_0} \cos(\delta) + i \frac{\sigma_0}{\varepsilon_0} \sin(\delta) = E' + iE''$$

$$D^* = \frac{\varepsilon^*}{\sigma^*} = \frac{\varepsilon_0}{\sigma_0} e^{-i\delta} = \frac{\varepsilon_0}{\sigma_0} \cos(\delta) - i \frac{\varepsilon_0}{\sigma_0} \sin(\delta) = D' - iD''$$

dynamic modulus en compliance

$$E_d = |E^*| = \sqrt{(E')^2 + (E'')^2} = \frac{\sigma_0}{\varepsilon_0}$$

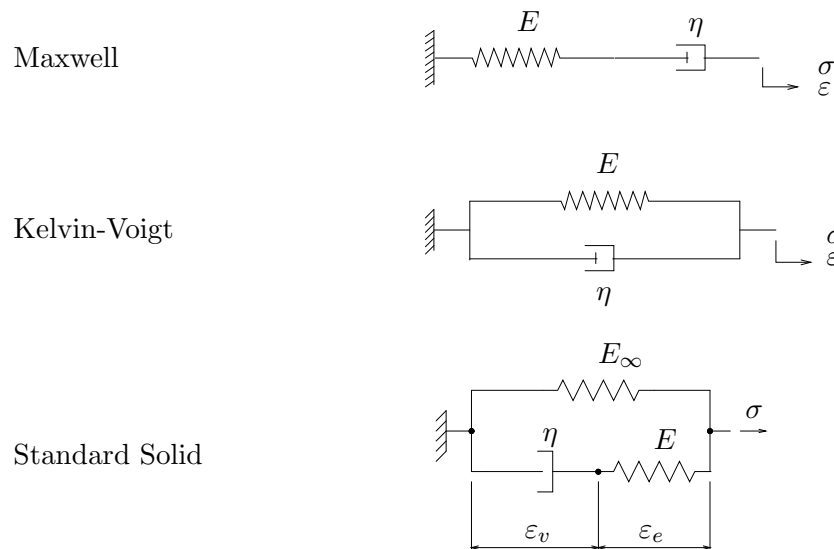
$$D_d = |D^*| = \sqrt{(D')^2 + (D'')^2} = \frac{\varepsilon_0}{\sigma_0}$$

6.20 Viscoelastic models

The response of a viscoelastic material is given by the Boltzmann integral and to calculate it we need the creep and/or relaxation functions $D(t)$ and $E(t)$. Mathematical expressions can be chosen for these functions taking into account some general requirements. The chosen functions can then be fitted onto data from creep and relaxation tests. Instead of choosing rather arbitrary functions, they are generally derived from the behavior of one-dimensional mechanical spring-dashpot systems. Simple systems like the Maxwell, Kelvin-Voigt and Standard Solid element, are not always useful, because the lack of parameters prohibits a good fit of experimental data. In practice Generalized Maxwell or Generalized Kelvin-Voigt models are used.

Because creep and relaxation tests may need a long experimental time period and accuracy is not high, harmonic excitation tests are carried out to determine $D'(\omega)$, $D''(\omega)$, $E'(\omega)$ and $E''(\omega)$. These parameters can be converted to $D(t)$ and $E(t)$. These experiments are generally known as D(ynamic) M(echanical) A(nalysis) or D(ynamic) M(echanical) T(hermal) A(nalysis), because time-temperature superposition is mostly used.

In the following we will study some mechanical models. Their behavior is described by a differential equation. Solving this for stress or strain excitations results in the viscoelastic material functions.



Generalized Maxwell

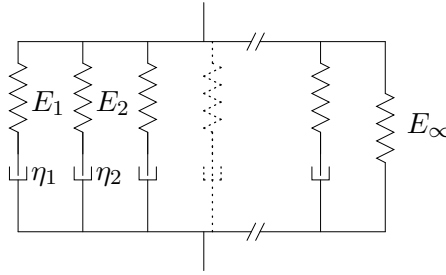


Fig. 6.62 : Discrete mechanical models for viscoelastic material behavior

6.21 Maxwell model

One of the simplest models to describe linear viscoelastic material behavior is the Maxwell model. It consists of a spring (modulus E) and a dashpot (viscosity η) in series.

The stress and strain in/of the Maxwell element is related by a first-order differential equation. For both stress and strain excitation, the differential equation can be solved, using appropriate initial conditions. General solutions – integrals for stress and strain – can be derived.



Fig. 6.63 : Maxwell model

$$\varepsilon = \varepsilon_E + \varepsilon_\eta \quad \rightarrow \quad \dot{\varepsilon} = \dot{\varepsilon}_E + \dot{\varepsilon}_\eta = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

6.22 Maxwell : step excitations

For step excitations of stress and strain the differential equation of the Maxwell model can be solved. The response represents the creep and relaxation functions, respectively.

$$\sigma(t) = \sigma_0 H(t, 0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t, 0)$$

$$\dot{\varepsilon}(t) = \frac{\sigma_0}{E} \delta(t, 0) + \frac{\sigma_0}{\eta}$$

$$\varepsilon(t) = \frac{\sigma_0}{E} H(t, 0) + \frac{\sigma_0}{\eta} t = \sigma_0 \left[\frac{1}{\eta} \left(t + \frac{\eta}{E} \right) \right] = \sigma_0 D(t)$$

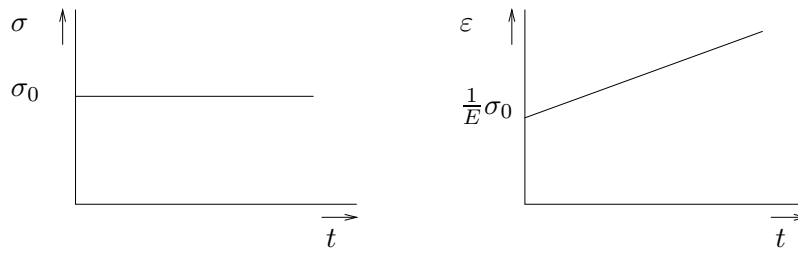


Fig. 6.64 : Creep for a Maxwell model

$$\varepsilon(t) = \varepsilon_0 H(t, 0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t, 0)$$

$$\sigma(t) = \varepsilon_0 E e^{-\frac{E}{\eta} t} = \varepsilon_0 E e^{-\frac{t}{\tau_m}} = \varepsilon_0 E(t)$$

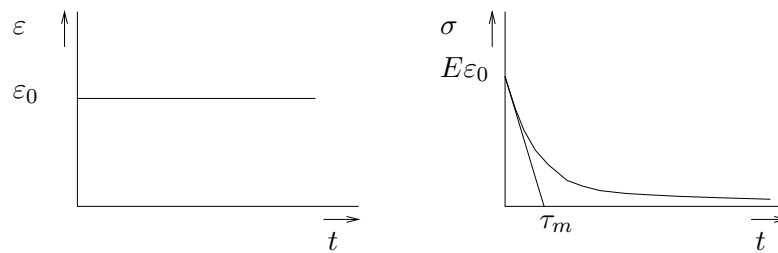


Fig. 6.65 : Relaxation for a Maxwell model

6.23 Maxwell : Boltzmann integrals

For a general stress and strain excitation the differential equation of the Maxwell model can also be solved. These general solutions are Boltzmann integrals, which can be used to calculate strain/stress responses to stress/strain excitations.

The creep and relaxation functions of the Maxwell model are readily recognized in the integrals. Response to step excitations reveals that the Maxwell model describes viscoelastic fluid behavior, characterized by a time constant $\tau = \frac{\eta}{E}$ [s].

$$\varepsilon(t) = \frac{1}{\eta} \int_{\tau=-\infty}^t \left\{ (t - \tau) + \frac{\eta}{E} \right\} \dot{\sigma}(\tau) d\tau = \int_{\tau=-\infty}^t D(t - \tau) \dot{\sigma}(\tau) d\tau$$

$$\sigma(t) = \int_{\tau=-\infty}^t \left\{ E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau) d\tau = \int_{\tau=-\infty}^t E(t - \tau) \dot{\varepsilon}(\tau) d\tau$$

6.24 Maxwell : harmonic stress excitation

The strain response of the Maxwell model to an harmonic stress excitation is readily calculated from the differential equation. Storage and loss compliances are thus determined.

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned} \dot{\varepsilon}(t) &= \frac{1}{E} \sigma_0 \omega \cos(\omega t) + \frac{1}{\eta} \sigma_0 \sin(\omega t) \\ \varepsilon(t) &= \sigma_0 \left[\frac{1}{E} \right] \sin(\omega t) - \sigma_0 \left[\frac{1}{\eta \omega} \right] \cos(\omega t) \\ &= \varepsilon_P(t) \qquad \qquad \qquad \varepsilon_H \text{ damps out} \\ &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t) \end{aligned}$$

dynamic quantities

$$D' = \frac{1}{E} \quad ; \quad D'' = \frac{1}{\eta \omega} \quad ; \quad \delta = \arctan \left(\frac{D''}{D'} \right) = \arctan \left(\frac{E}{\eta \omega} \right)$$

6.25 Maxwell : harmonic strain excitation

With the Boltzmann integral for the Maxwell model, the stress response to an harmonic strain excitation can be calculated. Storage and loss moduli are obtained as a function of ω . Comparing these functions with measured values reveals that the Maxwell model is generally not adequate to describe viscoelastic behavior of real materials.

$$\varepsilon(t) = \varepsilon_0 \sin(\omega t) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \omega \cos(\omega t)$$

stress response

$$\begin{aligned} \sigma(t) &= \int_{\tau=-\infty}^t E(t-\tau) \dot{\varepsilon}(\tau) d\tau \\ &= E \varepsilon_0 \omega e^{-\frac{E}{\eta} t} \int_{\tau=0}^t e^{\frac{E}{\eta} \tau} \cos(\omega \tau) d\tau \\ &= \left[\frac{E \varepsilon_0 \omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta} \right] e^{-\frac{E}{\eta} t} + \left[\frac{E \varepsilon_0 \omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \omega \right] \sin(\omega t) + \left[\frac{E \varepsilon_0 \omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta} \right] \cos(\omega t) \\ &= \varepsilon_0 \left[\frac{E \omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \omega \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E \omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta} \right] \cos(\omega t) \quad \text{for } t \geq 0 \\ &= \varepsilon_0 \left[\frac{E \omega^2 \tau_m^2}{1 + \omega^2 \tau_m^2} \right] \sin(\omega t) + \varepsilon_0 \left[\frac{E \omega \tau_m}{1 + \omega^2 \tau_m^2} \right] \cos(\omega t) \\ &= \varepsilon_0 E' \sin(\omega t) + \varepsilon_0 E'' \cos(\omega t) \end{aligned}$$

dynamic quantities

$$E' = \frac{E \omega^2}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \quad ; \quad E'' = \frac{E \omega \left(\frac{E}{\eta}\right)}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \quad ; \quad \tan(\delta) = \frac{E''}{E'} = \frac{1}{\omega \tau_m}$$

6.26 Kelvin-Voigt model

The Kelvin-Voigt model is a simple model for the description of linear viscoelastic material behavior. It consists of a spring (modulus E) parallel to a dashpot (viscosity η).

The stress and strain in/of the Kelvin-Voigt element is related by a first-order differential equation. For strain excitation, this equation directly describes the stress response. For stress excitation, a general integral solution of the differential equation can be derived.

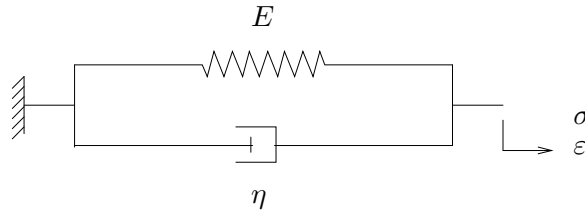


Fig. 6.66 : *Kelvin model*

$$\sigma = \sigma_E + \sigma_\eta = E\varepsilon + \eta\dot{\varepsilon}$$

6.27 Kelvin-Voigt : step excitations

Strain response to a step excitation of stress reveals that the Kelvin-Voigt model describes viscoelastic solid behavior, characterized by the time constant $\tau = \frac{\eta}{E}$ [s]. A stepwise strain excitation leads to infinite stress.

$$\sigma(t) = \sigma_0 H(t, 0) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \delta(t, 0)$$

$$\left. \begin{aligned} \eta\dot{\varepsilon}(t) + E\varepsilon(t) &= \sigma(t) = \sigma_0 H(t, 0) \\ \varepsilon(t) &= \varepsilon_H(t) + \varepsilon_P = C e^{-\frac{E}{\eta}t} + \frac{\sigma_0}{E} \\ \varepsilon(t=0) &= 0 \end{aligned} \right\} \rightarrow C = -\frac{\sigma_0}{E}$$

$$\varepsilon(t) = \frac{\sigma_0}{E} \left[1 - e^{-\frac{E}{\eta}t} \right] = \sigma_0 D(t)$$

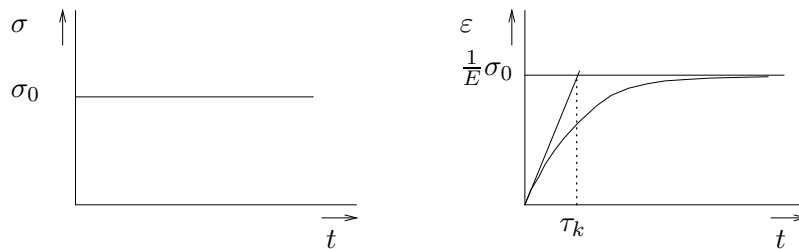


Fig. 6.67 : *Creep of a Kelvin model*

$$\varepsilon(t) = \varepsilon_0 H(t, 0) \quad \rightarrow \quad \dot{\varepsilon}(t) = \varepsilon_0 \delta(t, 0)$$

$$\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t)$$

$$\sigma(t) = E\varepsilon_0 + \eta\varepsilon_0\delta(t, 0) = \varepsilon_0 [E + \eta\delta(t, 0)] = \infty$$

6.28 Kelvin-Voigt : Boltzmann integral

The general solution for the strain response to a stress excitation is given by a Boltzmann integral, in which we recognize the creep function of the Kelvin-Voigt element. For a general strain excitation the stress response can be calculated directly from the Kelvin-Voigt element equation.

$$\varepsilon(t) = \frac{1}{E} \int_{\tau=-\infty}^t \left\{ 1 - e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\sigma}(\tau) d\tau = \int_{\tau=-\infty}^t D(t-\tau) \dot{\sigma}(\tau) d\tau$$

6.29 Kelvin-Voigt : harmonic stress excitation

For the Kelvin-Voigt model, the storage and loss compliance can be calculated. The Boltzmann integral with the Kelvin-Voigt creep function is used to calculate the strain response for an harmonic stress excitation.

$$\sigma(t) = \sigma_0 \sin(\omega t) \quad \rightarrow \quad \dot{\sigma}(t) = \sigma_0 \omega \cos(\omega t)$$

strain response

$$\begin{aligned} \varepsilon(t) &= \int_{\tau=0}^t D(t-\tau) \dot{\sigma}(\tau) d\tau \\ &= \sigma_0 \left[\frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} \right] \cos(\omega t) \\ &= \sigma_0 \left[\frac{1}{E(1 + \omega^2 \tau_k^2)} \right] \sin(\omega t) - \sigma_0 \left[\frac{\omega \tau_k}{E(1 + \omega^2 \tau_k^2)} \right] \cos(\omega t) \\ &= \sigma_0 D' \sin(\omega t) - \sigma_0 D'' \cos(\omega t) \end{aligned}$$

dynamic quantities

$$D'(\omega) = \frac{1}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{E}{\eta^2} = \frac{1}{E(1 + \omega^2 \tau_k^2)}$$

$$D''(\omega) = \frac{\omega}{\left(\frac{E}{\eta}\right)^2 + \omega^2} \frac{1}{\eta} = \frac{\omega \tau_k}{E(1 + \omega^2 \tau_k^2)}$$

$$\tan(\delta) = \frac{D''}{D'} = \omega \tau_k \quad \rightarrow \quad \delta = \arctan\left(\frac{\eta \omega}{E}\right)$$

6.30 Standard Solid model

The Standard Solid model consists of a parallel arrangement of a Maxwell element (modulus E , viscosity η) and a linear spring (modulus E_∞). This model incorporates the Maxwell model ($E_\infty = 0$) and the Kelvin-Voigt model ($E = 0$). The stress-strain relation is described by a differential equation, which can be solved resulting in Boltzmann integrals for strain and stress.

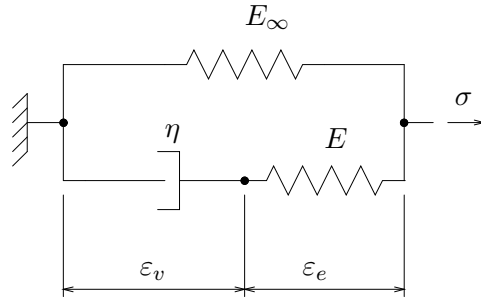


Fig. 6.68 : *Standard Solid model*

constitutive relations

- $\sigma = \sigma_\infty + \sigma_{ve}$
- $\dot{\sigma}_{ve} = E\varepsilon_e \rightarrow \dot{\varepsilon}_e = \frac{1}{E}\dot{\sigma}_{ve}$
- $\dot{\varepsilon} = \dot{\varepsilon}_v + \dot{\varepsilon}_e$
- $\varepsilon = \frac{1}{E_\infty}\sigma_\infty$
- $\dot{\varepsilon}_v = \frac{1}{\eta}\sigma_{ve}$

constitutive equation

$$\begin{aligned}
 \sigma &= \sigma_\infty + \sigma_{ve} = E_\infty\varepsilon + \eta\dot{\varepsilon}_v \\
 &= E_\infty\varepsilon + \eta(\dot{\varepsilon} - \dot{\varepsilon}_e) = E_\infty\varepsilon + \eta\dot{\varepsilon} - \eta\frac{\dot{\sigma}_{ve}}{E} \\
 &= E_\infty\varepsilon + \eta\dot{\varepsilon} - \frac{\eta}{E}(\dot{\sigma} - E_\infty\dot{\varepsilon}) \rightarrow \\
 \sigma + \frac{\eta}{E}\dot{\sigma} &= E_\infty\varepsilon + \frac{\eta(E + E_\infty)}{E}\dot{\varepsilon}
 \end{aligned}$$

6.31 Standard Solid : step excitations

Solutions for the differential equation when applying a step in the stress or a step in the strain can be derived. The time constant for creep is defined as $\tau_c = \frac{\eta}{E} + \frac{\eta}{E_\infty}$ and the time constant for relaxation as $\tau_r = \frac{\eta}{E}$. They represent the intersection point of the tangent to the creep/relaxation curve at $t = 0$ and the asymptote for strain ($\frac{\sigma_0}{E_\infty}$) and stress ($\varepsilon_0 E_\infty$), respectively.

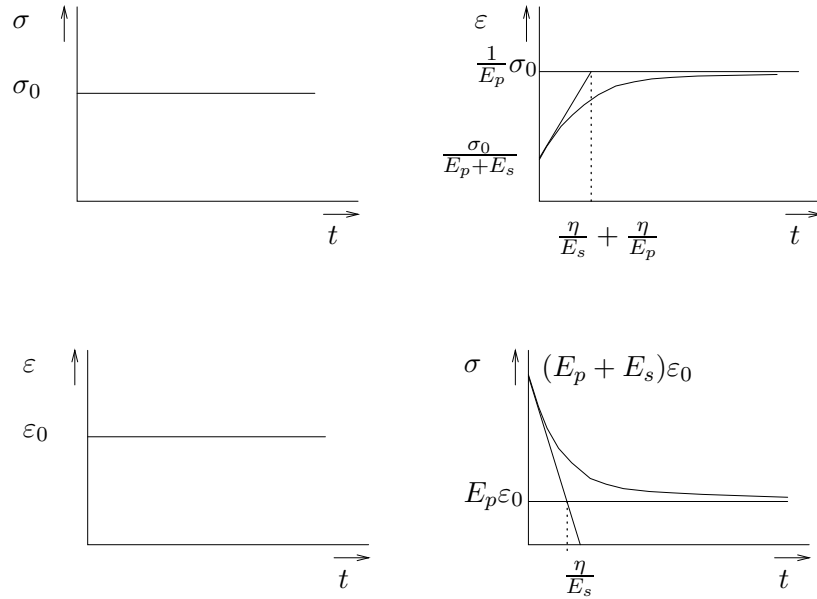


Fig. 6.69 : Creep and relaxation of a Standard Solid model

6.32 Standard Solid : Boltzmann integrals

In the Boltzmann integrals for strain and stress, the creep and relaxation functions of the Standard Solid element are readily recognized.

$$\begin{aligned}
 \varepsilon(t) &= \int_{\tau=-\infty}^t \left\{ \frac{1}{E_{\infty}} - \frac{E}{E_{\infty}(E_{\infty} + E)} e^{-\frac{E_{\infty}E}{\eta(E_{\infty}+E)}(t-\tau)} \right\} \dot{\sigma}(\tau) d\tau \\
 &= \int_{\tau=-\infty}^t D(t-\tau) \dot{\sigma}(\tau) d\tau \\
 \sigma(t) &= \int_{\tau=-\infty}^t \left\{ E_{\infty} + E e^{-\frac{E}{\eta}(t-\tau)} \right\} \dot{\varepsilon}(\tau) d\tau \\
 &= \int_{\tau=-\infty}^t E(t-\tau) \dot{\varepsilon}(\tau) d\tau
 \end{aligned}$$

6.33 Generalized Maxwell model

Both the Maxwell and the Kelvin-Voigt models are too simple to describe the viscoelastic behavior of real materials. Combining a number of Maxwell elements in a parallel configuration, leads to the generalized Maxwell model, which mostly also has an extra parallel spring for the correct description of long-term behavior of viscoelastic solid materials. Such a model is generally used for experimental characterization of the behavior of linear viscoelastic materials in a Dynamic Mechanical (Thermal) Analysis (DM(T)A) test.

The creep function $E(t)$ is easily determined and has a number of time constants to characterize the viscoelastic material response. A model like the generalized Maxwell model is therefore also referred to as *multi mode*.

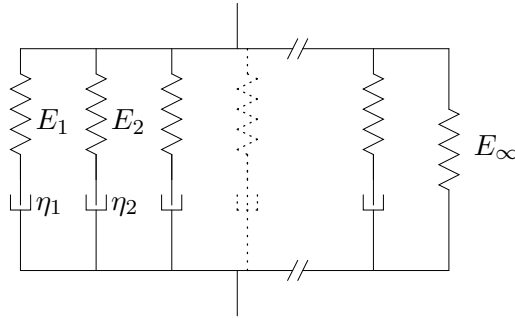


Fig. 6.70 : *Generalized Maxwell model*

$$E(t) = E_\infty + \sum_i E_i e^{-\frac{t}{\tau_i}} \quad ; \quad \tau_i = \frac{\eta_i}{E_i}$$

equilibrium modulus $E_\infty = \lim_{t \rightarrow \infty} E(t)$

glass modulus $E_g = \lim_{t \rightarrow 0} E(t) = E_\infty + \sum_i E_i$

6.34 Generalized Kelvin model

The generalized Kelvin model consists of a number of Kelvin-Voigt elements arranged in series. An extra spring – sometimes a dashpot – is also provided.

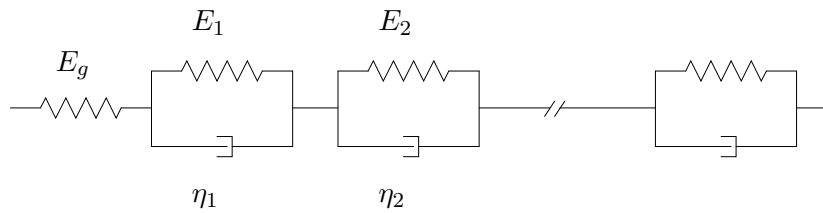


Fig. 6.71 : *Generalized Kelvin model*

$$D(t) = \frac{1}{E_g} + \sum_i \frac{1}{E_i} (1 - e^{-\frac{t}{\tau_i}}) \quad ; \quad \tau_i = \frac{\eta_i}{E_i}$$

$$= D_g + \sum_i D_i (1 - e^{-\frac{t}{\tau_i}})$$

$$\begin{aligned}
\text{glass compliance} & \quad D_g = \frac{1}{E_g} = \lim_{t \rightarrow 0} D(t) \\
\text{equilibrium compliance} & \quad D_\infty = \lim_{t \rightarrow \infty} D(t) = D_g + \sum_i D_i
\end{aligned}$$

6.35 Stress update

The current stress is given by a Boltzmann integral over the strain history.

Using a Generalized Maxwell model to specify the relaxation function $E(t)$, an expression for $\sigma(t)$ can be derived.

$$\left. \begin{aligned}
\sigma(t) &= \int_{\tau=0}^t E(t-\tau) \dot{\varepsilon}(\tau) d\tau \\
E(t) &= E_\infty + \sum_{i=1}^N E_i e^{-\frac{t}{\tau_i}}
\end{aligned} \right\} \rightarrow$$

$$\begin{aligned}
\sigma(t) &= \int_{\tau=0}^t \left[E_\infty + \sum_{i=1}^N E_i e^{-\frac{t-\tau}{\tau_i}} \right] \dot{\varepsilon}(\tau) d\tau = E_\infty \varepsilon(t) + \sum_{i=1}^N \int_{\tau=0}^t E_i e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau \\
&= E_\infty \varepsilon(t) + \sum_{i=1}^N \sigma_i(t)
\end{aligned}$$

6.36 Time discretization

In the numerical analysis of the time dependent behavior, the total time interval $[0, t]$ is discretized :

$$[0, t] \rightarrow [t_1 = 0, t_2, t_3, \dots, t_n, t_{n+1} = t]$$

The timespan between two discrete moments in the time interval is a time increment. It is assumed that these increments are of equal length :

$$\Delta t = t_{j+1} - t_j \quad ; \quad j = 1, \dots, n$$

The Boltzmann integral is now split in an integral over $[0, t_n]$ and an integral over the last or current increment $[t_n, t_{n+1} = t]$.

$$\begin{aligned}
\sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^N \sigma_i(t) = E_\infty \varepsilon(t) + \sum_{i=1}^N \int_{\tau=0}^t E_i e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau \\
&= E_\infty \varepsilon(t) + \sum_{i=1}^N \left[e^{-\frac{\Delta t}{\tau_i}} \int_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau + E_i \int_{\tau=t_n}^t e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau \right]
\end{aligned}$$

6.37 Linear incremental strain

For further evaluation of $\sigma(t)$ it is assumed that the strain is a linear function of time in each time increment. For the current increment we have :

$$\varepsilon(\tau) = \varepsilon(t_n) + (\tau - t_n) \frac{\Delta\varepsilon}{\Delta t} \quad \rightarrow \quad \dot{\varepsilon}(\tau) = \frac{\Delta\varepsilon}{\Delta t}$$

The integral over the current increment can now be evaluated very easily.

$$\int_{\tau=t_n}^t e^{-\frac{t-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau = \frac{\Delta\varepsilon}{\Delta t} \int_{\tau=t_n}^t e^{-\frac{t-\tau}{\tau_i}} d\tau = \frac{\Delta\varepsilon}{\Delta t} \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)$$

6.38 Stress

Calculating the current stress does not mean that the Boltzmann integral has to be evaluated over the total deformation history. When results are stored properly we can easily update the stress $\sigma(t)$.

$$\begin{aligned} \sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^N \sigma_i(t) \\ &= E_\infty \varepsilon(t) + \sum_{i=1}^N \left[e^{-\frac{\Delta t}{\tau_i}} \int_{\tau=0}^{t_n} E_i e^{-\frac{t_n-\tau}{\tau_i}} \dot{\varepsilon}(\tau) d\tau + E_i \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\varepsilon}{\Delta t} \right] \\ \sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^N \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right] \\ \text{with} \quad p_i &= \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \end{aligned}$$

6.39 Stiffness

The current stiffness of the material is the derivative of the stress with respect to the stretch ratio. Because the linear strain is used here, the derivative w.r.t. strain has the same value.

$$\begin{aligned} \sigma(t) &= E_\infty \varepsilon(t) + \sum_{i=1}^N \left[e^{-\frac{\Delta t}{\tau_i}} \sigma_i(t_n) + E_i p_i \Delta\varepsilon \right] \quad \rightarrow \\ \frac{\partial \sigma}{\partial \lambda} &= C_\lambda = C_\varepsilon = E_\infty + \sum_{i=1}^N E_i p_i \end{aligned}$$

6.40 Implementation

See `tr2dvie1.m` for the implementation.

6.41 Viscoelastic : differential formulation

The differential equation for a viscoelastic material model can be solved numerically. This is illustrated for the Standard Solid model.

constitutive equation

$$\begin{aligned}\sigma + \frac{\eta}{E} \dot{\sigma} &= E_{\infty} \varepsilon + \frac{\eta(E + E_{\infty})}{E} \dot{\varepsilon} \\ \sigma + A \dot{\sigma} &= B \varepsilon + C \dot{\varepsilon}\end{aligned}$$

stress update, implicit backward Euler

$$\begin{aligned}\Delta t \sigma + A \Delta \sigma &= \Delta t B \varepsilon + C \Delta \varepsilon \\ (\Delta t + A) \sigma &= A \sigma_n + \Delta t B \varepsilon + C \Delta \varepsilon \\ \sigma &= \frac{1}{\Delta t + A} [A \sigma_n + \Delta t B \varepsilon + C \Delta \varepsilon]\end{aligned}$$

6.42 Examples

A truss with length 100 mm and cross-sectional area 10 mm² is loaded with a time dependent axial strain. The stress is calculated as a function of time.

6.43 Strain step

A strain step with an amplitude of 0.1 is applied and the stress response is calculated for the Maxwell and the Standard-Solid models. Rather fictitious values for the material parameters are chosen. The initial stress can be verified, using the strain amplitude and the initial stiffness. The final stress value can be verified, using the strain amplitude and the equilibrium modulus.

Maxwell	$E_{\infty} = 0$	$E_1 = 1$	$\tau_1 = 0.01$
Standard-Solid	$E_{\infty} = 1$	$E_1 = 1$	$\tau_1 = 0.01$

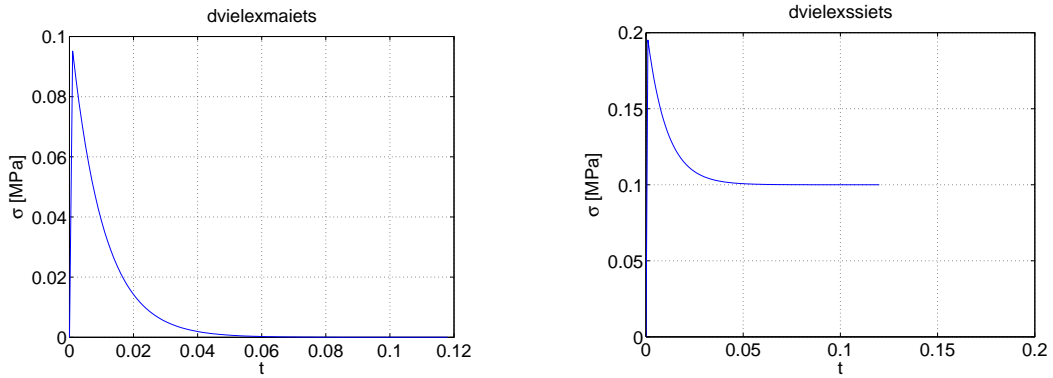


Fig. 6.72 : Stress response for Maxwell and Standard-Solid model

6.44 Linear viscoelastic models

When stress or strain is prescribed as a function of time, the strain or stress can be calculated. The examples show stress responses for a prescribed strain excitation, being a strain step ($\varepsilon_0 = 0.01$) followed by a constant strain rate ($\dot{\varepsilon} = 0.1 \text{ [s}^{-1}\text{]}$).

The stress response is calculated, using a Maxwell, a Kelvin-Voigt, a Standard-Solid and a 2-mode model. Parameter values are listed in the table below.

	E_∞	E_1	τ_1	E_2	τ_2	ν
Maxwell	0	100	0.1	0	0	0
Kelvin-Voigt	100	10^{10}	10^{-12}	0	0	0
Standard-Solid	100	100	0.1	0	0	0
2-mode	100	100	0.1	100	0.1	0

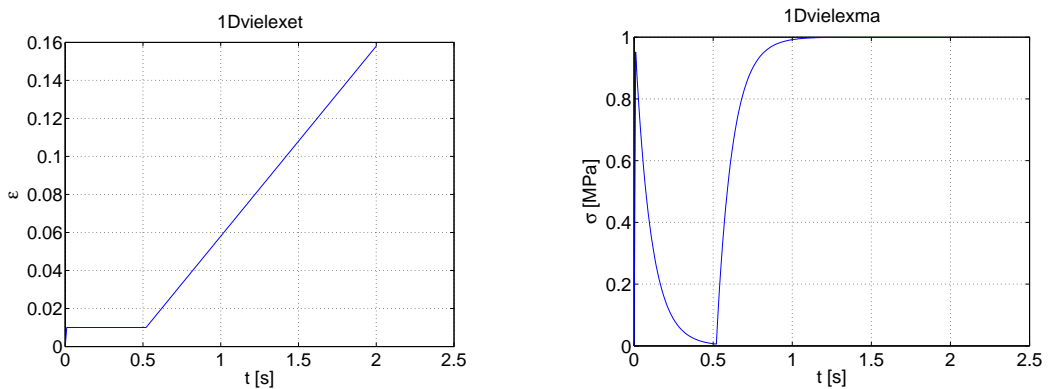


Fig. 6.73 : Prescribed strain and stress response for Maxwell model

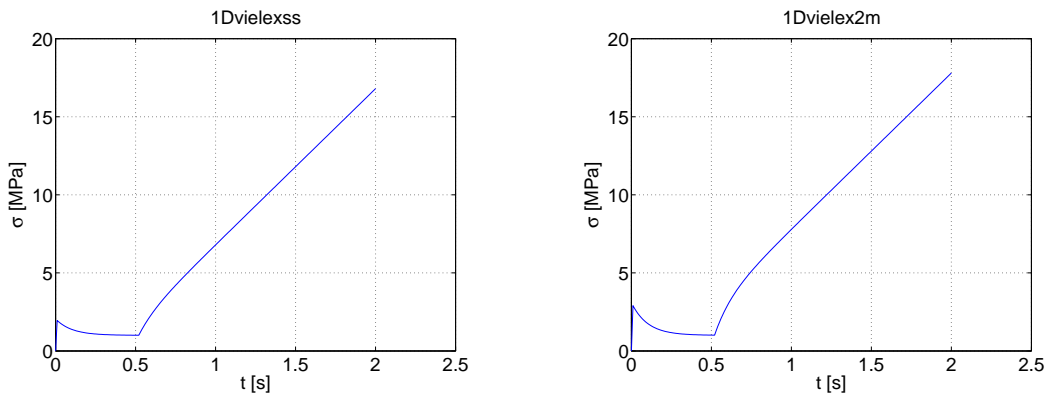


Fig. 6.74 : Stress response for Standard-Solid and 2-mode model

6.45 Multi-mode model response

An axial strain step with amplitude 0.01 is prescribed on an tensile bar with initial cross-sectional area $A_0 = 10 \text{ mm}^2$. The stress response is calculated for a 12-mode generalized Maxwell model. The modal parameters are listed in the table below.

	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3

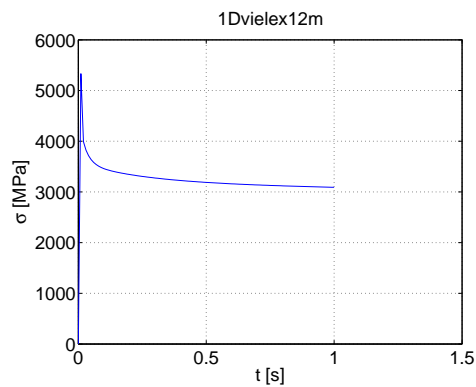


Fig. 6.75 : Tensile stress versus time

7 Creep behavior

The phenomenon called *creep* is the deformation under constant load. Viscoelastic material shows this behavior. The term *creep* however is especially reserved for deformation at temperatures, which are considered to be high with respect to the melting temperature T_m of the material, e.g. $T > 0.2T_m$. Such high temperatures are encountered in eg. (jet) engines and heat exchangers. Some materials with a low melting point, like lead (Pb), show creep at room temperature.

During the so-called stage I or primary creep, the strain rate decreases as a function of time. The strain rate is constant for stage II or secondary creep, also called steady state creep. Stage III or tertiary creep shows an increased strain rate and eventually leads to creep fracture or rupture.

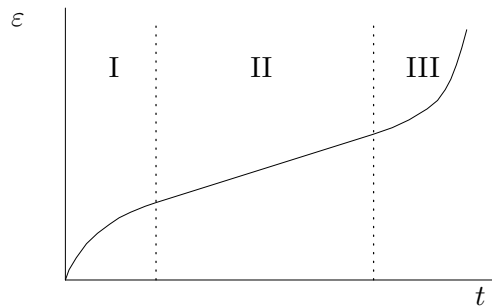


Fig. 7.76 : *Creep strain as a function of time at constant stress*

The creep behavior is influenced strongly by both stress and temperature, as is illustrated in the figure below. Obviously therefore, creep behavior is described by relating the creep strain rate $\dot{\epsilon}_c$ to stress, temperature and time. The temperature dependency could be included by making material parameters a function of temperature. It is more convenient, however, to implement the temperature dependency explicitly in the creep model. Much used are so-called power law models.

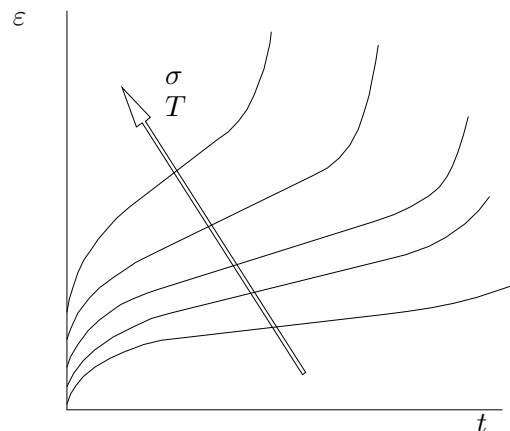


Fig. 7.77 : *The influence of stress and temperature on the creep strain rate*

general model	$\dot{\varepsilon}_c = A f_\sigma(\sigma) f_\varepsilon(\varepsilon_c) f_T(T) f_t(t)$
power law model	$\dot{\varepsilon}_c = A \sigma^m \varepsilon_c^n T^p (qt^{q-1})$

7.1 Primary creep

Primary creep – also referred to as stage I, transient creep or delayed elastic effect – is observed at $T < 0.4T_m$. Mechanisms, which are associated with this behavior are dislocation coalescence and dislocation entanglement, leading to slip steps (jogs). Dislocations may pile-up at grain boundaries and impurities. All this leads to macroscopic hardening. When temperature is higher, $0.4T_m < T < 0.5T_m$, the thermal activity of dislocations is higher and a transition to secondary creep is seen.

7.2 Secondary creep

Secondary creep – also referred to as stage II, steady-state creep or viscous flow – is observed at $0.5T_m < T < 0.6T_m$. The hardening, which is apparent in primary creep is balanced by recovery, leading to thermal softening. The thermal energy leads to vacancy movement (self diffusion) and this causes dislocation movement (climb). The moving dislocations can annihilate, align and/or pass obstacles. More drastic recovery may be caused by recrystallization, which can occur when internal stresses exist.

The temperature dependency is generally included with an Arrhenius function $\exp(Q_c/RT)$, where Q_c is the creep activation energy and R is Boltzmann's constant. The available evidence indicates that stage II creep is diffusion controlled, and so in the models the activation energy for creep, Q_c , can often be replaced by the activation energy for selfdiffusion Q_{sd} .

Most models for stage II creep are based on the *five-power-law* creep law. For temperatures below 0.5-0.6 T_m a transition toward primary creep is observed, which in reference to the modeling is called *power-law-breakdown*. Sometimes a threshold stress is introduced below which no creep can be measured.

7.3 Tertiary creep

Tertiary creep – also referred to as stage III or accelerating creep – is observed at $0.6T_m < T < 0.8T_m$ and is associated with geometric instabilities and damage.

One mechanism is grain boundary sliding and subsequent void initiation and coalescence, leading to inter granular cracks. Another mechanism is diffusional flow, which occurs mainly at higher temperatures and lower stresses. Two possibilities are : 1) diffusion through grains (*Nabarro-Herring creep*) with slow vacancy jump frequencies along many paths, and 2) diffusion along grain boundaries

(*Coble creep*) with high vacancy jump frequencies along a few paths.

Stage III creep is often modeled with continuum damage mechanics, where a damage

variable is used to model internal damage, which influences the creep strain rate. An evolution equation is required to control the damage growth as a function of stress and/or strain.

7.4 Stress functions

Several authors have reported various functions f_σ to implement the influence of the stress.

Norton; Bailey (1929)	$\dot{\epsilon}_c = K \sigma^n$
Hooke-Norton	$\dot{\epsilon}_c = \frac{\dot{\sigma}}{E} + K \sigma^n$
Johnson et.al. (1963)	$\dot{\epsilon}_c = D_1 \sigma^{n_1} + D_2 \sigma^{n_2}$
Dorn (1955)	$\dot{\epsilon}_c = B \exp(\beta \sigma)$
Soderberg (1936)	$\dot{\epsilon}_c = B \left[\exp\left(\frac{\sigma}{\sigma_0}\right) - 1 \right]$
Prandtl (1928)	$\dot{\epsilon}_c = A \sinh\left(\frac{\sigma}{\sigma_0}\right)$
Garofalo (1965)	$\dot{\epsilon}_c = A \left[\sinh\left(\frac{\sigma}{\sigma_0}\right) \right]^n$
Lemaitre, Chaboche (1985)	$\dot{\epsilon}_c = \left(\frac{\sigma}{\lambda_0}\right)^{N_0} \exp(\alpha \sigma^{N_0+1})$

7.5 Temperature functions

Several authors have reported various functions f_T to implement the influence of the temperature. These creep models also take into account the dependency of stress and (sometimes) time.

Kauzmann (1941)	$\dot{\epsilon}_c = A \exp\left(-\frac{\Delta H - \gamma \sigma}{RT}\right)$
Lifszic (1963)	$\dot{\epsilon}_c = \frac{\sigma}{T} \exp\left(-\frac{\Delta H}{RT}\right)$
Dorn, Tietz (1949/55)	$\epsilon_c = f\left(t \exp\left[-\frac{\Delta H}{RT}\right]\right) f_\sigma(\sigma)$
Penny, Marriott (1971)	$\epsilon_c = \left(t \exp\left[-\frac{\Delta H}{RT}\right]\right)^n f_\sigma(\sigma)$
Boyle, Spence (1983)	$\epsilon_c = C \exp\left(-\frac{\Delta H}{RT}\right) t^m \sigma^n$

7.6 Time functions

Several authors have reported various functions f_t to implement the influence of the time.

Andrade (1910) $\varepsilon_c = \ln \left(1 + \beta t^{\frac{1}{3}} \right) + kt$

Andrade (small ε) $\varepsilon_c = \beta t^{\frac{1}{3}} + kt \approx \beta t^{\frac{1}{3}}$

Bailey (1935) $\varepsilon_c = Ft^n$

Graham, Walles (1955) $\varepsilon_c = \sum_{j=1}^M a_j t^{m_j}$

McVetty (1934) $\varepsilon_c = G(1 - \exp(-qt)) + Ht$

Findley et.al. (1944) $\varepsilon_c = \varepsilon_1 + \varepsilon_2 t^n \quad (n < 1)$

Pugh (1975) $\varepsilon_c = \frac{a_1 t}{1 + b_1 t} + \frac{a_2 t}{1 + b_2 t} + \dot{\varepsilon}_m t$

Garofalo $\varepsilon_c = \varepsilon_t(1 - e^{-rt}) + \dot{\varepsilon}_s t$

7.7 Creep model

The discrete mechanical model for creep is a Maxwell element with a non-linear dashpot. The viscous or creep strain rate may be a function of stress σ , total creep strain ε_c , absolute temperature T and time t .

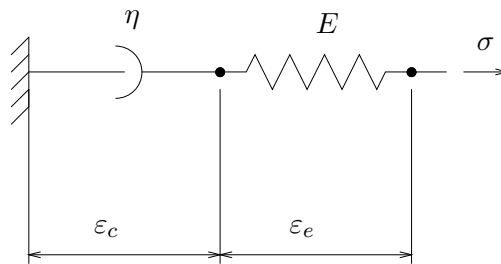


Fig. 7.78 : Creep model

constitutive relations

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_c$
- $\sigma = E\varepsilon_e \rightarrow \dot{\varepsilon}_e = \frac{1}{E} \dot{\sigma}$
- $\dot{\varepsilon}_c = A f_\sigma(\sigma) f_{\varepsilon_c}(\varepsilon_c) f_T(T) f_t(t) = f(\sigma, \varepsilon_c, T, t)$

constitutive equation

$$\dot{\sigma} = E\dot{\varepsilon}_e = E\dot{\varepsilon} - E\dot{\varepsilon}_c = E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t)$$

7.8 Stress update

The constitutive equation, can be solved explicitly or implicitly. For the latter case, a Newton iteration procedure must be implemented to calculate the stress.

$$\begin{aligned}\dot{\sigma} &= E\dot{\varepsilon} - Ef(\sigma, \varepsilon_c, T, t) \\ \Delta\sigma &= E\Delta\varepsilon - \Delta t Ef(\sigma, \varepsilon_c, T, t) \\ \sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t Ef(\sigma, \varepsilon_c, T, t)\end{aligned}$$

7.9 Implicit stress update

In the implicit procedure the end-increment stress is determined iteratively.

$$\begin{aligned}\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t Ef(\sigma, \varepsilon_c, T, t) \\ \sigma^* + \delta\sigma - \sigma_n &= E(\varepsilon - \varepsilon_n) - \Delta t E(f^* + \delta f) = E(\varepsilon - \varepsilon_n) - \Delta t Ef^* - \Delta t E\delta f \\ &= E(\varepsilon - \varepsilon_n) - \Delta t Ef^* - \Delta t E \frac{\partial f}{\partial \sigma} \delta\sigma \quad \rightarrow \\ \left[1 + \Delta t E \frac{\partial f}{\partial \sigma} \right] \delta\sigma &= -\sigma^* + \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t Ef^*\end{aligned}$$

7.10 Explicit stress update

In the explicit procedure the end-increment stress is determined directly.

$$\sigma = \sigma_n + E(\varepsilon - \varepsilon_n) - \Delta t Ef(\sigma_n, \varepsilon_{c_n}, T_n, t_n)$$

7.11 Stiffness

The material stiffness C_ε is the ratio of the variation of stress and strain.

implicit

$$\begin{aligned}\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t Ef(\sigma, \varepsilon_c, T, t) &= 0 \\ \delta\sigma + \Delta t E \left. \frac{\partial f}{\partial \sigma} \right|^* \delta\sigma - E\delta\varepsilon &= 0 \\ C_\varepsilon &= \left(1 + \Delta t E \left. \frac{\partial f}{\partial \sigma} \right|^* \right)^{-1} E\end{aligned}$$

explicit

$$\begin{aligned}\sigma - \sigma_n - E\varepsilon + E\varepsilon_n + \Delta t E f(\sigma_n, \varepsilon_{c_n}, T_n, t_n) &= 0 \\ \delta\sigma = E\delta\varepsilon &\rightarrow C_\varepsilon = E\end{aligned}$$

7.12 Implementation

See `tr2delvi.m` for the implementation.

7.13 Examples

In all examples a truss is subjected to an axial stress or strain.

7.14 Creep versus viscoelasticity

Linear viscoelastic behavior can be modeled with a multi-mode Maxwell model, represented by a mechanical system, which has a number of parallel Maxwell elements and one parallel spring. Springs and dashpots are linear.

Creep behavior is modeled with one Maxwell model with a nonlinear dashpot. The viscosity is a nonlinear function of stress, creep strain, temperature and time.

The Norton model for secondary creep can be made equivalent to the linear Maxwell model.

Maxwell model (E, η)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad E(t) = Ee^{t/\tau} \quad ; \quad \tau = \frac{\eta}{E} \quad ; \quad \dot{\varepsilon}_c = \frac{\sigma}{\eta} \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

Norton model (A, m)

$$\varepsilon = \varepsilon_e + \varepsilon_c \quad ; \quad \dot{\varepsilon}_c = f(\sigma, \varepsilon_c, T, t)\dot{\varepsilon}_c = A\sigma^m \quad ; \quad \varepsilon_e = \frac{\sigma}{E}$$

equivalence

Maxwell	$E = 10^9$	$\eta = 10^9$	$\tau = 1$
Norton	$E = 10^9$	$A = \frac{1}{\eta} = 10^{-9}$	$m = 1$

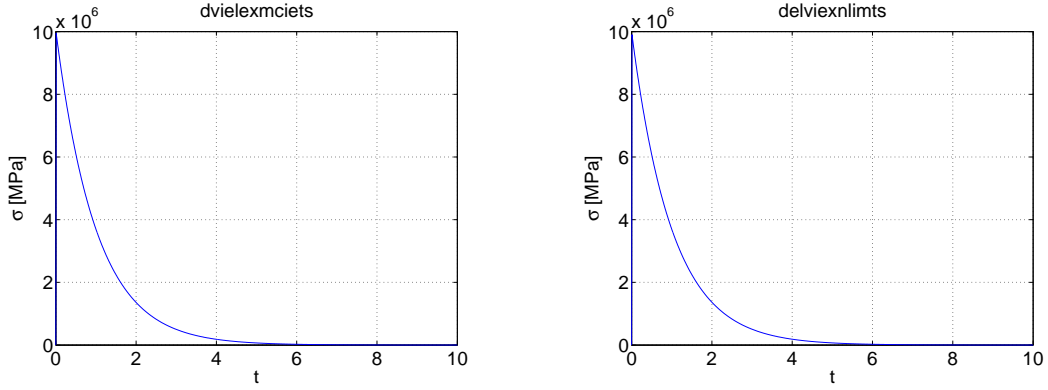


Fig. 7.79 : Stress response for Maxwell viscoelastic and Norton creep model

7.15 General creep model for SnAg-solder

Evans and Wilshire and later Maruyama and Oikawa proposed a general model, which describes the primary, secondary and tertiary creep of alloys. Parameters in the model must be fitted onto experimental data.

The creep strain at time t is described by two terms. The first one describes the hardening or primary creep stage and the second describes the weakening or tertiary creep stage. Combined, they characterize also the transition region, the secondary creep.

$$\begin{aligned}\varepsilon_c(t) &= \varepsilon_0 + A(\sigma) \left[1 - e^{-\alpha(\sigma, T)t} \right] + B(\sigma, T) \left[e^{\alpha(\sigma, T)t} - 1 \right] \\ \alpha(\sigma, T) &= c_1 [\sinh(\beta\sigma)]^{n_1} e^{-\frac{Q_1}{RT}} \\ A(\sigma) &= c_2 \sigma^{n_2} \quad ; \quad B(\sigma, T) = c_3 \sigma^{n_3} e^{-\frac{Q_2}{RT}}\end{aligned}$$

From the general model for the creep strain the creep strain rate $\dot{\varepsilon}_c$ can be calculated and subsequently the initial creep rate $\dot{\varepsilon}_{c,i}$, the time t_m for the minimum creep rate $\dot{\varepsilon}_{c,m}$ to occur and the strain $\varepsilon_{c,m}$ at that time.

With the universal gas constant $R = 8.314$ and stress in MPa and Q in kJ/mol, parameter values for SnAg-solder are fitted on experimental data and listed in the table below. The absolute temperature is assumed to be $T = 398$ [K].

$$\begin{aligned}\dot{\varepsilon}_c &= A\alpha e^{-\alpha t} + B\alpha e^{\alpha t} \quad ; \quad \dot{\varepsilon}_{c,i} = \dot{\varepsilon}_c(t=0) = \alpha(A+B) \quad ; \quad t_m = \frac{1}{2\alpha} \ln\left(\frac{A}{B}\right) \\ \dot{\varepsilon}_{c,m} &= \dot{\varepsilon}_c(t=t_m) = 2\alpha\sqrt{AB} \quad ; \quad \varepsilon_{c,m} = \varepsilon_c(t=t_m) = \varepsilon_0 + A - B\end{aligned}$$

ε_0	0		
c_1	1.73×10^5	n_1	4.66
β	0.095	Q_1	70
c_2	2.06×10^{-3}	n_2	1.1
c_3	9.65×10^{-4}	n_3	2.38
Q_2	17.8		

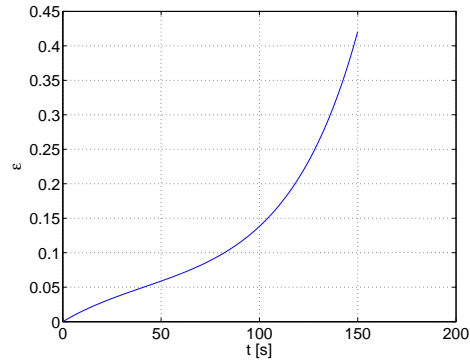


Fig. 7.80 : Creep strain at constant stress $\sigma = 20$ MPa

7.16 Special creep model for SnAg-solder

Several creep models for SnAg-solder have been published in literature. The 2-term model of Wiese (2005) is one of them and its parameter values have been fitted on experimental data for Sn4Ag0.5Cu solder material. Parameter values are listed in the table. Temperature (T) is in °K and equivalent stress (σ) is in MPa. The absolute temperature is assumed to be $T = 398$ [K].

$$\dot{\varepsilon}_c = A_1 \sigma^{m_1} e^{e_1/T} + A_2 \sigma^{m_2} e^{e_2/T}$$

$E = 59.533 - 66.667 T$		
$A_1 = 4.10^{-7}$	$m_1 = 3$	$e_1 = -3223$
$A_1 = 1.10^{-12}$	$m_1 = 12$	$e_1 = -7348$

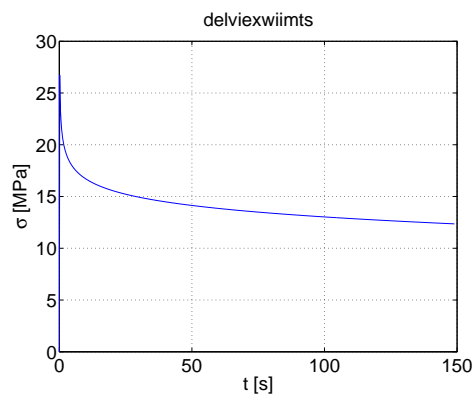


Fig. 7.81 : Relaxation stress for constant strain $\varepsilon = 0.001$

8 Viscoplastic material behavior

In many forming processes the deformation rates are small enough to consider the material behavior to be independent of strain rate and to use an elastoplastic material model. For high strain rates this assumption leads to faulty results. In a tensile test the yield stress is seen to increase with higher strain rates.

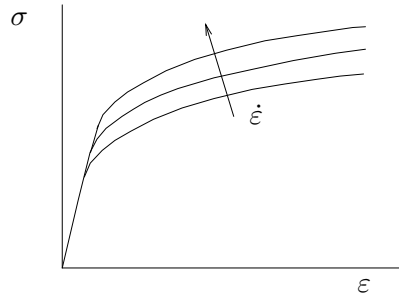


Fig. 8.82 : *Strain rate dependent plastic behavior*

Polymers and certain metallurgical alloys show softening behavior immediately after reaching the yield point. At larger strains the softening is followed by hardening. The complete stress-strain behavior is strain rate dependent, but the initial yield stress is constant.

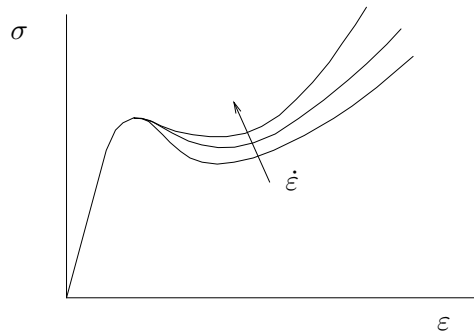


Fig. 8.83 : *Softening and resumed hardening*

8.1 Viscoplastic (Perzyna) model

The Perzyna model is a genuine viscoplastic model because it has a yield criterion, expressed with a yield function f . The model is called an "over-stress model" because $f > 0$ may occur. This is different compared to elastoplastic models, which always require $f \leq 0$. The rate of the *viscoplastic multiplier* λ , $\dot{\lambda}$, cannot be calculated from a consistency equation, but is given by a separate equation. We will only consider isotropic hardening.

The discrete mechanical model for viscoplastic material behavior consists of a spring

E in series with a parallel arrangement of a hardening spring H , a linear dashpot η and a friction slider, opening at $\sigma = \sigma_y$.

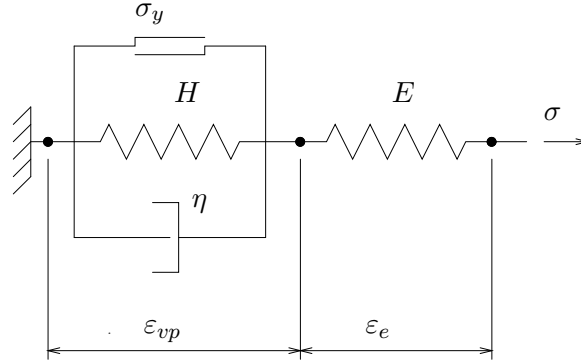


Fig. 8.84 : Discrete model for viscoplastic material behavior

- $f = \bar{\sigma} - \sigma_y$ with $f < 0 \rightarrow$ elastic
 $f \geq 0 \rightarrow$ viscoplastic
- $\sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_{vp})$ • $\bar{\sigma} = |\sigma|$
- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_{vp}$
- $\sigma = E\varepsilon_e \rightarrow \dot{\varepsilon}_e = \frac{1}{E} \dot{\sigma}$
- $\dot{\varepsilon}_{vp} = \dot{\lambda} \frac{\partial f}{\partial \sigma} = \dot{\lambda} \frac{\sigma}{\bar{\sigma}}$; $\dot{\varepsilon}_{vp} = |\dot{\varepsilon}_{vp}|$
- $\bar{\varepsilon}_{vp} = \int_{\tau=0}^t \dot{\varepsilon}_{vp} d\tau$
- $\dot{\lambda} = \gamma \phi(f) = \gamma (f/\sigma_{y0})^N$

8.2 Hardening laws

Various hardening laws, which were earlier described in section ??, could be used in the Perzyna model. The effective viscoplastic strain $\bar{\varepsilon}_{vp}$ is the history parameter used in the isotropic hardening models. Polymer materials may show softening, as is the case with Polycarbonate (PC). Parameters must be determined experimentally in a compression test, because the material softening poses problems of localization (necking) in a tensile test.

$$\sigma_y = \sigma_{y0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

8.3 Constitutive equations

From the constitutive relations a set of constitutive equations can be derived. The stress and the viscoplastic multiplier must be determined by integration of these equations.

$$\begin{cases} \dot{\sigma} = E\dot{\varepsilon}_e = E(\dot{\varepsilon} - \dot{\varepsilon}_{vp}) = E\{\dot{\varepsilon} - \dot{\lambda} \left(\frac{\sigma}{\bar{\sigma}}\right)\} \\ \dot{\lambda} = \gamma\phi \\ \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda \left(\frac{\sigma}{\bar{\sigma}}\right) \\ \Delta\lambda = \gamma\phi\Delta t \\ \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \gamma\phi\Delta t \end{cases}$$

8.4 Stress update

In the viscoplastic Perzyna model the stress and viscoplastic multiplier have to be solved from a set of differential equations. These equations are nonlinear although the viscosity in the model is constant.

Numerical analysis of mechanical behavior must be done iteratively, e.g. with a Newton-Raphson scheme. Following an incremental procedure the total loading time is subdivided into a discrete number of increments, which we assume to be of equal length Δt . All relevant variables $\{\sigma, \varepsilon, \varepsilon_{vp}, \bar{\varepsilon}_{vp}, \sigma_y\}$ are assumed to be known at the beginning t_n of the current increment.

8.5 Elastic stress predictor

Because it is not known a priori whether (ongoing) elastoviscoplastic deformation or elastic unloading will occur in the current increment $t_n \rightarrow t_{n+1}$, the stress calculation starts from the assumption that the strain increment is completely elastic. The *elastic stress predictor* σ_e is calculated and subsequently the yield criterion is evaluated with the yield function f .

$$\sigma_e = \sigma_n + E(\varepsilon - \varepsilon_n)$$

- $f = \bar{\sigma}_e - \sigma_{y_n} \leq 0 \quad \rightarrow \quad \text{elastic increment}$
- $f = \bar{\sigma}_e - \sigma_{y_n} > 0 \quad \rightarrow \quad \text{elastoviscoplastic increment}$

8.6 Elastic increment

When the increment is fully elastic, the end-increment stress equals the calculated elastic stress. As no viscoplastic deformation has occurred during the increment, the effective viscoplastic strain and the yield stress remain unchanged.

$$\begin{aligned} \sigma(t_{n+1}) &= \sigma_e \\ \bar{\varepsilon}_{vp}(t_{n+1}) &= \bar{\varepsilon}_{vp}(t_n) = \bar{\varepsilon}_{vp_n} \\ \sigma_y(t_{n+1}) &= \sigma_y(t_n) = \sigma_{y_n} \end{aligned}$$

8.7 Elastoviscoplastic increment

If the elastic stress predictor indicates that the yield criterion is violated, the increment is elastoviscoplastic. The end-increment stress has to be determined by integration of the constitutive equations. Integration of the stress can be carried out following an *explicit* or an *implicit* method.

$$\begin{cases} \Delta\sigma = E\Delta\varepsilon - E\Delta\lambda \left(\frac{\sigma}{\bar{\sigma}}\right) \\ \Delta\lambda = \gamma\phi\Delta t \\ \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\gamma\phi \end{cases}$$

8.8 Implicit stress update

When the increment appears to be elastoviscoplastic, the end-increment stress must be updated from the elastic trial stress. The viscoplastic multiplier λ and the stress σ are determined such that the constitutive equations are satisfied. Because λ and σ are not independent, an iterative procedure has to be used.

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}}\right) \\ \lambda - \lambda_n = \Delta t\gamma\phi \\ \sigma^* + \delta\sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda^* + \delta\lambda - \lambda_n) \left\{ \left(\frac{\sigma}{\bar{\sigma}}\right)^* + \delta \left(\frac{\sigma}{\bar{\sigma}}\right) \right\} \\ \lambda^* + \delta\lambda - \lambda_n = \Delta t\gamma(\phi^* + \delta\phi) \end{cases}$$

linearization and reorganization

$$\begin{cases} \delta\sigma + \left[E \left(\frac{\sigma}{\bar{\sigma}}\right)^* \right] \delta\lambda \\ \quad = -\sigma^* + \sigma_n + E\varepsilon - E\varepsilon_n - E(\lambda^* - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \left[-\Delta t\gamma \frac{\partial\phi}{\partial\sigma} \right] \delta\sigma + \left[1 - \Delta t\gamma \frac{\partial\phi}{\partial\lambda} \right] \delta\lambda \\ \quad = -\lambda^* + \lambda_n + \Delta t\gamma\phi^* \end{cases}$$

The variation of the function ϕ can be expressed in variations of σ and λ , using its definition and $\frac{\partial\bar{\varepsilon}_{vp}}{\partial\lambda} = \left(\frac{\sigma}{\bar{\sigma}}\right)$.

$$\begin{aligned} \frac{\partial\phi}{\partial\lambda} &= \frac{d\phi}{df} \frac{df}{d\sigma_y} \frac{d\sigma_y}{d\bar{\varepsilon}_{vp}} \frac{d\bar{\varepsilon}_{vp}}{d\lambda} = \frac{d\phi}{df} (-1) H \left(\frac{\sigma}{\bar{\sigma}}\right)^* = -\frac{d\phi}{df} H \left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \frac{\partial\phi}{\partial\sigma} &= \frac{d\phi}{df} \frac{df}{d\sigma} = \frac{d\phi}{df} \left(\frac{\sigma}{\bar{\sigma}}\right)^* \\ \frac{d\phi}{df} &= N \left(\frac{f}{\sigma_{y0}}\right)^{N-1} \frac{1}{\sigma_{y0}} \end{aligned}$$

In each iteration step, the stress, viscoplastic multiplier and other variables are updated.

8.9 Explicit stress update

In the explicit procedure, the end-increment values of λ and σ are determined directly.

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}} \right) \\ \lambda - \lambda_n = \Delta t \gamma \phi \end{cases}$$

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \\ \lambda - \lambda_n = \Delta t \gamma \phi_n \end{cases}$$

$$\begin{cases} \sigma + E \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \lambda = \sigma_n + E\varepsilon - E\varepsilon_n + E\lambda_n \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \\ \lambda = \lambda_n + \Delta t \gamma \phi_n \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \end{cases}$$

The total stress increment and other state variables can now be calculated.

8.10 Stiffness

The material stiffness is calculated as the ratio of the stress variation and the strain variation:

$$C_\varepsilon = \frac{\delta\sigma}{\delta\varepsilon}.$$

implicit

$$\begin{cases} \sigma - \sigma_n = E(\varepsilon - \varepsilon_n) - E(\lambda - \lambda_n) \left(\frac{\sigma}{\bar{\sigma}} \right) \\ \lambda - \lambda_n = \Delta t \gamma \phi \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon - E\delta\lambda \left(\frac{\sigma}{\bar{\sigma}} \right) - E(\lambda - \lambda_n) \left(\frac{1}{\bar{\sigma}} \right) \delta\sigma \\ \delta\lambda = \Delta t \gamma \delta\phi = \Delta t \gamma \frac{\partial\phi}{\partial\sigma} \delta\sigma + \Delta t \gamma \frac{\partial\phi}{\partial\lambda} \delta\lambda \end{cases}$$

$$\delta\sigma = E\delta\varepsilon - E \left(\frac{\sigma}{\bar{\sigma}} \right) \frac{\gamma \Delta t \frac{\partial\phi}{\partial\sigma}}{1 - \gamma \Delta t \frac{\partial\phi}{\partial\lambda}} \delta\sigma - E(\lambda - \lambda_n) \left(\frac{1}{\bar{\sigma}} \right) \delta\sigma$$

$$C_\varepsilon = \frac{E \left\{ 1 - \gamma \Delta t \frac{\partial\phi}{\partial\lambda} \right\}}{\left\{ 1 - \gamma \Delta t \frac{\partial\phi}{\partial\lambda} \right\} + E \left(\frac{\sigma}{\bar{\sigma}} \right) \gamma \Delta t \frac{\partial\phi}{\partial\sigma} + E(\lambda - \lambda_n) \frac{1}{\bar{\sigma}} \left\{ 1 - \gamma \Delta t \frac{\partial\phi}{\partial\lambda} \right\}}$$

explicit

$$\begin{cases} \sigma - \sigma_n = E\varepsilon - E\varepsilon_n - E(\lambda - \lambda_n) \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \\ \lambda - \lambda_n = \Delta t \gamma \phi_n \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon - E\delta\lambda \left(\frac{\sigma_n}{\bar{\sigma}_n} \right) \\ \delta\lambda = 0 \end{cases}$$

$$C_\varepsilon = E$$

8.11 Implementation

See `tr2dperz.m` for the implementation.

8.12 Examples

8.13 Tensile test at various strain rates

A truss is loaded axially with a prescribed elongation. In the initial state the length of the truss is $l_0 = 100$ mm and its cross-sectional area is $A_0 = 10$ mm². The axial force/elongation is calculated for various material models. The cross-sectional area will change as a function of the elongation.

The Perzyna model is used to describe the viscoplastic material behavior. The hardening model and tabulated data for polycarbonate are used. The strain rate is varied.

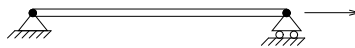


Fig. 8.85 : *Tensile loading of truss element*

$$\sigma_y = \sigma_{v0} + H\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

$$\dot{\varepsilon}_l = \{ 0.01, 0.1, 1 \}$$

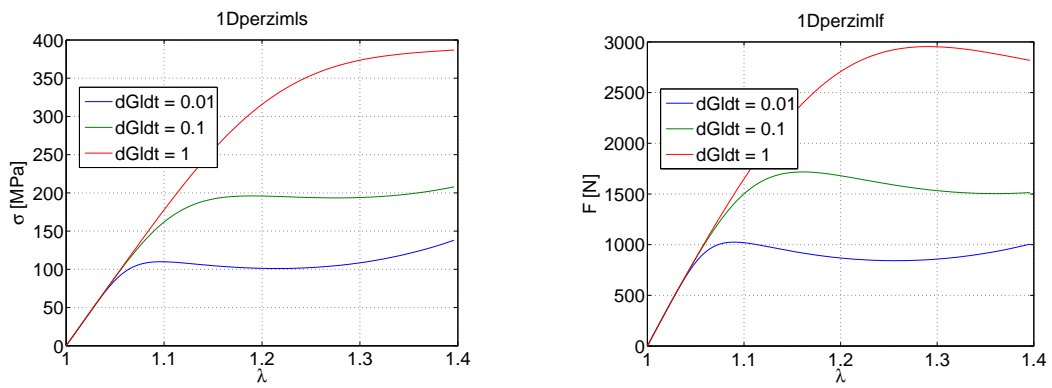


Fig. 8.86 : *Stress-stretch and force-elongation for PC*

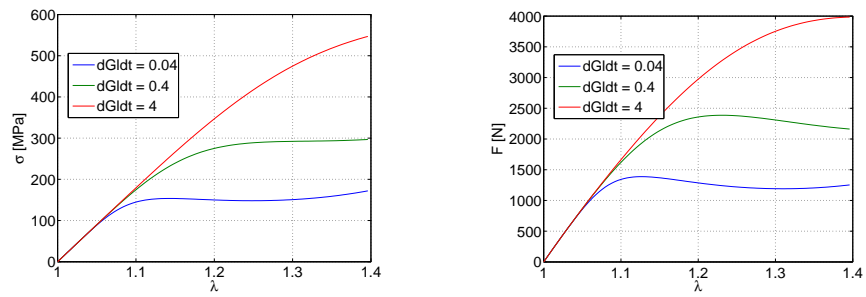


Fig. 8.87 : *Stress-stretch and force-elongation for PC; prescribed elongation*

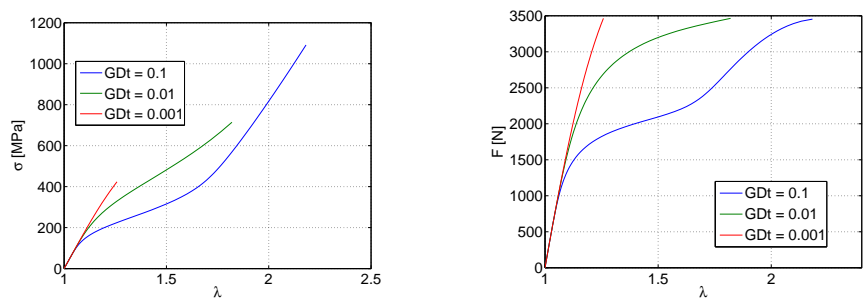


Fig. 8.88 : *Stress-stretch and force-elongation for PC; prescribed force*

9 Nonlinear viscoelastic material behavior

A polymeric material can be loaded in compression with a constant logarithmic strain rate. The true stress - absolute value - reaches a maximum value (B) after which softening occurs (BC) due to structural evolution. Subsequent hardening (CD) results in an increase of the stress, with increasing strain - absolute value - due to orientation of molecules.

Until the maximum stress level (B) is reached, the deformation is fully reversible. Initially the material behavior is linear viscoelastic (OA) but from a certain strain, nonlinear viscoelastic behavior (AB) is observed. After reaching the maximum stress (B), plastic flow occurs and therefore this stress is called the *yield stress* σ_y .

For a number of polymers, like polycarbonate (PC), polymethylmethacrylate (PMMA), polystyrene (PS) and polyetheneterfealtate (PET), the above typical stress-strain behavior is observed.

Two time-dependent processes can be observed, one related to the deformation kinetics (strain rate dependency) and another related to the aging kinetics.

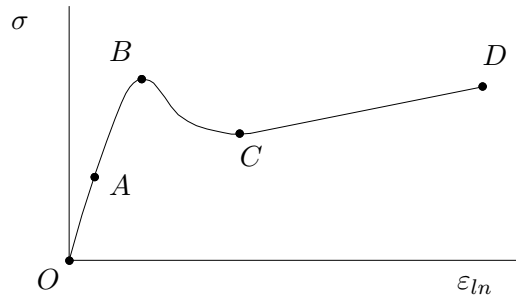


Fig. 9.89 : *Softening and resumed hardening*

When the uniaxial compression test for PC is carried out at a higher strain rate, the increase in stress is equal for each strain value.

This is shown in a graph, where the stress maximum σ_{yma} , the stress minimum after softening σ_{ymi} and the difference between those two values $\Delta\sigma_y$, are plotted against the logarithm of the true strain rate. The stress maximum is referred to as the *upper yield stress*, the stress minimum after softening is called the *lower yield stress* and the difference is the *yield drop*, which is constant for PC.

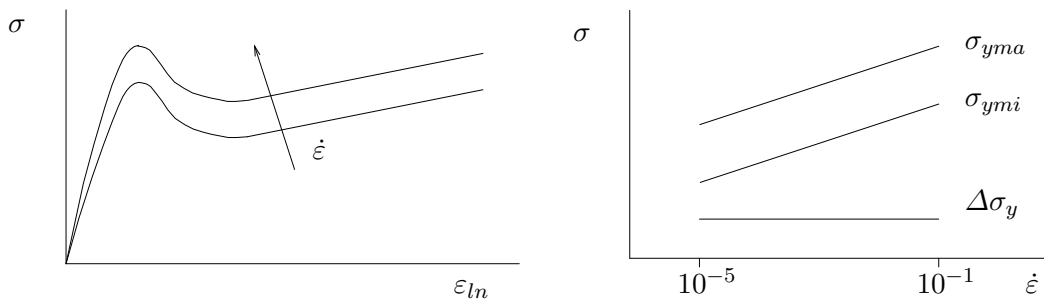


Fig. 9.90 : *Strain rate dependent stress-strain for PC*

The strain rate dependency of PMMA is different from that of PC. The increase of the stress with higher strain rates is not the same for each strain value. The upper yield stress increases more than the lower yield stress. The yield drop is a function of the strain rate.

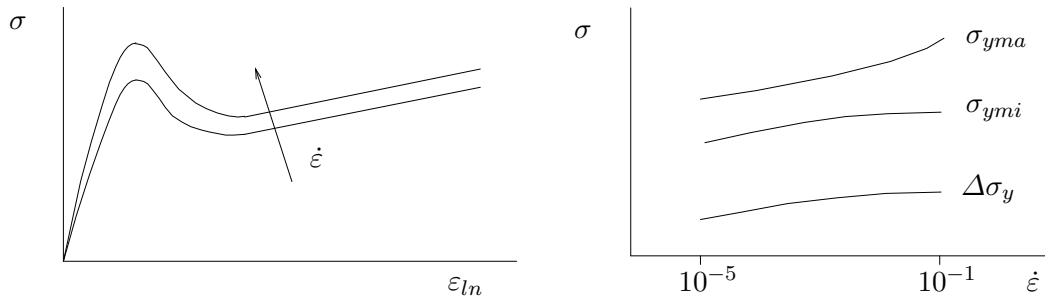


Fig. 9.91 : Strain rate dependent stress-strain for PMMA

The yield drop appears to be a function of the history of the material. When the specimen is quenched after processing, there is no yield drop. Softening is observed after a certain time, a phenomenon which is called *aging*. The time is characteristic for the polymer in question : 15 minutes for PS, 1 day for PMMA and about 3 weeks for PC. Aging and the resulting softening characteristic, can be neutralized by mechanical deformation, indicated as *rejuvenation*.

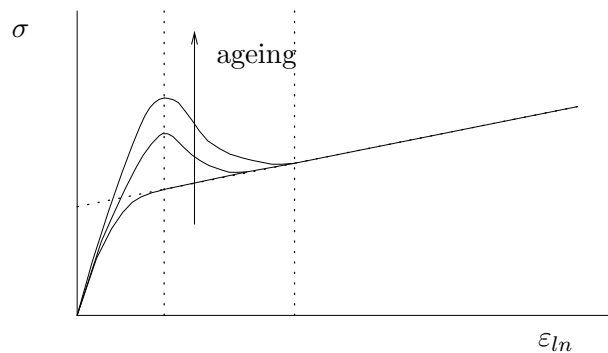


Fig. 9.92 : Aging

9.1 Nonlinear viscoelastic model

The complete model for nonlinear viscoelastic behavior is based on the models, which are used to describe the mechanical behavior at increasing stress level.

9.2 Linear viscoelastic behavior

For small strains the material behavior of polymers is linear viscoelastic and can be described by a Boltzmann integral with multi-mode Maxwell relaxation function. When more molecular processes are relevant, the relaxation functions for the separate processes can be added.

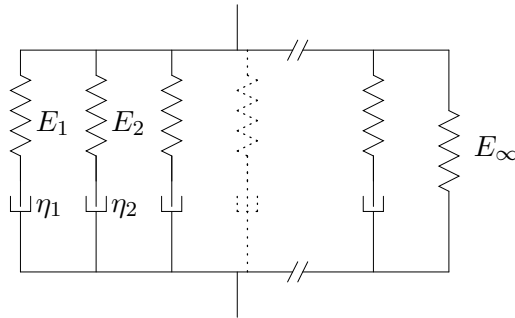


Fig. 9.93 : *Generalized Maxwell model for linear viscoelastic behavior*

$$\sigma(t) = \int_{\xi=-\infty}^t E(t-\xi)\dot{\epsilon}(\xi) d\xi \quad ; \quad E(x) = E_{\infty} + \sum_{i=1}^N E_i e^{-\frac{x}{\tau_i}} \quad ; \quad \tau_i = \frac{\eta_i}{E_i}$$

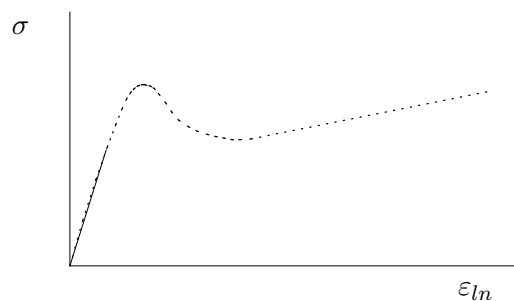


Fig. 9.94 : *Stress-strain relation in linear viscoelastic range*

9.3 Nonlinear viscoelastic behavior

For higher strains, but before yielding, the behavior is nonlinear viscoelastic and the relaxation function becomes a function of the stress. Fortunately this influence can generally be modeled by using time-stress superposition and adaptation of time variables using a time-stress shift factor.

$$\sigma(t) = \int_{\xi=-\infty}^t E(\psi - \psi')\dot{\epsilon}(\xi) d\xi$$

$$\psi = \int_{\zeta=-\infty}^t \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}} \quad ; \quad \psi' = \int_{\zeta=-\infty}^{\xi} \frac{d\zeta}{a_\sigma\{\sigma(\zeta)\}}$$

$$E(x) = E_\infty + \sum_{i=1}^N E_i e^{-\frac{x}{\tau_i(\sigma)}} = E_\infty + \sum_{i=1}^N E_i e^{-\frac{x}{\tau_i a_\sigma(\sigma)}} \quad ; \quad a_\sigma = \frac{\sigma/\sigma_0}{\sinh(\sigma/\sigma_0)}$$

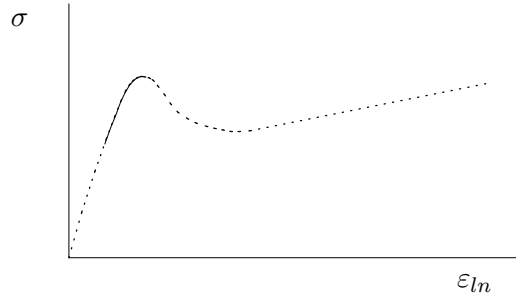


Fig. 9.95 : Stress-strain relation in nonlinear viscoelastic range

9.4 Creep

When the yield stress (= maximum stress) is reached, stress-activated plastic flow occurs, described by a semi-empirical relation for the viscous strain rate. The stress level depends on the strain rate and the temperature. When the stress is below the initial yield stress, the viscosity is very high and the material behavior is considered to be linear elastic with stiffness E . This behavior can be modeled with a Maxwell model with a linear spring (stiffness E) and a nonlinear dashpot (viscosity η). The total strain is additively decomposed in an elastic strain ε_e and a viscous strain ε_v . The viscous strain rate is given as a function of the equivalent viscoelastic stress \bar{s} and temperature T .

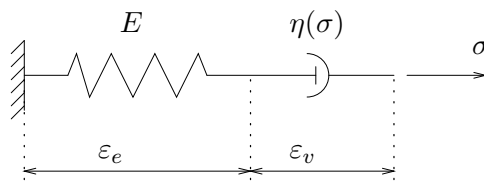


Fig. 9.96 : Nonlinear creep model

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = E\varepsilon_e \rightarrow \dot{\varepsilon}_e = \frac{1}{E} \dot{\sigma}$
- $\dot{\varepsilon}_v = f(\bar{s}, T) = \frac{\sigma}{\eta(\bar{s}, T)} \quad ; \quad \bar{s} = |s|$

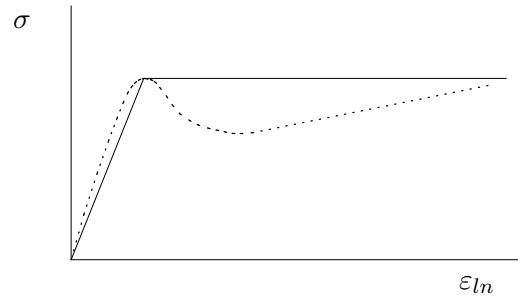


Fig. 9.97 : Stress-strain relation for creep range

9.5 Softening

After reaching the initial yield stress the stress decreases asymptotically toward a final value. Here, this softening is taken into account by decreasing the viscosity η with an internal (damage) variable D . Initially $D = D_0$ and finally D reaches a saturation value D_∞ . The value of D is determined by an evolution equation, which relates \dot{D} to the effective viscous strain rate $\dot{\bar{\epsilon}}_v$, with $\dot{\bar{\epsilon}}_v = |\dot{\epsilon}_v|$ for the one-dimensional case.

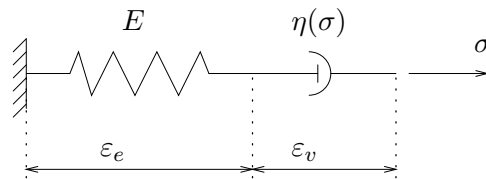


Fig. 9.98 : Model for nonlinear creep

- $\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_v$
- $\sigma = E\epsilon_e \rightarrow \dot{\epsilon}_e = \frac{1}{E} \dot{\sigma}$
- $\dot{\epsilon}_v = \frac{1}{\eta(\bar{s}, T, D)} \sigma \quad ; \quad \bar{s} = |s|$
- $\dot{D} = \left(1 - \frac{D}{D_\infty}\right) h \dot{\bar{\epsilon}}_v \quad ; \quad \dot{\bar{\epsilon}}_v = |\dot{\epsilon}_v|$

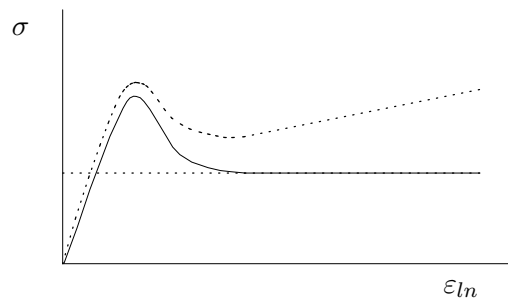


Fig. 9.99 : Softening

9.6 Hardening

In a compression test it is observed that the softening is followed by hardening. This can be modeled by decomposing the stress additively. In the discrete mechanical element a linear spring (stiffness H) is placed parallel to the Maxwell element with linear spring (stiffness E) and nonlinear dashpot (viscosity η).

The total axial stress σ is the sum of the viscoelastic stress s and the hardening stress w . The viscoelastic stress is related to the stiffness E , but also to the viscosity η .

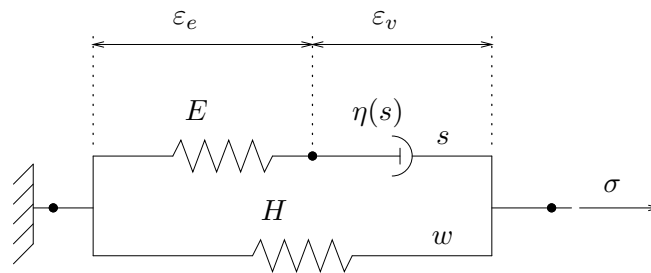


Fig. 9.100 : *Model for nonlinear viscoelastic behavior*

- $\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_v$
- $\sigma = s + w = E\epsilon_e + H\epsilon$
- $\dot{\epsilon}_v = \frac{1}{\eta(\bar{s}, T, D)} s$; $\bar{s} = |s|$
- $\dot{D} = \left(1 - \frac{D}{D_\infty}\right) h\dot{\epsilon}_v$; $\dot{\epsilon}_v = |\dot{\epsilon}_v|$

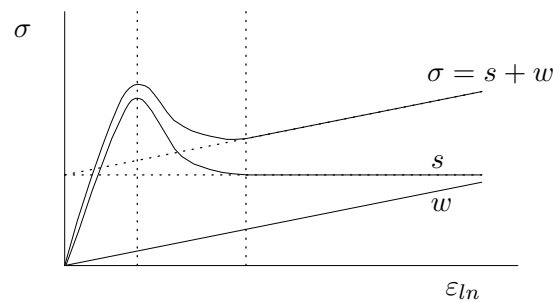


Fig. 9.101 : *Stress-strain curve for nonlinear viscoelastic behavior*

9.7 Aging and hardening

A different model to describe aging and softening is based on additive decomposition of the stress, where the total stress σ is the sum of the flow stress s , the hardening stress w and the aging stress $\Delta\sigma_y$, which is determined by an aging characteristic function $S(t, \bar{\varepsilon}_v)$. This function is taken to be the product of a time-dependent function $S_a(t)$, which describes the aging kinetics and a softening function $R_\gamma(\bar{\varepsilon}_v)$ which describes the softening kinetics. The viscosity is now a function of this function $S(t, \bar{\varepsilon}_v)$.

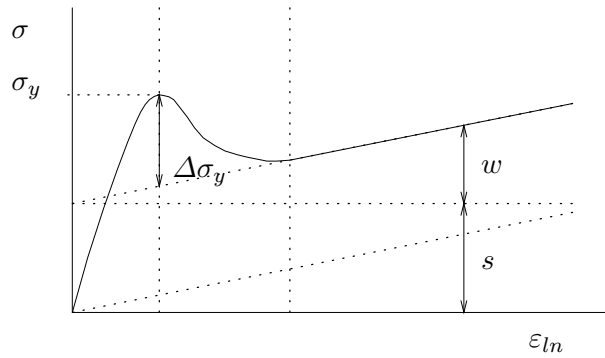


Fig. 9.102 : Aging and hardening

- $\dot{\varepsilon} = \dot{\varepsilon}_e + \dot{\varepsilon}_v$
- $\sigma = s + \Delta\sigma_y + w = E\varepsilon_e + \Delta\sigma_y + H\varepsilon$
- $\dot{\varepsilon}_v = \frac{1}{\eta(\bar{s}, T, S)} s \quad ; \quad \bar{s} = |s|$

- $S(t, \bar{\varepsilon}_v) = S_a(t)R_\gamma(\bar{\varepsilon}_v)$
- $R_\gamma(\bar{\varepsilon}_v) = \left[\frac{1 + (r_0 e^{\bar{\varepsilon}_v})^{r_1}}{1 + r_0^{r_1}} \right]^{\frac{r_2 - 1}{r_1}} \quad ; \quad 0 < R < 1$
- $S_\alpha(t) = S_\alpha(t_{eff}) = c_0 + c_1 \ln \left[\frac{t_{eff} + t_a}{t_0} \right]$
- $t_{eff}(T, \bar{s}) = \int_{\xi=0}^t \frac{d\xi}{\alpha_T(T(\xi))\alpha_\sigma(\bar{s}(\xi))}$
- $t_a = \exp \left(\frac{S_\alpha(0) - c_0}{c_1} \right)$
- $\Delta\sigma_y = \sigma_y(t) - \sigma_{y0} = \frac{c}{c_1} \{S_\alpha(t) - c_0\}$

9.8 Viscosity

For each material the proper relation for the viscosity has to be chosen. For polymers the *Eyring viscosity* function is used.

$$\eta = A_0 \frac{\bar{s}}{\sqrt{3} \sinh(\bar{s}/(\sqrt{3}\tau_0))} \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\bar{s} = |s| \quad ; \quad p = -\frac{1}{3}s \quad ; \quad \tau_0 = \frac{RT}{V}$$

For metals the *Bodner-Partom viscosity* function is used.

$$\eta = \frac{\bar{s}}{\sqrt{12I_0}} \exp\left[\frac{1}{2} \left(\frac{Z}{\bar{s}}\right)^{2n}\right]$$

$$Z = Z_1 + (Z_0 - Z_1) \exp[-m\bar{\epsilon}_p]$$

9.9 Nonlinear viscoelastic model

The nonlinear viscoelastic material behavior is described by some relations, which can be combined. The resulting constitutive equations must be solved simultaneously.

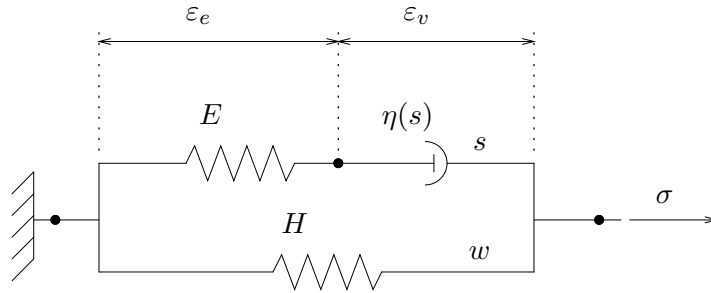


Fig. 9.103 : Model for nonlinear viscoelastic behavior

- $\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_v$
- $\sigma = s + w = E\epsilon_e + H\epsilon$
- $\dot{\epsilon}_v = \frac{1}{\eta(\bar{s}, T, D)} s \quad ; \quad \bar{s} = |s|$
- $\dot{D} = \left(1 - \frac{D}{D_\infty}\right) h\dot{\epsilon}_v \quad ; \quad \dot{\epsilon}_v = |\dot{\epsilon}_v|$

constitutive equations

$$\left. \begin{aligned} \dot{\epsilon}_e &= \dot{\epsilon} - \dot{\epsilon}_v = \dot{\epsilon} - \frac{1}{\eta(\bar{s}, T, D)} s = \dot{\epsilon} - \frac{E}{\eta(\bar{s}, T, D)} \epsilon_e \\ \sigma &= s + w = E\epsilon_e + H\epsilon \\ \dot{D} &= \left(1 - \frac{D}{D_\infty}\right) h\dot{\epsilon}_v \end{aligned} \right\}$$

9.10 Stress update

The stress is related to the strain rate by a differential equation, which has to be solved together with the damage evolution equation. After updating ε_e the stress is calculated directly. In the following we use $\zeta = \frac{1}{\eta}$. An implicit or explicit procedure can be used to determine the elastic strain and the damage parameter.

$$\begin{cases} \dot{\varepsilon}_e = \dot{\varepsilon} - E\zeta(\bar{s}, T, D)\varepsilon_e \\ \dot{D} = \left(1 - \frac{D}{D_\infty}\right) h\dot{\bar{\varepsilon}}_v \end{cases} \quad \begin{cases} \Delta\varepsilon_e = \Delta\varepsilon - \Delta t E\zeta(\bar{s}, T, D)\varepsilon_e \\ \Delta D = \left(1 - \frac{D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases}$$

9.11 Implicit stress update

In the implicit update procedure the elastic strain ε_e and the damage parameter D are updated iteratively.

$$\begin{cases} \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}, T, D)\varepsilon_e \\ D - D_n = \left(1 - \frac{D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases} \quad \begin{cases} \varepsilon_e^* + \delta\varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}, T, D^* + \delta D)(\varepsilon_e^* + \delta\varepsilon_e) \\ D^* + \delta D - D_n = \left(1 - \frac{D^* + \delta D}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases}$$

$$\begin{cases} \delta\varepsilon_e + \Delta t E\zeta(\bar{s}, T, D^*)\delta\varepsilon_e + \Delta t E \frac{\partial\zeta}{\partial D} \delta D \varepsilon_e^* \\ \quad = -\varepsilon_e^* + \varepsilon_{en} + \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}, T, D^*)\varepsilon_e^* \\ \left[1 + \frac{h\Delta\bar{\varepsilon}_v}{D_\infty}\right] \delta D \\ \quad = -D^* + D_n + \left(1 - \frac{D^*}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases}$$

9.12 Explicit stress update

In the explicit update procedure the elastic strain ε_e and the damage parameter D are determined directly.

$$\begin{cases} \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E\zeta(\bar{s}_n, T, D_n)\varepsilon_{en} \\ D - D_n = \left(1 - \frac{D_n}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases} \quad \begin{cases} \varepsilon_e = \varepsilon - \varepsilon_n + \{1 - \Delta t E\zeta(\bar{s}_n, T, D_n)\}\varepsilon_{en} \\ D = D_n + \left(1 - \frac{D_n}{D_\infty}\right) h\Delta\bar{\varepsilon}_v \end{cases}$$

9.13 Stiffness

The material stiffness is calculated as $C_\varepsilon = \frac{\delta\sigma}{\delta\varepsilon}$.

implicit

$$\begin{cases} \sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}, T, D) \varepsilon_e \\ D - D_n = \left(1 - \frac{D}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E \frac{\partial\zeta}{\partial D} \delta D \varepsilon_e - \Delta t E \zeta(\bar{s}, T, D) \delta\varepsilon_e \\ \delta D = -\frac{\delta D}{D_\infty} h \Delta \bar{\varepsilon}_v \quad \rightarrow \quad \delta D = 0 \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E \zeta(\bar{s}, T, D) \delta\varepsilon_e \\ \delta D = 0 \end{cases}$$

$$C_\varepsilon = \frac{E + H\{1 + \Delta t E \zeta(\bar{s}, T, D)\}}{1 + \Delta t E \zeta(\bar{s}, T, D)}$$

explicit

$$\begin{cases} \sigma = s + w = E\varepsilon_e + H\varepsilon \\ \varepsilon_e - \varepsilon_{en} = \varepsilon - \varepsilon_n - \Delta t E \zeta(\bar{s}_n, T, D_n) \varepsilon_e \\ D = D_n + \left(1 - \frac{D_n}{D_\infty}\right) h \Delta \bar{\varepsilon}_v \end{cases}$$

$$\begin{cases} \delta\sigma = E\delta\varepsilon_e + H\delta\varepsilon \\ \delta\varepsilon_e = \delta\varepsilon - \Delta t E \zeta(\bar{s}_n, T, D_n) \delta\varepsilon_e \\ \delta D = 0 \end{cases}$$

$$\delta\sigma = \frac{E}{1 + \Delta t E \zeta(\bar{s}_n, T, D_n)} \delta\varepsilon + H\delta\varepsilon$$

$$C_\varepsilon = \frac{E + H\{1 + \Delta t E \zeta(\bar{s}_n, T, D_n)\}}{1 + \Delta t E \zeta(\bar{s}_n, T, D_n)}$$

9.14 Implementation

See `tr2degp1.m` for the implementation.

9.15 Examples

Polymer materials are characterized by an Eyring viscosity. Parameters for various materials are experimentally determined and listed in the table. For the values in the table the temperature is chosen to be $T = 285$ K. The universal gas constant is $R = 8.314$ J/(mol.K).

	PET	PC	PS	PP	
E	2400	2305	3300	1092	MPa
ν	0.35	0.37	0.37	0.4	-
H	15	29	13	3	MPa
h	13	270	100	0	-
D_∞	11	19	14	-	-
A_0	3.8568E-27	9.7573E-27	4.2619E-34	2.0319E-29	s
ΔH	2.617E+05	2.9E+05	2.6E+5	2.2E+5	J/mol
μ	0.0625	0.06984	0.294	0.23	-
τ_0	0.927	0.72	2.1	1.0	MPa

9.16 Tensile test at various strain rates

A truss is loaded axially with a prescribed elongation. In the initial state the length of the truss is $l_0 = 100$ mm and its cross-sectional area is $A_0 = 10$ mm². The axial force/elongation is calculated for various material models. The cross-sectional area will change as a function of the elongation. The hardening model and tabulated data for polycarbonate are used. The strain rate is varied.

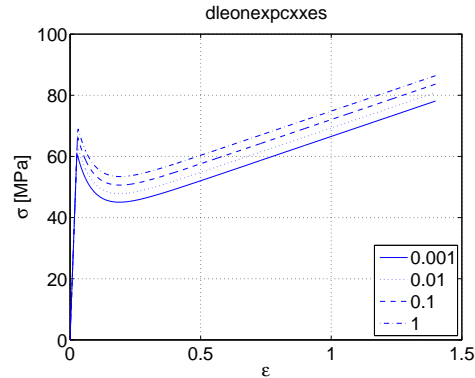


Fig. 9.104 : Stress versus strain in polycarbonate for different strain rates

9.17 Tensile test for various polymers

The truss is also loaded with a strain rate $\dot{\epsilon} = 10^{-1}$ s⁻¹, using the tabulated parameter values for polycarbonate, polypropylene, polystyrene and PET.

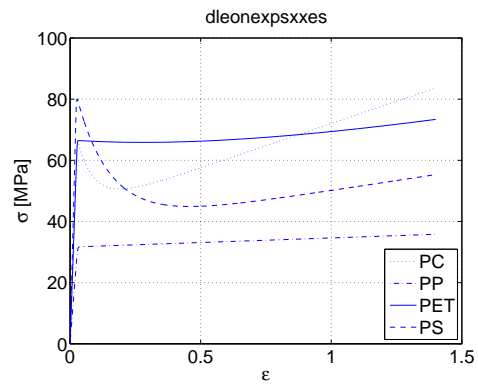


Fig. 9.105 : *Stress versus strain for different polymers at strain rate 0.1*