THREE-DIMENSIONAL MATERIAL MODELS

Contents

1 Three-dimensional material models

In this chapter we consider three-dimensional material models for various material behavior. Implementation in finite element software is the main focus, which means that the calculation of the stress and the stiffness during the iterative solution procedure is paramount.

In the following sections we consider models for elastic, elastomeric, elastoplastic, linear viscoelastic, viscoplastic and nonlinear viscoelastic materials behavior. Implementation in FEM modules is explained and simple examples are calculated.

2 Elastic material behavior

For elastic materials the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is related to the Green-Lagrange strain tensor $\boldsymbol{E}$. The Cauchy stress tensor can be written as a function of the right Cauchy-Green strain tensor $\boldsymbol{B}$. To assure the stress to be zero when there is no deformation, it is more suitable to relate the Cauchy stress to the Finger tensor $\boldsymbol{A}$.

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{G}(\boldsymbol{E}) && \text{with} \quad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}) \\
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1}\boldsymbol{F} \cdot \boldsymbol{G}(\boldsymbol{E}) \cdot \boldsymbol{F}^c && \text{with} \quad J = \det(\boldsymbol{F}) \\
&= \boldsymbol{K}(\boldsymbol{A}) && \text{with} \quad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{F} \cdot \boldsymbol{F}^c - \boldsymbol{I})
\end{aligned}$$

2.1 Isotropic elastic material models

For an isotropic material a principal strain deformation of a material cube can only result in normal stresses on its faces. Using its definition it can be shown that the principal directions of the 2nd-Piola-Kirchhoff stress tensor coincide with the principal strain directions. It is easily seen that the principal directions of the Cauchy stress tensor coincide with those of the Finger tensor.

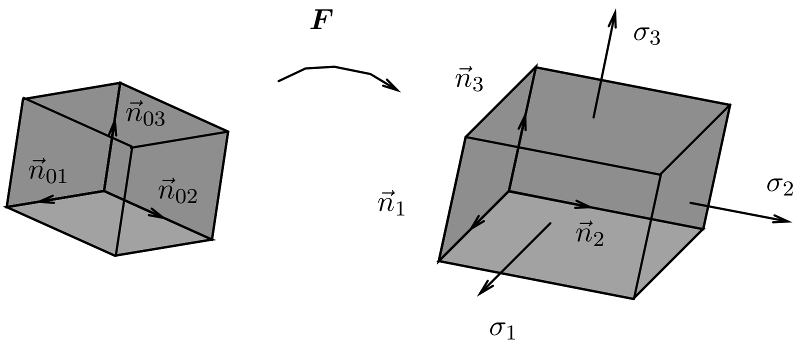


Fig. 2.1 : *Deformation in principal directions*

$$\begin{aligned}
\boldsymbol{U} &= \lambda_1 \vec{n}_{01}\vec{n}_{01} + \lambda_2 \vec{n}_{02}\vec{n}_{02} + \lambda_3 \vec{n}_{03}\vec{n}_{03} \\
\boldsymbol{R} &= \vec{n}_1\vec{n}_{01} + \vec{n}_2\vec{n}_{02} + \vec{n}_3\vec{n}_{03} \\
\boldsymbol{F} &= \lambda_1 \vec{n}_1\vec{n}_{01} + \lambda_2 \vec{n}_2\vec{n}_{02} + \lambda_3 \vec{n}_3\vec{n}_{03} \\
\boldsymbol{P} &= J\boldsymbol{F}^{-1} \cdot (\sigma_1 \vec{n}_1\vec{n}_1 + \sigma_2 \vec{n}_2\vec{n}_2 + \sigma_3 \vec{n}_3\vec{n}_3) \cdot \boldsymbol{F}^{-c} \\
&= J\left\{\sigma_1 \lambda_1^{-2} \vec{n}_{01}\vec{n}_{01} + \sigma_2 \lambda_2^{-2} \vec{n}_{02}\vec{n}_{02} + \sigma_3 \lambda_3^{-2} \vec{n}_{03}\vec{n}_{03}\right\} \\
&= s_1 \vec{n}_{01}\vec{n}_{01} + s_2 \vec{n}_{01}\vec{n}_{01} + s_3 \vec{n}_{01}\vec{n}_{01}
\end{aligned}$$

$\boldsymbol{P} - \boldsymbol{E}$ model

For a general isotropic material the principal directions of the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$, coincide. As a result, $\boldsymbol{P}$ can be written as a polynomial function of $\boldsymbol{E}$.

Applying Cayley-Hamilton's theorem, a second-order polynomial relation remains. The coefficients α_i in this relation are not constant. For the isotropic material they are a function of the invariants of $\boldsymbol{E}$ and have to be determined experimentally.

$$\begin{aligned}\boldsymbol{P} &= s_1\vec{n}_{01}\vec{n}_{01} + s_2\vec{n}_{02}\vec{n}_{02} + s_3\vec{n}_{03}\vec{n}_{03} \\ \boldsymbol{E} &= \varepsilon_1\vec{n}_{01}\vec{n}_{01} + \varepsilon_2\vec{n}_{02}\vec{n}_{02} + \varepsilon_3\vec{n}_{03}\vec{n}_{03}\end{aligned}$$

$$\boldsymbol{P} = \sum s_i\vec{n}_{0i}\vec{n}_{0i} = \boldsymbol{G}(\boldsymbol{E}) = \sum G(\varepsilon_i)\vec{n}_{0i}\vec{n}_{0i} = a_0\boldsymbol{I} + a_1\boldsymbol{E} + a_2\boldsymbol{E}^2 + a_3\boldsymbol{E}^3 + ...$$

Cayley-Hamilton's theorem $\qquad \boldsymbol{E}^3 = J_1(\boldsymbol{E})\boldsymbol{E}^2 - J_2(\boldsymbol{E})\boldsymbol{E} + J_3(\boldsymbol{E})\boldsymbol{I}$

$$\boldsymbol{P} = \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{E} + \alpha_2\boldsymbol{E}^2 \qquad \text{with} \qquad \alpha_i = \alpha_i\left\{J_1(\boldsymbol{E}), J_2(\boldsymbol{E}), J_3(\boldsymbol{E})\right\}$$

$\boldsymbol{\sigma} - \boldsymbol{A}$ model

For an isotropic material the principal directions of $\boldsymbol{\sigma}$ and $\boldsymbol{A}$ coincide, which implies that $\boldsymbol{\sigma}$ can be written as a polynomial function of $\boldsymbol{A}$. Applying Cayley-Hamilton's theorem results in a second-order polynomial with coefficients depending on the invariants of $\boldsymbol{A}$.

$$\begin{aligned}\boldsymbol{\sigma} &= \sigma_1\vec{n}_1\vec{n}_1 + \sigma_2\vec{n}_2\vec{n}_2 + \sigma_3\vec{n}_3\vec{n}_3 \\ \boldsymbol{A} &= A_1\vec{n}_1\vec{n}_1 + A_2\vec{n}_2\vec{n}_2 + A_3\vec{n}_3\vec{n}_3\end{aligned}$$

$$\boldsymbol{\sigma} = \sum \sigma_i\vec{n}_i\vec{n}_i = \boldsymbol{K}(\boldsymbol{A}) = \sum K(A_i)\vec{n}_i\vec{n}_i = b_0\boldsymbol{I} + b_1\boldsymbol{A} + b_2\boldsymbol{A}^2 + b_3\boldsymbol{A}^3 + ...$$

Cayley-Hamilton's theorem $\qquad \boldsymbol{A}^3 = J_1(\boldsymbol{A})\boldsymbol{A}^2 - J_2(\boldsymbol{A})\boldsymbol{A} + J_3(\boldsymbol{A})\boldsymbol{I}$

$$\boldsymbol{\sigma} = \beta_0\boldsymbol{I} + \beta_1\boldsymbol{A} + \beta_2\boldsymbol{A}^2 \qquad \text{with} \qquad \beta_i = \beta_i\left\{J_1(\boldsymbol{A}), J_2(\boldsymbol{A}), J_3(\boldsymbol{A})\right\}$$

The constitutive equation for the Cauchy stress tensor can also be derived from the expression for the second Piola-Kirchhoff stress tensor, which reveals that the coefficients β_i are related to the coefficients α_i.

$$\begin{aligned}\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = J^{-1}\boldsymbol{F}\cdot\left[\alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{E} + \alpha_2\boldsymbol{E}^2\right]\cdot\boldsymbol{F}^c \\ &= J^{-1}\boldsymbol{F}\cdot\left[(\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2)\boldsymbol{I} + (\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2)\boldsymbol{C} + \tfrac{1}{4}\alpha_2\boldsymbol{C}^2\right]\cdot\boldsymbol{F}^c \\ &= \{J_3(\boldsymbol{B})\}^{-1/2}\left[(\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2)\boldsymbol{B} + (\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2)\boldsymbol{B}^2 + \tfrac{1}{4}\alpha_2\boldsymbol{B}^3\right] \\ &\qquad\qquad \boldsymbol{B}^3 = J_1(\boldsymbol{B})\boldsymbol{B}^2 - J_2(\boldsymbol{B})\boldsymbol{B} + J_3(\boldsymbol{B})\boldsymbol{I} \\ &= J_3^{-1/2}\left[(\tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 + \tfrac{1}{4}\alpha_2J_1)\boldsymbol{B}^2 + (\alpha_0 - \tfrac{1}{2}\alpha_1 + \alpha_2 - \tfrac{1}{4}\alpha_2J_2)\boldsymbol{B} + \tfrac{1}{4}\alpha_2J_3\boldsymbol{I}\right] \\ &\qquad\qquad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) \quad\rightarrow\quad \boldsymbol{B} = 2\boldsymbol{A} + \boldsymbol{I} \\ &\qquad\qquad \boldsymbol{A}^2 = \tfrac{1}{4}\boldsymbol{B}^2 - \tfrac{1}{2}\boldsymbol{B} + \tfrac{1}{4}\boldsymbol{I} \quad\rightarrow\quad \boldsymbol{B}^2 = 4\boldsymbol{A}^2 + 2\boldsymbol{B} - \boldsymbol{I} \\ &= J_3^{-1/2}\left[(2\alpha_1 - 2\alpha_2 + \alpha_2J_1)\boldsymbol{A}^2 + (\alpha_0 + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2J_1 - \tfrac{1}{4}\alpha_2J_2)\boldsymbol{A} + \right. \\ &\qquad\qquad \left.(\alpha_0 + \alpha_1 - \tfrac{1}{2}\alpha_2 + \tfrac{3}{4}\alpha_2J_1 - \tfrac{1}{4}\alpha_2J_2 + \tfrac{1}{4}\alpha_2J_3)\boldsymbol{I}\right] \\ &= \beta_2\boldsymbol{A}^2 + \beta_1\boldsymbol{A} + \beta_0\boldsymbol{I}\end{aligned}$$

2.1.1 Linear elastic material models

The above isotropic elastic material models are nonlinear. The polynomial functions have a quadratic tensor term and, moreover, the coefficients are functions of the three invariants of the strain tensor. The first invariant is a linear, the second a quadratic and the third a cubic function of the tensor.

Simplification towards purely linear models is possible and allowed if it suits the experimental observations.

$\boldsymbol{P} - \boldsymbol{E}$ model

When experiments show that the relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ is linear, conclusions can be drawn concerning the coefficients α_i. The coefficient of the quadratic term, α_2, must be zero. The coefficient of the linear term, α_1, must be a constant. The coefficient of the unit tensor may be a linear, isotropic function of the tensor $\boldsymbol{E}$, which means it can be written as a constant times the trace of $\boldsymbol{E}$. No constant tensor is contained in the linear models, because stress has to be zero at zero strain.

Substituting an (experimentally motivated) linear relation between $\boldsymbol{P}$ and $\boldsymbol{E}$ in the definition relation of $\boldsymbol{\sigma}$, results in a nonlinear relation between $\boldsymbol{\sigma}$ and $\boldsymbol{B}$ and vice versa.

$$\boldsymbol{P} = c_0 \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + c_1 \boldsymbol{E}$$

Tensile test

For a tensile test only the axial component of $\boldsymbol{P}$ is non-zero and can be expressed in the axial and cross-sectional stretch ratios λ and μ. Because stresses perpendicular to the axial direction are zero, μ can be eliminated and the axial stress can be expressed in the axial stretch λ.

The parameters c_0 and c_1 can be expressed in the more commonly used Young's modulus E and Poisson's ratio ν.

$$\left.\begin{aligned} P &= c_0\tfrac{1}{2}(\lambda^2-1) + 2c_0\tfrac{1}{2}(\mu^2-1) + c_1\tfrac{1}{2}(\lambda^2-1) \\ 0 &= c_0\tfrac{1}{2}(\lambda^2-1) + 2c_0\tfrac{1}{2}(\mu^2-1) + c_1\tfrac{1}{2}(\mu^2-1) \end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2-1) = -\frac{c_0}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = -\nu\,\tfrac{1}{2}(\lambda^2-1)$$

$$P = \frac{c_1(3c_0+c_1)}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = E\,\tfrac{1}{2}(\lambda^2-1)$$

$$F = \sigma A = \frac{\lambda}{\mu^2}\,P\mu^2 A_0 = \lambda P A_0 = \tfrac{1}{2}\lambda(\lambda^2-1)\,E A_0$$

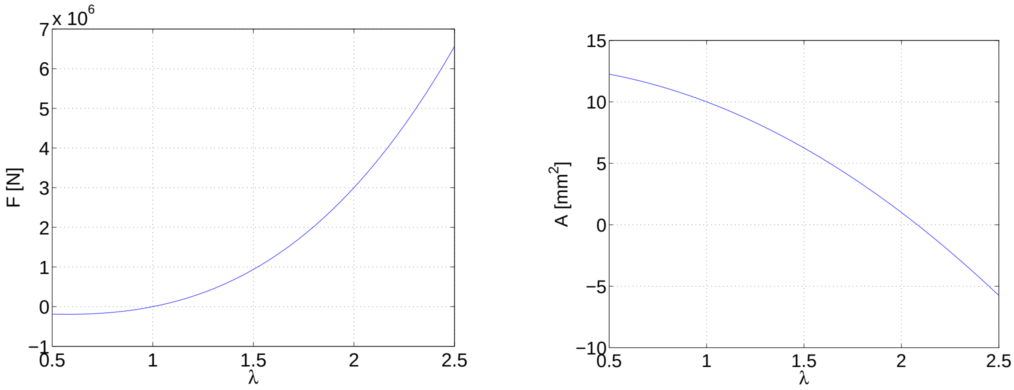


Fig. 2.2 : *Tensile test: force and cross-sectional area against stretch ratio*

Simple shear test : plane strain

For plane strain $F_{33} = 1$ holds and thus for the simple shear test $\det \boldsymbol{F} = 1$. To calculate the shear and normal force, the Cauchy stress has to be derived from the material model.

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{I} + \gamma \vec{e}_1 \vec{e}_2 \rightarrow J = \det(\boldsymbol{F}) = 1 \\
\boldsymbol{E} &= \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}\gamma^2 \vec{e}_2 \vec{e}_2 + \tfrac{1}{2}\gamma(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1) \\
\boldsymbol{P} &= c_0 \tfrac{1}{2}\gamma^2 \boldsymbol{I} + c_1 \tfrac{1}{2}\gamma^2 \vec{e}_2 \vec{e}_2 + c_1 \tfrac{1}{2}\gamma(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1) \\
\boldsymbol{\sigma} &= J^{-1} \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c \\
&= \tfrac{1}{2} \left\{ (c_0 \gamma^2 + c_0 \gamma^4 + c_1 2\gamma^2 + c_1 \gamma^4) \vec{e}_1 \vec{e}_1 + (c_0 \gamma^2 + c_1 \gamma^2) \vec{e}_2 \vec{e}_2 + c_0 \gamma^2 \vec{e}_3 \vec{e}_3 + \right. \\
&\qquad \left. (c_0 \gamma^3 + c_1 \gamma + c_1 \gamma^3)(\vec{e}_1 \vec{e}_2 + \vec{e}_2 \vec{e}_1) \right\} \\
p_n &= \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \tfrac{1}{2}(c_0 + c_1)\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \tfrac{1}{2} c_1 \gamma + \tfrac{1}{2}(c_0 + c_1)\gamma^3 \\
F_n &= p_n d_0 w_0 \quad ; \quad F_s = p_s d_0 w_0
\end{aligned}
$$

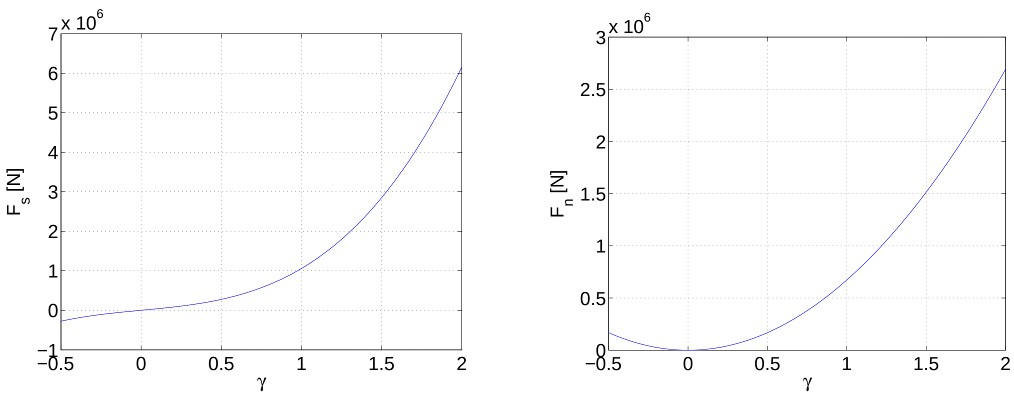


Fig. 2.3 : *Shear test plane strain: shear force and normal force against shear*

Simple shear test : plane stress

For plane strain $Gs_{33} = P_{33} = 0$ holds and thus for the simple shear test E_{33} can be expressed in E_{11} and E_{22}. To calculate the shear and normal force, the Cauchy stress has to be derived from the material model.

$$\sigma_{33} = P_{33} = 0 \quad \rightarrow \quad c_0(E_{11} + E_{22} + E_{33}) + c_1 E_{33} = 0 \quad \rightarrow \quad E_{33} = -\frac{c_0}{c_0 + c_1}(E_{11} + E_{22})$$

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{I} + (F_{33} - 1)\vec{e}_3\vec{e}_3 + \gamma\vec{e}_1\vec{e}_2 \\
\boldsymbol{E} &= \tfrac{1}{2}(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}) = \tfrac{1}{2}\left[\gamma^2\vec{e}_2\vec{e}_2 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + \left\{2(F_{33} - 1) + (F_{33} - 1)^2\right\}\vec{e}_3\vec{e}_3\right] \\
F_{33} &= \sqrt{2E_{33} + 1} \quad \rightarrow \quad J = \det(\boldsymbol{F}) = F_{33} = \sqrt{2E_{33} + 1} \\
\boldsymbol{P} &= \frac{c_0 c_1}{c_0 + c_1}(E_{11} + E_{22}) + c_1\boldsymbol{E} \\
&= \frac{c_0 c_1}{c_0 + c_1}\tfrac{1}{2}\gamma^2\boldsymbol{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_2\vec{e}_2 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = J^{-1}\left[\boldsymbol{P} + (\gamma P_{12} + \gamma P_{21} + \gamma^2 P_{22})\vec{e}_1\vec{e}_1 + \gamma P_{22}(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)\right] \\
p_n &= \vec{e}_2\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = \sigma_{22} \quad ; \quad p_s = \vec{e}_1\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = \sigma_{12} \\
F_n &= p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0
\end{aligned}$$

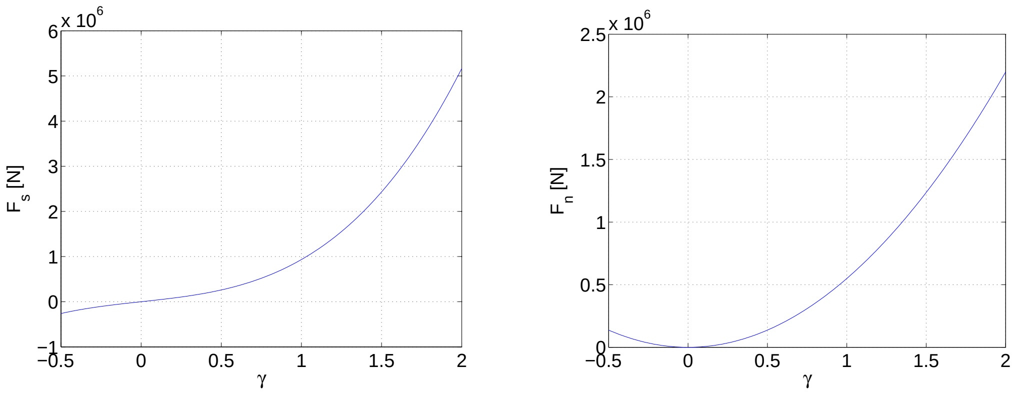


Fig. 2.4 : *Shear test plane stress: shear force and normal force against shear*

$\boldsymbol{\sigma} - \boldsymbol{A}$ model

For linear isotropic behavior the relation between $\boldsymbol{\sigma}$ and $\boldsymbol{A}$ is also characterised by two material constants.

$$\boldsymbol{\sigma} = c_0\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} + c_1\boldsymbol{A}$$

Tensile test

For a tensile test only the axial component of $\boldsymbol{\sigma}$ is non-zero and can be expressed in the axial and cross-sectional stretch ratios λ and μ. Because stresses perpendicular to the axial direction are zero, μ can be eliminated and the axial stress can be expressed in the axial stretch λ. The parameters c_0 and c_1 can be expressed in the more commonly used Young's modulus E and Poisson's ratio ν.

$$\left.\begin{aligned}\sigma &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\lambda^2-1)\\ 0 &= c_0\,\tfrac{1}{2}(\lambda^2-1) + 2c_0\,\tfrac{1}{2}(\mu^2-1) + c_1\,\tfrac{1}{2}(\mu^2-1)\end{aligned}\right\} \quad \rightarrow$$

$$\tfrac{1}{2}(\mu^2-1) = -\frac{c_0}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = -\nu\,\tfrac{1}{2}(\lambda^2-1)$$

$$\sigma = \frac{c_0(3c_0+c_1)}{2c_0+c_1}\,\tfrac{1}{2}(\lambda^2-1) = E\,\tfrac{1}{2}(\lambda^2-1)$$

$$F = \sigma A = \sigma\mu^2 A_0 = \tfrac{1}{2}(\lambda^2-1)\{1-\nu(\lambda^2-1)\}\,EA_0$$

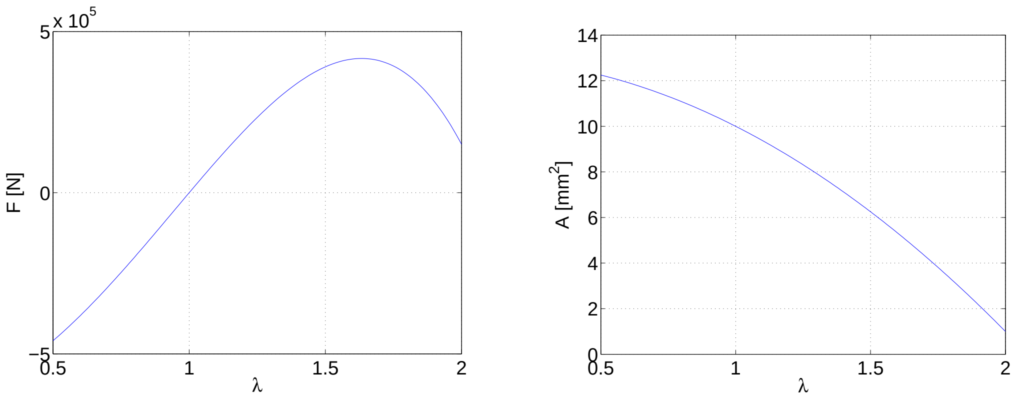


Fig. 2.5 : *Tensile test: force and cross-sectional area against stretch ratio*

Simple shear test : plane strain

A simple shear test for plane strain can also be calculated straightforwardly.

$$\boldsymbol{F} = \boldsymbol{I} + \gamma\,\vec{e}_1\vec{e}_2$$
$$\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c = \boldsymbol{I} + \gamma^2\vec{e}_1\vec{e}_1 + \gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$
$$\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B}-\boldsymbol{I}) = \tfrac{1}{2}\,\gamma^2\vec{e}_1\vec{e}_1 + \tfrac{1}{2}\,\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$

$$\boldsymbol{\sigma} = c_0\tfrac{1}{2}\gamma^2\boldsymbol{I} + c_1\tfrac{1}{2}\gamma^2\vec{e}_1\vec{e}_1 + c_1\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$
$$\sigma_{33} = c_0\tfrac{1}{2}\gamma^2$$
$$p_n = \vec{e}_2\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = c_0\tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1\cdot\boldsymbol{\sigma}\cdot\vec{e}_2 = c_1\tfrac{1}{2}\gamma$$
$$F_n = p_n d_0 w_0 \quad ; \quad F_s = p_s d_0 w_0$$

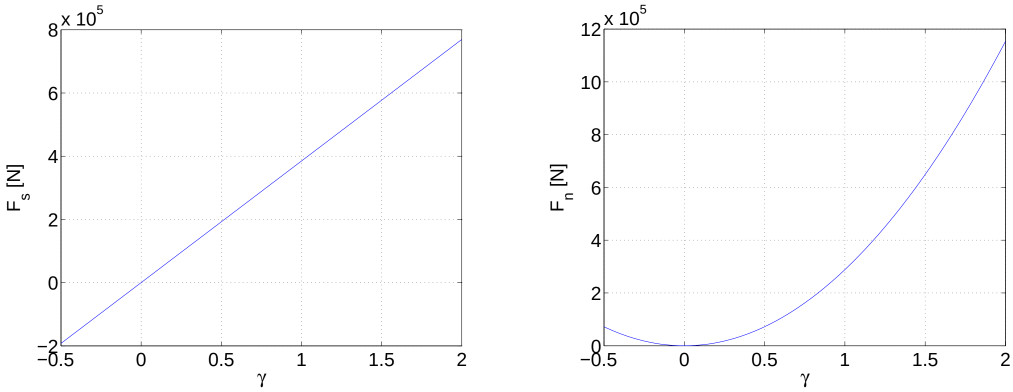


Fig. 2.6 : *Shear test plane strain: shear force and normal force against shear*

Simple shear test : plane stress

A simple shear test for plane stress needs some more consideration. For plane stress, it is assumed here that $\sigma_{33} = 0$. This assumption results in a relation for $F_{33} = \boldsymbol{F} \cdot \vec{e}_3$. For simple shear, we can than calculate the normal and shear stress. The normal and shear forces must be calculated, taking the deformed area into account.

$$
\begin{aligned}
&\sigma_{33} = c_0(A_{11} + A_{22} + A_{33}) + c_1 A_{33} = 0 \quad \rightarrow \\
&A_{33} = -\frac{c_0}{c_0 + c_1}(A_{11} + A_{22}) \quad \rightarrow \quad F_{33} = \sqrt{2A_{33} + 1} \\
&\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}(A_{11} + A_{22})\boldsymbol{I} + c_1 \boldsymbol{A}
\end{aligned}
$$

$$
\begin{aligned}
&\boldsymbol{A} = \tfrac{1}{2}\gamma^2 \vec{e}_1\vec{e}_1 + \tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&\boldsymbol{\sigma} = \frac{c_0 c_1}{c_0 + c_1}\,\tfrac{1}{2}\gamma^2 \boldsymbol{I} + c_1\,\tfrac{1}{2}\gamma^2 \vec{e}_1\vec{e}_1 + c_1\,\tfrac{1}{2}\gamma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) \\
&p_n = \vec{e}_2 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = \frac{c_0 c_1}{c_0 + c_1}\,\tfrac{1}{2}\gamma^2 \quad ; \quad p_s = \vec{e}_1 \cdot \boldsymbol{\sigma} \cdot \vec{e}_2 = c_1\,\tfrac{1}{2}\gamma \\
&F_n = p_n d w_0 = p_n F_{33} d_0 w_0 \quad ; \quad F_s = p_s d w_0 = p_s F_{33} d_0 w_0
\end{aligned}
$$

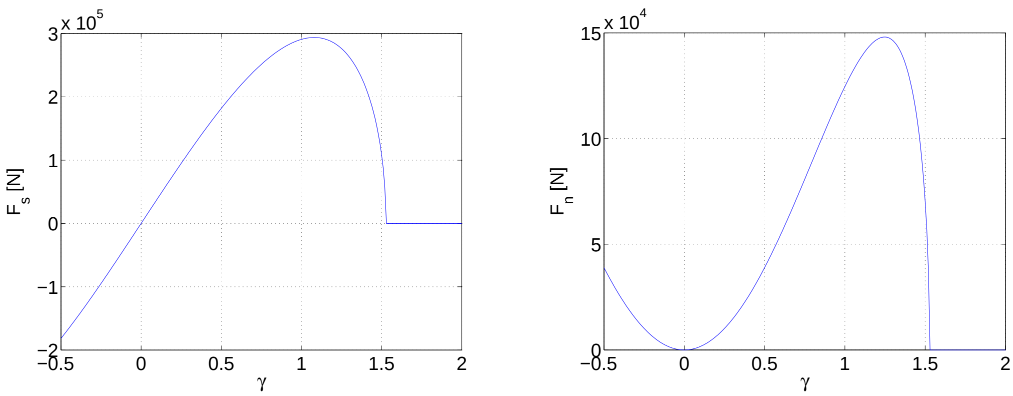


Fig. 2.7 : *Shear test plane stress: shear force and normal force against shear*

2.2 Hyper-elastic material models

When an explicit stored energy function is available for an elastic material, it is called hyper-elastic. The stress tensor can then be calculated as the derivative of the energy function with respect to the associated strain tensor.

When the stress-strain relation is not derived from a stored energy function, the elastic model is called hypo-elastic. For large strains such a model predicts the elastic behavior not correctly. In a closed cycle deformation loop residual stresses and elastic energy will remain. A hyper-elastic material model describes large elastic strains correctly.

The elastic energy must of course always become zero when there is no deformation. The function can be formulated with various strain tensors. The stress tensor can be derived by differentiation of the stored energy function with respect to the strain tensor.

The second Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is derived from an energy function $\phi(\boldsymbol{E})$, depending on the Green-Lagrange strain tensor. Instead of ϕ, a function $W(\boldsymbol{C})$ is usually specified. Although the stress tensor can now still be derived by differentiation – in this case W w.r.t. $\boldsymbol{C}$ – an additional requirement must be formulated or incorporated, namely that stress must be zero ($\boldsymbol{P} = \boldsymbol{O}$) when there is no deformation ($\boldsymbol{C} = \boldsymbol{I}$).

$$\phi = \phi(\boldsymbol{E}) \quad \rightarrow \quad W = W(\boldsymbol{C}) \qquad \rightarrow$$

$$\boldsymbol{P} = \frac{d\phi(d\boldsymbol{E})}{\boldsymbol{E}} = \frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} : \frac{d\boldsymbol{C}}{d\boldsymbol{E}} = 2\frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} = \boldsymbol{G}(\boldsymbol{E})$$

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = \frac{2}{J}\boldsymbol{F} \cdot \frac{dW(\boldsymbol{C})}{d\boldsymbol{C}} \cdot \boldsymbol{F}^c$$

2.2.1 Isotropic hyper-elastic material models

For isotropic material the elastic energy function can be written as a function of the invariants of $\boldsymbol{E}$ or $\boldsymbol{C}$.

Isotropic hyper-elastic model : $\boldsymbol{P} - \boldsymbol{E}$

For isotropic material the energy function $\phi(\boldsymbol{E})$ is only depending on the invariants of the strain tensor. Again we see that $\boldsymbol{P}$ can be written as a second-order polynomial of $\boldsymbol{E}$. The coefficients α_i are now functions of the derivatives of ϕ w.r.t. the invariants of $\boldsymbol{E}$.

$$\phi = \phi(\boldsymbol{E}) = \phi\{J_1(\boldsymbol{E}), J_2(\boldsymbol{E}), J_3(\boldsymbol{E})\} \quad \rightarrow \quad \boldsymbol{P} = \frac{\partial\phi}{\partial J_1}\frac{dJ_1}{d\boldsymbol{E}} + \frac{\partial\phi}{\partial J_2}\frac{dJ_2}{d\boldsymbol{E}} + \frac{\partial\phi}{\partial J_3}\frac{dJ_3}{d\boldsymbol{E}}$$

derivatives of invariants

$$\frac{dJ_1}{d\boldsymbol{E}} = \boldsymbol{I} \quad ; \quad \frac{dJ_2}{d\boldsymbol{E}} = J_1\boldsymbol{I} - \boldsymbol{E} \quad ; \quad \frac{dJ_3}{d\boldsymbol{E}} = J_2\boldsymbol{I} - J_1\boldsymbol{E} + \boldsymbol{E}^2 \qquad \rightarrow$$

stress tensor

$$
\begin{aligned}
\boldsymbol{P} &= \left(\frac{\partial\phi}{\partial J_1} + \frac{\partial\phi}{\partial J_2}J_1 + \frac{\partial\phi}{\partial J_3}J_2\right)\boldsymbol{I} + \left(-\frac{\partial\phi}{\partial J_2} - \frac{\partial\phi}{\partial J_3}J_1\right)\boldsymbol{E} + \frac{\partial\phi}{\partial J_3}\boldsymbol{E}^2 \\
&= \alpha_0\boldsymbol{I} + \alpha_1\boldsymbol{E} + \alpha_2\boldsymbol{E}^2
\end{aligned}
$$

Isotropic hyper-elastic model : $\boldsymbol{P}-\boldsymbol{C}$

For isotropic material the energy function $\phi(\boldsymbol{E})$ is only depending on the invariants of the strain tensor. Again we see that $\boldsymbol{P}$ can be written as a second-order polynomial of $\boldsymbol{E}$. The coefficients α_i are now functions of the derivatives of ϕ w.r.t. the invariants of $\boldsymbol{E}$.

$$
W = W(\boldsymbol{C}) = W\{J_1(\boldsymbol{C}), J_2(\boldsymbol{C}), J_3(\boldsymbol{C})\} \quad \rightarrow \quad \boldsymbol{P} = 2\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_3}\frac{dJ_3}{d\boldsymbol{C}}\right)
$$

derivatives of invariants

$$
\frac{dJ_1}{d\boldsymbol{C}} = \boldsymbol{I} \quad ; \quad \frac{dJ_2}{d\boldsymbol{C}} = J_1\boldsymbol{I} - \boldsymbol{C} \quad ; \quad \frac{dJ_3}{d\boldsymbol{C}} = J_2\boldsymbol{I} - J_1\boldsymbol{C} + \boldsymbol{C}^2 \quad \rightarrow
$$

stress tensor

$$
\begin{aligned}
\boldsymbol{P} &= 2\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1 + \frac{\partial W}{\partial J_3}J_2\right)\boldsymbol{I} + 2\left(-\frac{\partial W}{\partial J_2} - \frac{\partial W}{\partial J_3}J_1\right)\boldsymbol{C} + 2\frac{\partial W}{\partial J_3}\boldsymbol{C}^2 \\
&= \bar{\alpha}_0\boldsymbol{I} + \bar{\alpha}_1\boldsymbol{E} + \bar{\alpha}_2\boldsymbol{E}^2
\end{aligned}
$$

Isotropic hyper-elastic model : $\boldsymbol{\sigma}-\boldsymbol{A}$

The Cauchy stress tensor is a function of the 2nd-Piola-Kirchhoff stress tensor and can thus be derived from the elastic energy function $W(\boldsymbol{C})$.

$$
\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{J}\boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = \frac{2}{J}\boldsymbol{F}\cdot\frac{dW(\boldsymbol{C})}{d\boldsymbol{C}}\cdot\boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}}\boldsymbol{F}\cdot\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_3}\frac{dJ_3}{d\boldsymbol{C}}\right)\cdot\boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}}\boldsymbol{F}\cdot\left\{\left(\frac{\partial W}{\partial J_1} + J_1\frac{\partial W}{\partial J_2} + J_2\frac{\partial W}{\partial J_3}\right)\boldsymbol{I} + \left(-\frac{\partial W}{\partial J_2} - J_1\frac{\partial W}{\partial J_3}\right)\boldsymbol{C} + \left(\frac{\partial W}{\partial J_3}\right)\boldsymbol{C}^2\right\}\cdot\boldsymbol{F}^c \\
&= \frac{2}{\sqrt{J_3}}\boldsymbol{F}\cdot\left(\gamma_0\boldsymbol{I} + \gamma_1\boldsymbol{C} + \gamma_2\boldsymbol{C}^2\right)\cdot\boldsymbol{F}^c = \frac{2}{\sqrt{J_3}}\left(\gamma_0\boldsymbol{B} + \gamma_1\boldsymbol{B}^2 + \gamma_2\boldsymbol{B}^3\right) \\
&\qquad\qquad \boldsymbol{B}^3 = J_1\boldsymbol{B}^2 - J_2\boldsymbol{B} + J_3\boldsymbol{I} \\
&= \frac{2}{\sqrt{J_3}}\left[(\gamma_1 + \gamma_2 J_1)\boldsymbol{B}^2 + (\gamma_0 - \gamma_2 J_2)\boldsymbol{B} + (\gamma_2 J_3)\boldsymbol{I}\right] \\
&\qquad\qquad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B} - \boldsymbol{I}) \quad \rightarrow \quad \boldsymbol{B} = 2\boldsymbol{A} + \boldsymbol{I} \quad \rightarrow \quad \boldsymbol{B}^2 = 4\boldsymbol{A}^2 + 2\boldsymbol{B} - \boldsymbol{I} \\
&= \frac{2}{\sqrt{J_3}}\left[(4\gamma_1 + 4\gamma_2 J_1)\boldsymbol{A}^2 + (\gamma_0 + 2\gamma_1 + 2\gamma_2 J_1 - \gamma_2 J_2)\boldsymbol{A} + \right. \\
&\qquad\qquad\qquad \left.(\gamma_0 - \gamma_1 + \gamma_2 J_1 - \gamma_2 J_2 + \gamma_2 J_3)\boldsymbol{I}\right] \\
&= \beta_2\boldsymbol{A}^2 + \beta_1\boldsymbol{A} + \beta_0\boldsymbol{I}
\end{aligned}
$$

2.2.2 Incompressibility

For a hyper-elastic material model the stress-strain relation is derived from an energy function $W(\boldsymbol{C})$. For an isotropic material W is a function of the invariants of $\boldsymbol{C}$. Due to the incompressibility, the energy function cannot depend on the third invariant, which has always the value 1.

From a given function $W(\boldsymbol{C})$, the 2nd-Piola-Kirchhoff stress tensor can be determined by differentiation. Subsequently the Cauchy stress tensor can be calculated from $\boldsymbol{P}$.

$$
\begin{aligned}
& J = \det(\boldsymbol{F}) = 1 \quad \rightarrow \quad \det(\boldsymbol{C}) = J_3(\boldsymbol{C}) = 1 \quad \rightarrow \quad W(\boldsymbol{C}) = W\{J_1(\boldsymbol{C}), J_2(\boldsymbol{C})\} \\
& \boldsymbol{P} = 2\left(\frac{\partial W}{\partial J_1}\frac{dJ_1}{d\boldsymbol{C}} + \frac{\partial W}{\partial J_2}\frac{dJ_2}{d\boldsymbol{C}}\right) = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{I} - \frac{\partial W}{\partial J_2}\boldsymbol{C}\right\} \\
& \boldsymbol{\sigma} = \boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{B} - \frac{\partial W}{\partial J_2}\boldsymbol{B}^2\right\}
\end{aligned}
$$

Elastic material behavior can be described by a relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c$. When the material is incompressible and isotropic, the deformation will not be affected by the addition of a hydrostatic stress $p\boldsymbol{I}$.

When the deformation is known, the stress cannot be determined, because the hydrostatic stress remains arbitrary. Only the so-called *extra stress tensor* $\boldsymbol{\tau}$ depends solely on $\boldsymbol{B}$ and can be calculated.

To determine the unknown hydrostatic stress $p\boldsymbol{I}$ the incompressibility condition must be used.

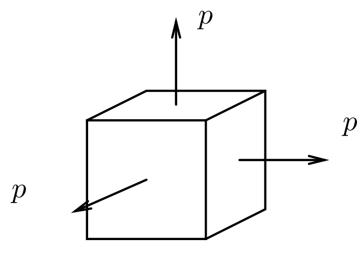


Fig. 2.8 : *Hydrostatic stress state*

$$
\begin{aligned}
\boldsymbol{\sigma} &= -\,p\boldsymbol{I} + \boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c = -\,p\boldsymbol{I} + 2\left\{\left(\frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}J_1\right)\boldsymbol{B} - \frac{\partial W}{\partial J_2}\boldsymbol{B}^2\right\} \\
&= -\,p\boldsymbol{I} + \boldsymbol{\tau}
\end{aligned}
$$

2.3 Rivlin models

Energy functions $W(\boldsymbol{C})$ are generally written as polynomials of $(J_1 - 3)$ and $(J_2 - 3)$ such that $W = 0$ when there is no deformation ($\boldsymbol{C} = \boldsymbol{I} \rightarrow J_1 = J_2 = 3$). The invariants of $\boldsymbol{C}$ can be expressed in the principal stretch ratios λ_1, λ_2 and λ_3. The polynomial energy function

$W(\boldsymbol{C})$ can then also be written as a polynomial function of these stretch ratios. This way of denoting these functions is often referred to as the Rivlin formulation.

$$W(\boldsymbol{C}) = \sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij}\{J_1(\boldsymbol{C})-3\}^i\{J_2(\boldsymbol{C})-3\}^j \qquad \text{with} \quad C_{00}=0$$

$$\begin{aligned}
J_1 &= \mathrm{tr}(\boldsymbol{C}) = \lambda_1^2+\lambda_2^2+\lambda_3^2 \\
J_2 &= \tfrac{1}{2}\left\{\mathrm{tr}^2(\boldsymbol{C}) - \mathrm{tr}(\boldsymbol{C}^2)\right\} \\
&= \tfrac{1}{2}\left\{\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^2 - \left(\lambda_1^4+\lambda_2^4+\lambda_3^4\right)\right\} = \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2 \\
J_3 &= \det(\boldsymbol{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 = 1
\end{aligned}$$

$$W(\boldsymbol{C}) = \sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij}\left(\lambda_1^2+\lambda_2^2+\lambda_3^2-3\right)^i\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}-3\right)^j$$

Neo-Hookean model

The Neo-Hookean energy function has only one material parameter : C_{10}. The model describes the mechanical behavior of natural rubbers rather well.

$$\begin{aligned}
W &= C_{10}(J_1-3) \\
\boldsymbol{\sigma} &= -p\boldsymbol{I} + 2C_{10}\boldsymbol{B}
\end{aligned}$$

Tensile test

In a tensile test, the axial stress is σ and stresses perpendicular to the axial are zero. The hydrostatic pressure can then be eliminated and the axial stress and force can be expressed in the axial stretch ratio λ. The incompressibility condition $\lambda\mu^2=1$ is taken into account.

$$\begin{aligned}
&\boldsymbol{B} = \lambda^2\vec{e}_1\vec{e}_1 + \mu^2(\vec{e}_2\vec{e}_2+\vec{e}_3\vec{e}_3) = \lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2+\vec{e}_3\vec{e}_3) \\
&\sigma = -p\boldsymbol{I} + 2C_{10}\lambda^2\vec{e}_1\vec{e}_1 + 2C_{10}\frac{1}{\lambda}(\vec{e}_2\vec{e}_2+\vec{e}_3\vec{e}_3) \\
&\left.\begin{aligned} \sigma &= -p + 2C_{10}\lambda^2 \\ 0 &= -p + 2C_{10}\frac{1}{\lambda} \end{aligned}\right\} \quad \rightarrow \quad \sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) \\
&F = \sigma A = \sigma\mu^2 A_0 = \sigma\frac{1}{\lambda}A_0 = 2C_{10}A_0(\lambda - \frac{1}{\lambda^2})
\end{aligned}$$

Mooney-Rivlin model

The mechanical behavior of industrial rubbers cannot be captured well with the one-parameter Neo-Hookean model. Instead the Mooney-Rivlin model is often used, which has two parameters.

$$\begin{aligned} W &= C_{10}(J_1 - 3) + C_{01}(J_2 - 3) \\ \boldsymbol{\sigma} &= -\,p\boldsymbol{I} + 2\{C_{10} + C_{01}\mathrm{tr}(\boldsymbol{B})\}\boldsymbol{B} - 2C_{01}\boldsymbol{B}^2 \end{aligned}$$

Tensile test

In a tensile test the hydrostatic pressure can be eliminated and the axial stress and force can be expressed in the axial stretch ratio λ.

$$\boldsymbol{B} = \lambda^2\vec{e}_1\vec{e}_1 + \mu^2(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) = \lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3) \quad ; \quad \mathrm{tr}(\boldsymbol{B}) = \lambda^2 + \frac{2}{\lambda}$$

$$\boldsymbol{B}^2 = \lambda^4\vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\{\lambda^2\vec{e}_1\vec{e}_1 + \frac{1}{\lambda}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\} - 2C_{01}\{\lambda^4\vec{e}_1\vec{e}_1 + \frac{1}{\lambda^2}(\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3)\}$$

$$\left.\begin{aligned} \sigma &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\lambda^2 - 2C_{01}\lambda^4 \\ 0 &= -p + 2\{C_{10} + C_{01}(\lambda^2 + \frac{2}{\lambda})\}\frac{1}{\lambda} - 2C_{01}\frac{1}{\lambda^2} \end{aligned}\right\} \quad \rightarrow \quad \sigma = 2C_{10}(\lambda^2 - \frac{1}{\lambda}) + 2C_{01}(\lambda - \frac{1}{\lambda^2})$$

$$F = \sigma A = \sigma\mu^2 A_0 = \sigma\frac{1}{\lambda}A_0 = 2A_0\{C_{10}(\lambda - \frac{1}{\lambda^2}) + C_{01}(1 - \frac{1}{\lambda^3})\}$$

2.3.1 Other energy functions

There are a lot of energy functions used for different elastomeric materials. They all belong to the polynomial energy functions.

3-term Mooney-Rivlin $\quad W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3)$

Signiorini $\quad W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2$

Yeoh $\quad W = c_{10}(J_1 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$

2nd-order invariant model

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2$$

Kloaner-Segal

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{03}(J_2 - 3)^3$$

James, Green, Simpson (3rd-order deformation model)

$$W = c_{10}(J_1 - 3) + c_{01}(J_2 - 3) + c_{11}(J_1 - 3)(J_2 - 3) + c_{20}(J_1 - 3)^2 + c_{30}(J_1 - 3)^3$$

2.4 Ogden models

For slightly compressible materials Ogden models are used. The strain energy function is written in terms of the principal stretch ratios. The first part of the Ogden function can be shown to be purely deviatoric/ The second part accounts for the volumetric deformation. Because the volumetric behavior is characterized by a constant bulk modulus, the model is confined to slightly compressible deformation.

To describe the mechanical behavior of elastomeric materials, which show large volumetric deformations, the foam model can be used. The first part of the energy function is not purely deviatoric.

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} J^{\frac{-\alpha_n}{3}} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) + 4.5K \left(1 - J^{\frac{1}{3}}\right)^2$$

with

μ_n	:	moduli
α_n	:	exponents
K	:	bulk modulus
J	:	volume ratio $= \det(\boldsymbol{F})$

foam model

$$W = \sum_{n=1}^{N} \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right) + \sum_{n=1}^{N} \frac{\mu_n}{\beta_n} \left(1 - J^{\beta_n}\right)$$

2.5 Incremental analysis

In nonlinear analysis, the load is applied in a number of steps, the *load increments*.

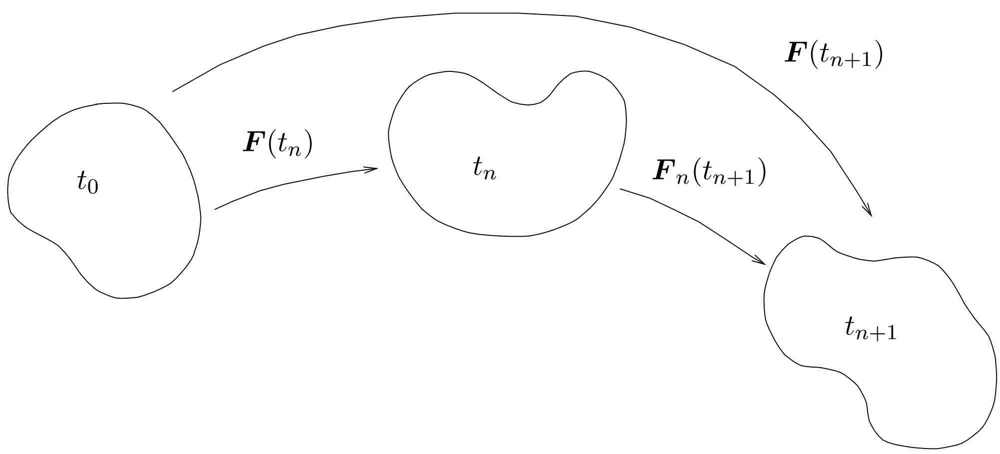


Fig. 2.9 : *Incremental deformation*

The end-increment state, i.e. deformation and stresses, must be determined such that equilibrium equations, material relations and boundary conditions are satisfied. Due to the nonlinear character of deformation and material behavior, the equations must be solved iteratively. In each iteration step both the stress and the material stiffness must be updated.

2.5.1 Linear P-E model

2.5.2 Stress update

Elastic material behavior may be described by a linear relation between the second Piola-Kirchhoff stress tensor $\boldsymbol{P}$ and the Green-Lagrange strain tensor $\boldsymbol{E}$. This relation can be derived from an elastic energy function and that is why this model is called hyper-elastic.

For a given deformation the stress in the material can be calculated directly for an elastic material. The Cauchy stress tensor $\boldsymbol{\sigma}$ can be calculated from $\boldsymbol{P}$.

$$
\begin{aligned}
\boldsymbol{P} &= c_0 \mathrm{tr}(\boldsymbol{E})\boldsymbol{I} + c_1 \boldsymbol{E} && \text{with} \quad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
&= \tfrac{1}{2}c_0 \boldsymbol{C} : \boldsymbol{II} + \tfrac{1}{2}c_1 \boldsymbol{C} - \tfrac{1}{2}(3c_0 + c_1)\boldsymbol{I} && \text{with} \quad \boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F}
\end{aligned}
$$

$$
\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c = J^{-1}\boldsymbol{F} \cdot (\boldsymbol{P} \cdot \boldsymbol{F}^c) = J^{-1}\boldsymbol{F} \cdot (\boldsymbol{F} \cdot \boldsymbol{P}^c)^c
$$

2.5.3 Stiffness

In the Newton-Raphson iterative solution procedure, the variation of the stress tensor must be expressed in the iterative displacement of material points.

Starting from the $\boldsymbol{P} \sim \boldsymbol{E}$ elastic model, the relation between $\delta\boldsymbol{P}$ and $\delta\boldsymbol{F}$ is calculated. The variation of the deformation tensor $\boldsymbol{F}$ can be expressed in the gradient of the iterative displacement vector $\vec{u} = \delta\vec{x}$:

$$
\delta\boldsymbol{F} = \boldsymbol{L} \cdot \boldsymbol{F} = (\boldsymbol{F}^c \cdot \boldsymbol{L}^c)^c \qquad \text{with} \qquad \boldsymbol{L}^c = \vec{\nabla}\vec{u}
$$

Combining the variations $\delta\boldsymbol{P}$ and $\delta\boldsymbol{\sigma}$ leads to a relation between $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}$.

$$
\begin{aligned}
\delta\boldsymbol{P} &= \tfrac{1}{2}c_0 \delta\boldsymbol{C} : \boldsymbol{II} + \tfrac{1}{2}c_1 \delta\boldsymbol{C} \\
&\qquad \boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \quad \rightarrow \quad \delta\boldsymbol{C} = \delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F} \\
&= \tfrac{1}{2}c_0 \left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) : \boldsymbol{II} + \tfrac{1}{2}c_1 \left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) \\
&= c_0(\boldsymbol{F}^c \cdot \delta\boldsymbol{F}) : \boldsymbol{II} + \tfrac{1}{2}c_1 \left(\delta\boldsymbol{F}^c \cdot \boldsymbol{F} + \boldsymbol{F}^c \cdot \delta\boldsymbol{F}\right) \\
&= c_0\boldsymbol{I}(\boldsymbol{F}^c : \delta\boldsymbol{F}) + \tfrac{1}{2}c_1 \left\{(\boldsymbol{F}^c \cdot \delta\boldsymbol{F})^c + (\boldsymbol{F}^c \cdot \delta\boldsymbol{F})\right\}
\end{aligned}
$$

$$
\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c \quad \rightarrow
$$

$$
\delta\boldsymbol{\sigma} = J^{-1}\left[-\delta J \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + \delta\boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c + \boldsymbol{F} \cdot \delta\boldsymbol{P} \cdot \boldsymbol{F}^c + \boldsymbol{F} \cdot \boldsymbol{P} \cdot \delta\boldsymbol{F}^c\right]
$$

$$
\begin{aligned}
\delta J = J\mathrm{tr}(\boldsymbol{L}) = J\boldsymbol{L} : \boldsymbol{I} \quad ; \quad \delta\boldsymbol{F} = \boldsymbol{L}\cdot\boldsymbol{F} \\
= J^{-1}\left[-(\boldsymbol{L}:\boldsymbol{I})\boldsymbol{F}\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c + (\boldsymbol{L}\cdot\boldsymbol{F})\cdot\boldsymbol{P}\cdot\boldsymbol{F}^c + \right. \\
\left. \boldsymbol{F}\cdot\delta\boldsymbol{P}\cdot\boldsymbol{F}^c + \boldsymbol{F}\cdot\boldsymbol{P}\cdot(\boldsymbol{F}^c\cdot\boldsymbol{L}^c)\right] \\
= -(\boldsymbol{L}:\boldsymbol{I})\boldsymbol{\sigma} + \boldsymbol{L}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\boldsymbol{L}^c + J^{-1}\boldsymbol{F}\cdot\delta\boldsymbol{P}\cdot\boldsymbol{F}^c \\
= -\boldsymbol{\sigma}(\boldsymbol{I}:\boldsymbol{L}) + (\boldsymbol{\sigma}^c\cdot\boldsymbol{L}^c)^c + \boldsymbol{\sigma}\cdot\boldsymbol{L}^c + J^{-1}\boldsymbol{F}\cdot(\boldsymbol{F}\cdot\delta\boldsymbol{P}^c)^c
\end{aligned}
$$

2.5.4 Matrix/column notation

The tensorial expression is transferred to matrix/column notation.

$$
\begin{aligned}
\boldsymbol{P} &= \tfrac{1}{2}c_0\boldsymbol{C}:\boldsymbol{II} + \tfrac{1}{2}c_1\boldsymbol{C} - \tfrac{1}{2}(3c_0+c_1)\boldsymbol{I} \qquad \text{with} \qquad \boldsymbol{C} = \boldsymbol{F}^c\cdot\boldsymbol{F} \\
\underset{\approx}{P} &= \tfrac{1}{2}c_0\underset{\approx}{C}^T\underset{\approx}{I}_t\underset{\approx}{I} + \tfrac{1}{2}c_1\underset{\approx}{C} - \tfrac{1}{2}(3c_0+c_1)\underset{\approx}{I} \qquad \text{with} \qquad \underset{\approx}{C} = \underline{\underline{F}}_t\underset{\approx}{F}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F}\cdot(\boldsymbol{F}\cdot\boldsymbol{P}^c)^c \\
\underset{\approx}{\sigma} &= J^{-1}\underline{\underline{F}}\left(\underline{\underline{F}}_r\underset{\approx}{P}_t\right) = J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_r\underset{\approx}{P}_t
\end{aligned}
$$

$$
\begin{aligned}
\delta\boldsymbol{P} &= c_0\boldsymbol{I}(\boldsymbol{F}^c:\delta\boldsymbol{F}) + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}^c\cdot\delta\boldsymbol{F})^c + (\boldsymbol{F}^c\cdot\delta\boldsymbol{F})\right\} \\
\delta\underset{\approx}{P} &= c_0\underset{\approx}{I}\underset{\approx}{F}_t^T\delta\underset{\approx}{F}_t + \tfrac{1}{2}c_1\left\{(\underline{\underline{F}}_t\delta\underset{\approx}{F})_r + (\underline{\underline{F}}_t\delta\underset{\approx}{F})\right\} \\
&= c_0\underset{\approx}{I}\underset{\approx}{F}_{tc}^T\delta\underset{\approx}{F} + \tfrac{1}{2}c_1\left(\underline{\underline{F}}_{tr}\delta\underset{\approx}{F} + \underline{\underline{F}}_t\delta\underset{\approx}{F}\right) \\
&= \underline{\underline{\underline{M}}}_0\delta\underset{\approx}{F} = \underline{\underline{\underline{M}}}_0\left(\underset{\approx}{L}_0\right)_t = \underline{\underline{\underline{M}}}_0\underline{\underline{F}}_{tr}\underset{\approx}{L}_t = \underline{\underline{\underline{M}}}_1\,\underset{\approx}{L}_t
\end{aligned}
$$

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= -\boldsymbol{\sigma}(\boldsymbol{I}:\boldsymbol{L}) + (\boldsymbol{\sigma}^c\cdot\boldsymbol{L}^c)^c + \boldsymbol{\sigma}\cdot\boldsymbol{L}^c + J^{-1}\boldsymbol{F}\cdot(\boldsymbol{F}\cdot\delta\boldsymbol{P}^c)^c \\
\delta\underset{\approx}{\sigma} &= -\underset{\approx}{\sigma}\underset{\approx}{I}^T\underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr}\underset{\approx}{L}_t + \underline{\underline{\sigma}}\underset{\approx}{L}_t + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_r\delta\underset{\approx}{P}_t \\
&= -\underset{\approx}{\sigma}\underset{\approx}{I}^T\underset{\approx}{L}_t + \underline{\underline{\sigma}}_{tr}\underset{\approx}{L}_t + \underline{\underline{\sigma}}\underset{\approx}{L}_t + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_{rc}\delta\underset{\approx}{P} \\
&= \left[-\underset{\approx}{\sigma}\underset{\approx}{I}^T + \underline{\underline{\sigma}}_{tr} + \underline{\underline{\sigma}} + J^{-1}\underline{\underline{F}}\,\underline{\underline{F}}_{rc}\underline{\underline{\underline{M}}}_1\right]\underset{\approx}{L}_t = \underline{\underline{\underline{M}}}\,\underset{\approx}{L}_t
\end{aligned}
$$

2.5.5 Linear s - A model

2.5.6 Stress update

Elastic material behavior may be described by a linear relation between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the Finger tensor $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{B}-\boldsymbol{I})$ with $\boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c$. The above relation cannot be derived from an elastic energy function and is thus referred to as hypo-elastic.

$$
\begin{aligned}
\boldsymbol{\sigma} &= c_0\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} + c_1\boldsymbol{A} \qquad \text{with} \qquad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{B}-\boldsymbol{I}) \\
&= \tfrac{1}{2}c_0\boldsymbol{B}:\boldsymbol{II} + \tfrac{1}{2}c_1\boldsymbol{B} - \tfrac{1}{2}(3c_0+c_1)\boldsymbol{I} \qquad \text{with} \qquad \boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c
\end{aligned}
$$

2.5.7 Stiffness

The variation of the Cauchy stress tensor can be related to $\delta\boldsymbol{F}$, and consequently to the gradient of the iterative displacement vector $\delta\vec{u}$.

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \tfrac{1}{2}c_0\delta\boldsymbol{B} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\delta\boldsymbol{B} \\
&= \tfrac{1}{2}c_0\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&= c_0(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c) : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&= c_0\boldsymbol{I}\boldsymbol{F} : \delta\boldsymbol{F}^c + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \\
&\quad \text{with} \qquad \delta\boldsymbol{F} = \boldsymbol{L}\cdot\boldsymbol{F} = (\boldsymbol{F}^c\cdot\boldsymbol{L}^c)^c \qquad \text{and} \quad \boldsymbol{L}^c = \vec{\nabla}\vec{u}
\end{aligned}
$$

2.5.8 Matrix/column notation

All tensor equations can be trasferred to matrix equations.

$$
\begin{aligned}
\boldsymbol{\sigma} &= \tfrac{1}{2}c_0\boldsymbol{B} : \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2}c_1\boldsymbol{B} - \tfrac{1}{2}(3c_0 + c_1)\boldsymbol{I} && \text{with} \qquad \boldsymbol{B} = \boldsymbol{F}\cdot\boldsymbol{F}^c \\
\underset{\approx}{\sigma} &= \tfrac{1}{2}c_0\underset{\approx}{B}^T\underset{\approx}{I}_t\underset{\approx}{I} + \tfrac{1}{2}c_1\underset{\approx}{B} - \tfrac{1}{2}(3c_0 + c_1)\underset{\approx}{I} && \text{with} \qquad \underset{\approx}{B} = \underline{\underline{F}}\,\underset{\approx}{F}_t
\end{aligned}
$$

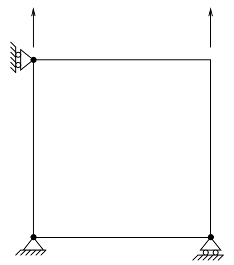
$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= c_0\boldsymbol{I}\boldsymbol{F} : \delta\boldsymbol{F}^c + \tfrac{1}{2}c_1\left\{(\boldsymbol{F}\cdot\delta\boldsymbol{F}^c)^c + \boldsymbol{F}\cdot\delta\boldsymbol{F}^c\right\} \quad \text{with} \qquad \delta\boldsymbol{F} = \boldsymbol{L}\cdot\boldsymbol{F} = (\boldsymbol{F}^c\cdot\boldsymbol{L}^c)^c \\
\delta\underset{\approx}{\sigma} &= c_0\underset{\approx}{I}\,\underset{\approx}{F}^T\delta\underset{\approx}{F} + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_r\,\delta\underset{\approx}{F}_t + \underline{\underline{F}}\,\delta\underset{\approx}{F}_t\right\} \\
&= \left[c_0\underset{\approx}{I}\,\underset{\approx}{F}^T + \tfrac{1}{2}c_1\left\{\underline{\underline{F}}_{rc} + \underline{\underline{F}}_c\right\}\right]\delta\underset{\approx}{F} \qquad \text{with} \qquad \delta\underset{\approx}{F} = \left(\underline{\underline{F}}_t\,\underset{\approx}{L}_t\right)_r = \underline{\underline{F}}_{tr}\,\underset{\approx}{L}_t \\
&= \left[c_0\underset{\approx}{I}\,\underset{\approx}{F}^T\underline{\underline{F}}_{tr} + \tfrac{1}{2}c_1\left(\underline{\underline{F}}_{rc}\underline{\underline{F}}_{tr} + \underline{\underline{F}}_c\underline{\underline{F}}_{tr}\right)\right]\underset{\approx}{L}_t = \underline{\underline{M}}\,\underset{\approx}{L}_t
\end{aligned}
$$

2.6 Examples

A square plate is subjected to a tensile and a shear deformation. The two linear elastic models, described before, are used to model the elastic behavior. Both plane stress and plane strain states are considered.

2.6.1 Tensile test

A square plate or cylindrical bar is loaded uniaxially using different elastic material models. Dimensions are listed in the table. For plane stress and axisymmetry, the loading is equivalent to a tensile test.



Cartesian			
initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

cylindrical			
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

The axial elongation is prescribed and the resulting axial force is calculated for various elastic material models. Material parameter values are $C = 100000$ MPa and $\nu = 0.3$.

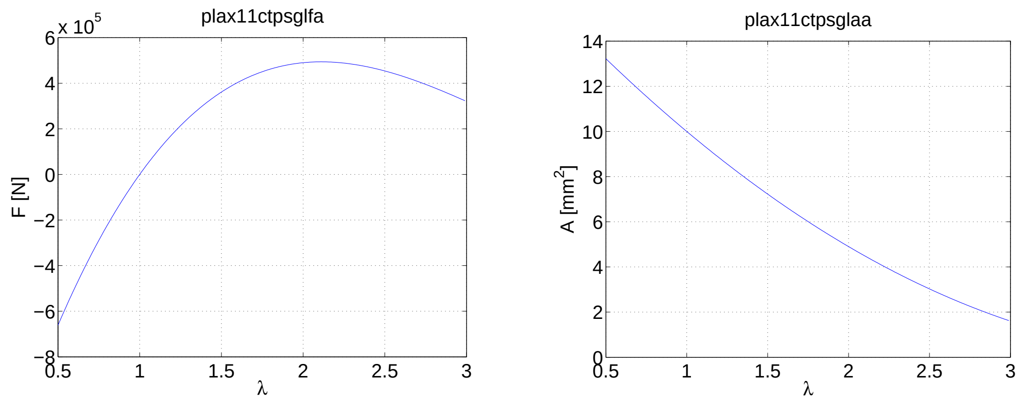


Fig. 2.10 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{\varepsilon}$ *model*

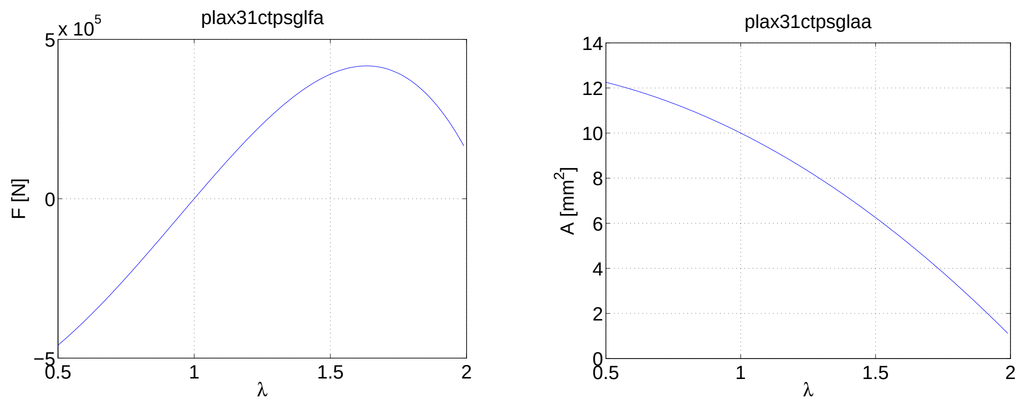


Fig. 2.11 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{A}$ *model*

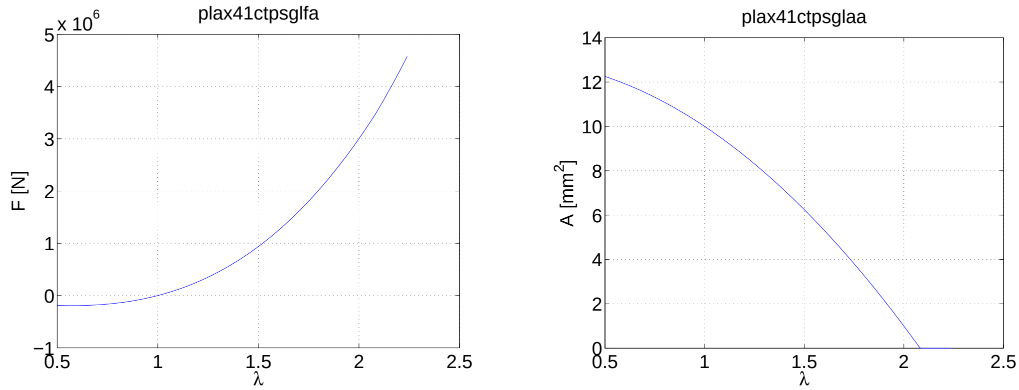


Fig. 2.12 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model.*

The latter model is also used in a plane strain tensile test. Both Updated Lagrange and Total Lagrange formulation are used. The results are the same, which should be the case.

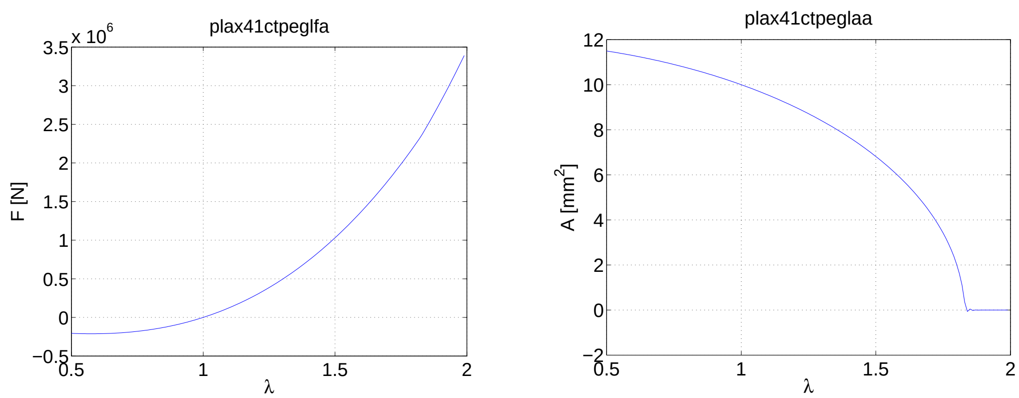


Fig. 2.13 : *Tensile force and cross-sectional area versus elongation; plane strain;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model; Updated Lagrange formulation*

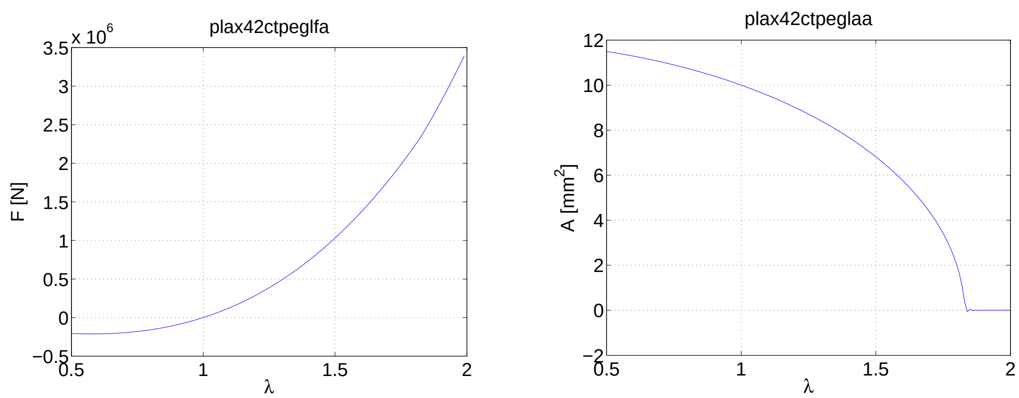
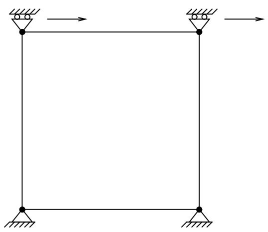


Fig. 2.14 : *Tensile force and cross-sectional area versus elongation; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model; Total Lagrange formulation*

2.6.2 Shear test

The simple shear test is analyzed with one element, where the horizontal displacement/force in the upper nodes is prescribed. Dimensions are listed in the table.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

Subsequently the material model $\boldsymbol{\sigma} \sim \boldsymbol{A}$ and $\boldsymbol{P} \sim \boldsymbol{E}$ are used in the analysis.

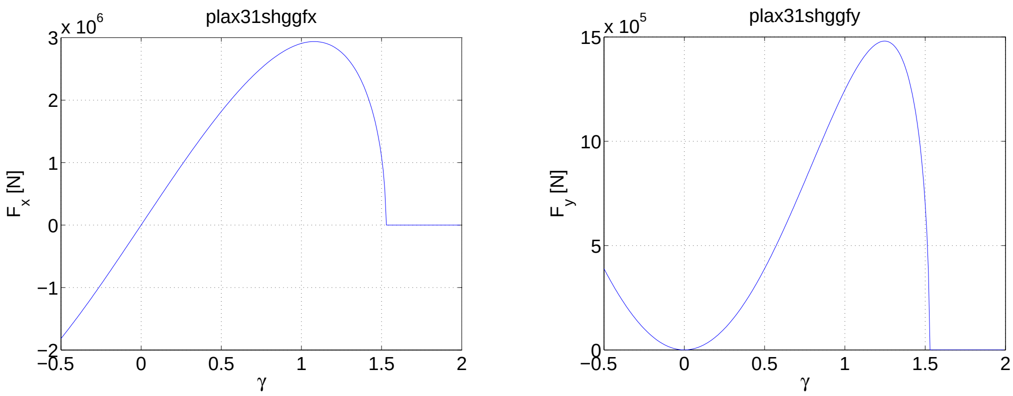


Fig. 2.15 : *Shear and normal force versus shear strain; plane stress;* $\boldsymbol{\sigma} \sim \boldsymbol{A}$ *model*

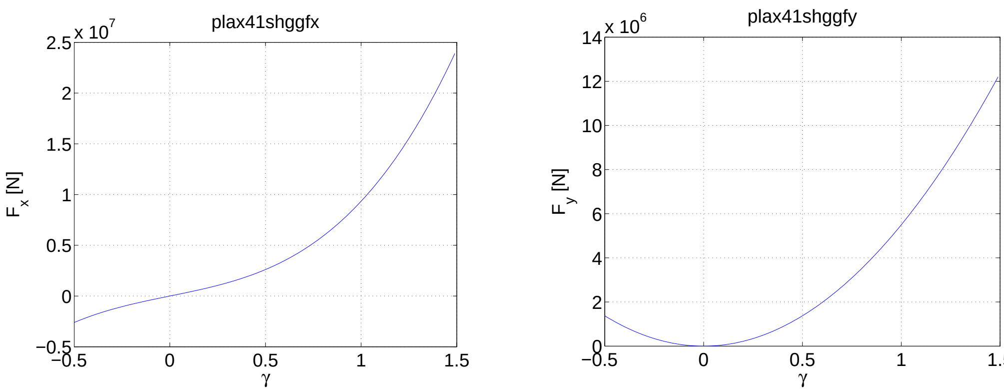
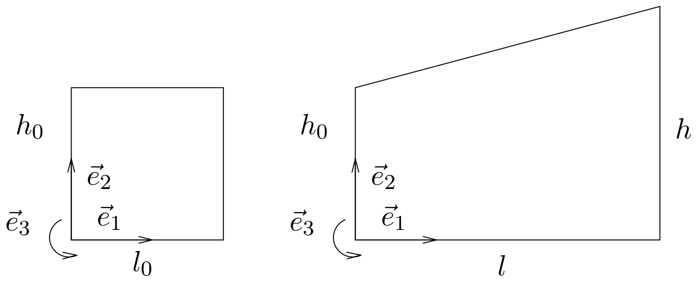


Fig. 2.16 : *Shear and normal force versus shear strain; plane stress;* $\boldsymbol{P} \sim \boldsymbol{E}$ *model*

2.6.3 Inhomogeneous deformation



$$x_1 = \frac{l}{l_0}\,x_{01} \quad ; \quad x_2 = x_{02} + \frac{h-h_0}{h_0 l_0}\,x_{01}x_{02} \quad ; \quad x_3 = x_{03}$$

$$\boldsymbol{F} = \left(\frac{l}{l_0}\right)\vec{e}_1\vec{e}_{01} + \left(\frac{h-h_0}{h_0 l_0}x_{02}\right)\vec{e}_2\vec{e}_{01} + \left(1 + \frac{h-h_0}{h_0 l_0}x_{01}\right)\vec{e}_2\vec{e}_{02} + \vec{e}_3\vec{e}_{03}$$

$$\begin{aligned}\boldsymbol{\sigma} &= c_0\mathrm{tr}(\boldsymbol{A})\boldsymbol{I} + c_1\boldsymbol{A} \qquad \text{with} \quad \boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{F}\cdot\boldsymbol{F}^c - \boldsymbol{I}) \\ &= \tfrac{c_0}{2}[x_{01}^2 + x_{02}^2 + 2x_{02} + 3]\boldsymbol{I} + \\ &\quad \tfrac{c_1}{2}[(x_{01}^2 + x_{02}^2 + 2x_{02})\vec{e}_1\vec{e}_1 + 3\vec{e}_2\vec{e}_2 + 2x_{01}(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)]\end{aligned}$$

$$p_n = \vec{p}\cdot\vec{e}_2 = c_0(6 + x_{01}^2) + 3c_1 \quad ; \quad p_s = \vec{p}\cdot\vec{e}_1 = 2c_1 x_{01}$$

3 Elastoplastic material behavior

The one-dimensional mechanical representation of an elastoplastic model consists of a spring in series with a parallel arrangement of a spring and friction slider. The series-spring represents the purely elastic part of the deformation, when stress is below the yield stress. The elastoplastic response becomes manifest when the stress exceeds the yield stress σ_y.

After yielding the total strain rate $\dot{\varepsilon}$ is the sum of the elastic strain rate $\dot{\varepsilon}_e$ and the plastic strain rate $\dot{\varepsilon}_p$. It is only for small strains that we can also add strains. The rate is a fictitious time derivative as for this material model the stress is not influenced by the strain rate.

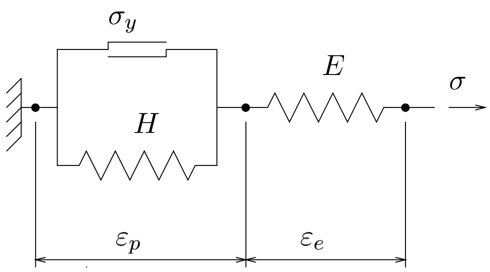


Fig. 3.17 : *Discrete model for elastoplastic material behavior*

3.1 Kinematics

Transformation from the undeformed configuration at time t_0 (position vector $\vec{x}_0$) to the current configuration at time t (position vector $\vec{x}$) is described by the deformation tensor $\boldsymbol{F} = (\vec{\nabla}_0\vec{x})^c$, where $\vec{\nabla}_0$ is the gradient operator with respect to the undeformed state.

The right and left Cauchy-Green strain tensors, $\boldsymbol{C}$ and $\boldsymbol{B}$, are functions of $\boldsymbol{F}$ as is the Green-Lagrange strain tensor $\boldsymbol{E}$. The deformation rate is described by the velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla}\vec{v})^c$, where $\vec{\nabla}$ is the gradient operator with respect to the current state and $\vec{v}$ is the velocity of the material volume.

The total deformation $\boldsymbol{F}$ is multiplicatively decomposed into an elastic and a plastic contribution. For the velocity gradient tensor an additive decomposition into the symmetric deformation rate tensor $\boldsymbol{D}$ and the skew-symmetric spin tensor $\boldsymbol{\Omega}$ is used. Both $\boldsymbol{D}$ and $\boldsymbol{\Omega}$ can be split into an elastic and a plastic part.

To make the decomposition unique it is commonly assumed that the plastic rotation rate during the current increment is zero, i.e. $\boldsymbol{\Omega}_p = 0$. Superimposed material rotations are thus fully represented in $\boldsymbol{F}_e$.

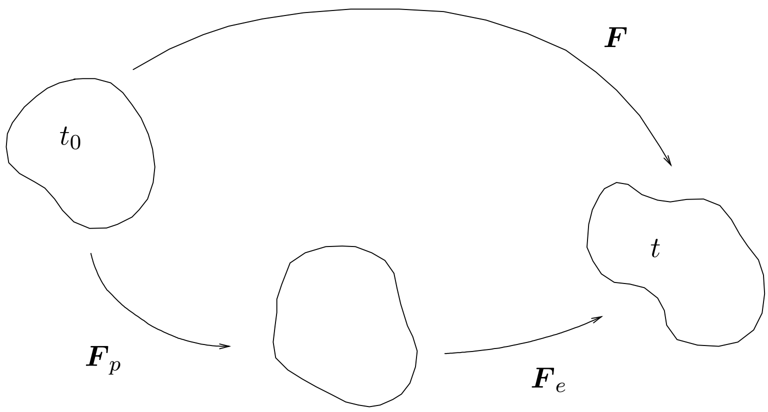


Fig. 3.18 : *Multiplicative decomposition of total deformation*

$$
\begin{aligned}
&\boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_p \\
&\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \qquad ; \qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
&\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla} \vec{v})^c \\
&\quad = \boldsymbol{L}_e + \boldsymbol{L}_p = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_p + \boldsymbol{\Omega}_p) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_p
\end{aligned}
$$

3.2 Constitutive relations

Elastic deformation

The stress is related to the elastic strain with an elastic material model. In elastoplastic deformation problems, it can often be assumed that elastic strains are small, which allows the use of a hypo-elastic generalized Hooke's law, relating the Cauchy stress tensor $\boldsymbol{\sigma}$ to the logarithmic strain tensor $\boldsymbol{\Lambda}$.

The material is assumed to be isotropic in which case the elastic material behavior is characterized by two material constants : the bulk modulus K and the shear modulus G. The fourth-order unity tensor is defined as : ${}^4\boldsymbol{I} = \vec{e}_i \vec{e}_j \delta_{il} \delta_{jk} \vec{e}_k \vec{e}_l$ and ${}^4\boldsymbol{I}^{rc}$ is its right conjugate.

The current stress state must be determined from the elastoplastic constitutive model, which is necessarily a rate formulation, i.e. a relation between a time derivative of the stress and the deformation rate. To avoid problems with large rigid rotations, the constitutive relations are formulated in invariant variables. A general invariant stress tensor $\boldsymbol{\sigma}_A$ is introduced first. It can be proved that both $\boldsymbol{\sigma}_A$ and $\dot{\boldsymbol{\sigma}}_A$ are invariant, when rigid body rotation (rotation tensor $\boldsymbol{Q}$) transforms $\boldsymbol{A}$ into $\boldsymbol{A}^*$ according to : $\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c$.

The elastic material law can then be reformulated, such that it obeys the objectivity requirement.

$$
\left.
\begin{aligned}
&\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_e \\[1ex]
&{}^4\boldsymbol{C} = c_0 \boldsymbol{I}\boldsymbol{I} + \tfrac{1}{2} c_1 ({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}) = K \boldsymbol{I}\boldsymbol{I} + 2G \left({}^4\boldsymbol{I} - \tfrac{1}{3} \boldsymbol{I}\boldsymbol{I} \right)
\end{aligned}
\right\}
$$

invariant tensors

$$\boldsymbol{\sigma}_A = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c = \boldsymbol{\sigma}_A^* \qquad \text{with} \quad \boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c \quad \forall \quad \boldsymbol{Q}$$
$$\dot{\boldsymbol{\sigma}}_A = \boldsymbol{A} \cdot \left\{ (\boldsymbol{A}^{-1} \cdot \dot{\boldsymbol{A}}) \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot (\boldsymbol{A}^{-1} \cdot \dot{\boldsymbol{A}})^c + \dot{\boldsymbol{\sigma}} \right\} \cdot \boldsymbol{A}^c = \boldsymbol{A} \cdot \overset{\odot}{\boldsymbol{\sigma}}_A \cdot \boldsymbol{A}^c = \dot{\boldsymbol{\sigma}}_A^*$$

objective elastic law $\qquad \overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C} : \boldsymbol{D}_e$

Yield criterion and hardening

A yield function F is used to evaluate the stress state and to check whether the deformation is purely elastic ($F < 0$) or elastoplastic ($F = 0$). The current stress state, represented by the equivalent stress $\bar{\sigma}$, is compared to a yield stress σ_y. Its initial value is σ_{y0}. This yield stress changes with plastic deformation and is therefore related to the effective plastic strain $\bar{\varepsilon}_p$. The relation between σ_y and $\bar{\varepsilon}_p$ is described by the hardening law. To decide whether elastic or elastoplastic deformation occurs, the Kuhn-Tucker relations are used.

yield criterion $\qquad F = \bar{\sigma}^2 - \sigma_y^2(\bar{\varepsilon}_p)$

effective plastic strain $\qquad \bar{\varepsilon}_p = \int\limits_{\tau=0}^{t} \dot{\bar{\varepsilon}}_p \, d\tau$

hardening law $\qquad \sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad \text{with} \quad \dfrac{\partial \sigma_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$

Kuhn-Tucker relations $\qquad \{(F < 0) \vee (F = 0 \wedge \dot{F} < 0)\} \quad \rightarrow \quad$ elastic
$\qquad \{(F = 0) \wedge (\dot{F} = 0)\} \quad \rightarrow \quad$ elastoplastic

3.2.1 Von Mises plasticity

For the Von Mises yield criterion, the yield surface is a circular cylinder in principal stress space. The equivalent Von Mises stress can be expressed in the deviatoric stress tensor $\boldsymbol{\sigma}^d$. It is required that the disspated plastic energy per unit of time is the product of the equivalent stress and the effective or equivalent plastic strain rate :

$$\boldsymbol{\sigma} : \boldsymbol{D}_p = \bar{\sigma} \, \dot{\bar{\varepsilon}}_p$$

which leads to the definition of the effective plastic strain rate as a function of the plastic deformation rate tensor $\boldsymbol{D}_p$.

$$\bar{\sigma} = \sqrt{\tfrac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}$$

$$\dot{\bar{\varepsilon}}_p = \sqrt{\tfrac{2}{3} \boldsymbol{D}_p : \boldsymbol{D}_p}$$

$$F = \tfrac{3}{2} \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d - \sigma_y^2(\bar{\varepsilon}_p)$$

$$
\begin{aligned}
\dot{F} &= 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y\dot{\sigma}_y = 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y H\dot{\bar{\varepsilon}}_p \\
&= 3\boldsymbol{\sigma}^d : \dot{\boldsymbol{\sigma}} - 2\sigma_y H\dot{\bar{\varepsilon}}_p = 3\boldsymbol{\sigma}^d_A : \dot{\boldsymbol{\sigma}}_A - 2\sigma_y H\dot{\bar{\varepsilon}}_p = 0
\end{aligned}
$$

Elastoplastic deformation

During elastoplastic deformation ($F = 0$) the plastic deformation rate $\boldsymbol{D}_p$ is related to the stress by the flow rule. For a so-called *normality* or *associative* flow rule the direction of $\boldsymbol{D}_p$ is perpendicular to the yield surface in stress space. The *length* of $\boldsymbol{D}_p$ is characterized by the plastic multiplier $\dot{\lambda}$. The normal to the yield surface can be expressed in the deviatoric stress $\boldsymbol{\sigma}^d$.

The value of the plastic multiplier $\dot{\lambda}$ can be determined from the requirement that the stress state must always reside on the yield surface during elastoplastic deformation, so : $\dot{F} = 0$. This relation is referred to as the *consistency condition*.

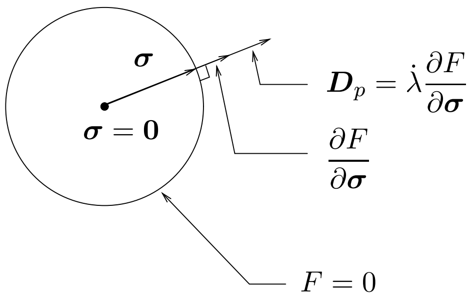


Fig. 3.19 : *Associative flow rule*

$$
\begin{aligned}
\boldsymbol{D}_p &= \dot{\lambda}\frac{\partial F}{\partial \boldsymbol{\sigma}} = \dot{\lambda}\boldsymbol{a} \\
\boldsymbol{a} &= \frac{\partial F}{\partial \boldsymbol{\sigma}^d} : \frac{\partial \boldsymbol{\sigma}^d}{\partial \boldsymbol{\sigma}} \\
&= \left[3\boldsymbol{\sigma}^d : {}^4\boldsymbol{I}\right] : \frac{\partial}{\partial \boldsymbol{\sigma}}\left\{\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}\right\} = 3\boldsymbol{\sigma}^d : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) = 3\boldsymbol{\sigma}^d \\
\dot{\bar{\varepsilon}}_p &= \dot{\lambda}\sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

3.3 Constitutive model

The material model can be summarized as a set of constitutive relations. In accordance with $\boldsymbol{\sigma}_A$, invariant tensors $\boldsymbol{D}_A$ and $\boldsymbol{a}_A$ are defined. Also a new fourth-order material tensor ${}^4\boldsymbol{C}_A$ is introduced according to the requirement :

$$
{}^4\boldsymbol{C}_A : \boldsymbol{D}_A = \boldsymbol{A} \cdot {}^4\boldsymbol{C} : \boldsymbol{D} \cdot \boldsymbol{A}^c \qquad \forall \quad \boldsymbol{A}
$$

The set of differential equations must be integrated over the deformation history to determine the current stress $\boldsymbol{\sigma}(t)$ when the current deformation $\boldsymbol{F}(t)$ is known. It is also used to derive a relation between the variation of stress and deformation, which is an essential part of the element stiffness matrix.

$$
\begin{aligned}
&\{(F<0) \vee (F=0 \wedge \dot{F}<0)\} \quad \rightarrow \quad \boldsymbol{D}=\boldsymbol{D}_e \quad \rightarrow \quad \dot{\bar{\varepsilon}}_p=0 \\
&\qquad \overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}:\boldsymbol{D} \quad \rightarrow \quad \dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A:\boldsymbol{D}_A
\end{aligned}
$$

$$
\begin{aligned}
&\{(F=0) \wedge (\dot{F}=0)\} \quad \rightarrow \quad \boldsymbol{D}=\boldsymbol{D}_e+\boldsymbol{D}_p \\
&\qquad \left.\begin{array}{l} \overset{\odot}{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}:(\boldsymbol{D}-\dot{\lambda}\boldsymbol{a}) \\ 2\bar{\sigma}\dot{\bar{\sigma}} - 2\sigma_y H \dot{\bar{\varepsilon}}_p = 0 \end{array}\right\} \quad \rightarrow \\
&\qquad \left.\begin{array}{l} \dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A : \left(\boldsymbol{D}_A - \dot{\lambda}\boldsymbol{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d : \dot{\boldsymbol{\sigma}}_A - 2\sigma_y H \dot{\lambda}\sqrt{\frac{2}{3}\boldsymbol{a}_A:\boldsymbol{a}_A} = 0 \end{array}\right\} \\
&\qquad \left.\begin{array}{l} \dot{\boldsymbol{\sigma}}_A = {}^4\boldsymbol{C}_A : \left(\boldsymbol{D}_A - \dot{\lambda}\boldsymbol{a}_A\right) \\ 3\boldsymbol{\sigma}_A^d : {}^4\boldsymbol{C}_A : \boldsymbol{D}_A - \dot{\lambda}\left(3\boldsymbol{\sigma}_A^d : {}^4\boldsymbol{C}_A : \boldsymbol{a}_A + 2\sigma_v H\sqrt{\frac{2}{3}\boldsymbol{a}_A:\boldsymbol{a}_A}\right) = 0 \end{array}\right\} \\
&\qquad \sigma_y = \sigma_y(\sigma_{y0}, \bar{\varepsilon}_p) \quad ;
\end{aligned}
$$

3.4 Incremental analysis

The figure shows the relevant configurations in a large strain plastic deformation process. Although the time t is used to identify various configurations, it is noted that the material behavior is considered to be time independent. The variable t is thus a pseudo-time.

Starting from the undeformed configuration at t_0 the external load is applied and the deformation leads to the current configuration t. During a numerical analysis of this deformation process the state of the material is determined at a finite number of discrete moments $t_i,\ i=0,1,..,n+1$. The period between two subsequent moments is an increment : $\Delta t_i = t_{i+1} - t_i$.

It is assumed that the analysis has brought us to $t = t_n$, the beginning of the last increment and that all relevant variables are known and satisfying all governing equations (balance laws, boundary conditions, constitutive relations). The state at the current time $t = t_{n+1}$, the end of the current increment has to be determined.

The incremental deformation is described by the deformation tensor $\boldsymbol{F}_n$. The incremental principle elongation factors and directions, λ_{ni} and $\vec{n}_{ni}$ $(i=1,2,3)$, respectively, with respect to the begin increment state, can be determined from $\boldsymbol{C}_n = \boldsymbol{F}_n^c \cdot \boldsymbol{F}_n$. The incremental stretch tensor $\boldsymbol{U}_n$ and logarithmic strain tensor $\boldsymbol{\Lambda}_n$ can be expressed in λ_{ni} and $\vec{n}_{ni}$.

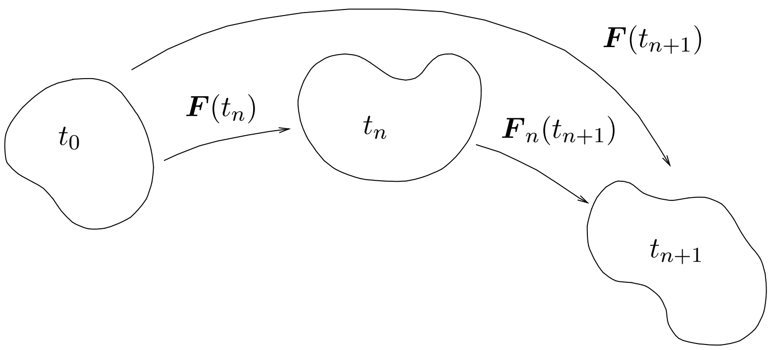


Fig. 3.20 : *Incremental deformation*

$$\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = (\vec{\nabla}_n \vec{x})^c = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n)$$

$$\boldsymbol{D} = \tfrac{1}{2}\left(\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} + \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c\right) = \tfrac{1}{2}\boldsymbol{R}_n \cdot \left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} + \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n\right) \cdot \boldsymbol{R}_n^c$$

$$\boldsymbol{\Omega} = \tfrac{1}{2}\left\{\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} - \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c\right\} = \dot{\boldsymbol{R}}_n \cdot \boldsymbol{R}_n^c + \tfrac{1}{2}\boldsymbol{R}_n \cdot \left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} - \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n\right) \cdot \boldsymbol{R}_n^c$$

$$\boldsymbol{U}_n = \sum_{i=1}^{3} \lambda_{ni} \vec{n}_{ni} \vec{n}_{ni} \quad ; \quad \boldsymbol{\Lambda}_n = \sum_{i=1}^{3} \ln(\lambda_{ni}) \vec{n}_{ni} \vec{n}_{ni}$$

3.4.1 Elastic stress predictor

The stress integration procedure is always started with the calculation of an elastic stress predictor. It is assumed that the increment is fully elastic and that the begin-increment elasticity tensor can be used to calculate the rotation neutralized Cauchy stress tensor. Subsequently the elastic Cauchy stress tensor is calculated and used to evaluate the yield criterion with two possible outcomes :

1. the increment is indeed fully elastic,
2. the yield criterion is violated which implies that during the increment further elastoplastic deformation has taken place.

elastic trial stress $\quad \boldsymbol{\sigma}_e = \boldsymbol{\sigma}(t_n) + {}^4\boldsymbol{C} : (\boldsymbol{\Lambda} - \boldsymbol{\Lambda}(t_n))$

yield criterion $\quad F = \tfrac{3}{2}\boldsymbol{\sigma}_e^d : \boldsymbol{\sigma}_e^d - \sigma_y^2(\sigma_{y0}, \bar{\varepsilon}_p(t_n))$

$F \leq 0 \quad \rightarrow \quad$ elastic increment

$F > 0 \quad \rightarrow \quad$ elastoplastic increment

matrix/column notation

$$
\begin{array}{lllll}
\underline{\underline{C}} = K \underset{\approx}{I} \underset{\approx}{I}^T + 2G \left(\underline{\underline{I}} - \frac{1}{3} \underset{\approx}{I} \underset{\approx}{I}^T \right) & ; & \underline{\Lambda}_n & \rightarrow & \underset{\approx}{\Lambda}_n \\
\underset{\approx}{\sigma}_{D_e} = \underset{\approx}{\sigma}(t_n) + \underline{\underline{C}}_c \underset{\approx}{\Lambda}_n & \rightarrow & \underline{\sigma}_{D_e} & \rightarrow & \underline{\sigma}_e = \underline{R}_n \, \underline{\sigma}_{D_e} \, \underline{R}_n^T \\
F = \frac{3}{2} \left(\underset{\approx}{\sigma}_{D_{tr}} \right)^T \left(\underset{\approx}{\sigma}_{D_{tr}} \right) - \sigma_y^2(\bar{\varepsilon}_p) & & & &
\end{array}
$$

Elastic increment

When the increment is fully elastic the end-increment Cauchy stress equals the calculated elastic Cauchy stress. As no plastic deformation has occurred during the increment, the effective plastic strain and the yield stress have not changed.

$$
\begin{array}{lll}
\boldsymbol{\sigma}(t_{n+1}) = \boldsymbol{\sigma}_e & ; & \Delta\lambda = 0 \\
\bar{\varepsilon}_p(t_{n+1}) = \bar{\varepsilon}_p(t_n) & ; & \sigma_y(t_{n+1}) = \sigma_y(t_n)
\end{array}
$$

Elastoplastic increment

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations must be solved. Before this is possible the invariant tensors $(\ )_A$ must be specified. It is also necessary to make some assumptions about the incremental deformation. Because the rigid rotation during the increment is not uniquely known, rotation neutralized quantities are used. This implies the specification of the invariant tensors by choosing $\boldsymbol{A} = \boldsymbol{R}_n^c$ resulting in Dienes tensors and Dienes rates. The complete elastoplastic model can now be formulated in rotation neutralized quantities $\boldsymbol{\sigma}_D$, $\boldsymbol{D}_D$ and $\boldsymbol{a}_D$.

$$
\left.
\begin{array}{l}
\boldsymbol{\sigma}_D = \boldsymbol{R}_n^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R}_n \quad \rightarrow \quad \dot{\boldsymbol{\sigma}}_D = \boldsymbol{R}_n^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_D \cdot \boldsymbol{R}_n \\
\boldsymbol{D}_D = \boldsymbol{R}_n^c \cdot \boldsymbol{D} \cdot \boldsymbol{R}_n = \frac{1}{2} \left(\dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} + \boldsymbol{U}_n^{-1} \cdot \dot{\boldsymbol{U}}_n \right)
\end{array}
\right\}
$$

$$
\left.
\begin{array}{l}
\dot{\boldsymbol{\sigma}}_D = {}^4\boldsymbol{C}_D : \left(\boldsymbol{D}_D - \dot{\lambda} \boldsymbol{a}_D \right) \\
3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C}_D : \boldsymbol{D}_D - \dot{\lambda} \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C}_D : \boldsymbol{a}_D + 2\sigma_y H \sqrt{\frac{2}{3} \boldsymbol{a}_D : \boldsymbol{a}_D} \right) = 0
\end{array}
\right\}
$$

Rotation neutralized elastoplastic increment

It is assumed that there is no rigid body rotation during the increment. All rigid body rotation will be taken into account at the end-increment time t_{n+1}. The integrated stress tensor is the so-called rotation neutralized stress tensor $\boldsymbol{\sigma}_D$.

When it is also assumed that the incremental principal strain directions are constant during the increment, the tensors $\dot{\boldsymbol{U}}_n$ and $\boldsymbol{U}_n^{-1}$ are commuting. With this assumption, the constitutive equations for the rotation neutralized Dienes stress $\boldsymbol{\sigma}_D$ can now be used for integration.

$$t_n \le \tau < t_{n+1} \quad : \quad \boldsymbol{R}_n = I \quad ; \quad \boldsymbol{D}_D = \boldsymbol{D} \quad ; \quad \boldsymbol{a}_D = \boldsymbol{a} \quad ; \quad {}^4\boldsymbol{C}_D = {}^4\boldsymbol{C}$$

$$\tau = t_{n+1} \quad : \quad \boldsymbol{R}_n(t_{n+1}) = \boldsymbol{F}(t_{n+1}) \cdot \boldsymbol{U}^{-1}(t_{n+1})$$

$$\boldsymbol{U}_n(\tau) = \sum_{i=1}^{3} \lambda_{ni}(\tau)\vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n)$$
$$\boldsymbol{D} = \dot{\boldsymbol{U}}_n \cdot \boldsymbol{U}_n^{-1} = \sum_{i=1}^{3} \left(\frac{\dot{\lambda}_{ni}(\tau)}{\lambda_{ni}(\tau)} \right) \vec{n}_{ni}(t_n)\vec{n}_{ni}(t_n) = \dot{\boldsymbol{\Lambda}}_n$$

constitutive equations

$$\left.\begin{aligned} &\dot{\boldsymbol{\sigma}}_D = {}^4\boldsymbol{C} : \left\{ \dot{\boldsymbol{\Lambda}}_n - \dot{\lambda}\boldsymbol{a} \right\} \\ &3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \dot{\boldsymbol{\Lambda}}_n - \dot{\lambda} \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \right) = 0 \end{aligned}\right\}$$

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations are integrated using an implicit Euler integration scheme.

The derivative of the incremental logarithmic strain tensor is the end-increment value divided by the time increment, because $\boldsymbol{\Lambda}_n(t_n) = \boldsymbol{O}$.

$$\left.\begin{aligned} &\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\boldsymbol{C} : (\boldsymbol{\Lambda}_n - \Delta\lambda\boldsymbol{a}) \\ &3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \right) = 0 \end{aligned}\right\}$$

3.4.2 Iterative stress update

The set of coupled nonlinear equations is solved iteratively following a Newton-Raphson procedure. The derivative of $\boldsymbol{a}$ is :

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\sigma}_D} = \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\sigma}_D^d} : \frac{\partial \boldsymbol{\sigma}_D^d}{\partial \boldsymbol{\sigma}_D} = \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\sigma}_D^d} : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\,\boldsymbol{II} \right) = 3\,{}^4\boldsymbol{I}$$

From the coupled set of iterative equations $\delta\boldsymbol{\sigma}_D$ and $\delta\lambda$ leading to new values of $\boldsymbol{\sigma}_D$ and $\Delta\lambda$. The iteration process is stopped when the residuals $\boldsymbol{s}_1$ and s_2 are small enough.

For a plane stress situation, the deformation tensor must be adapted during the stress update procedure. This implies that the elastic trial stress will change as well. Excluding plane stress situations, the elastic trial stress is constant in the stress update procedure, so $\delta\boldsymbol{\sigma}_{D_e} = \boldsymbol{O}$.

$$\left.\begin{aligned} {}^4\boldsymbol{R} : \delta\boldsymbol{\sigma}_D + \boldsymbol{t}\delta\lambda &= -\boldsymbol{s}_1 \\ \boldsymbol{u} : \delta\boldsymbol{\sigma}_D + v\delta\lambda &= -s_2 \end{aligned}\right\}$$

$$\begin{aligned}
{}^4\boldsymbol{R} &= {}^4\boldsymbol{I} + 3\Delta\lambda\,{}^4\boldsymbol{C} : {}^4\boldsymbol{I} \\
\boldsymbol{t} &= {}^4\boldsymbol{C} : \boldsymbol{a} \\
\boldsymbol{u} &= \left(3\,{}^4\boldsymbol{C} - \boldsymbol{II} : {}^4\boldsymbol{C}\right) : \boldsymbol{\Lambda}_n - \Delta\lambda \left\{ \left(3\,{}^4\boldsymbol{C} - \boldsymbol{II} : {}^4\boldsymbol{C}\right) : \boldsymbol{a} + 4\sigma_y H \left(\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}\right)^{-\frac{1}{2}} \boldsymbol{a} : {}^4\boldsymbol{I} \right\} \\
v &= 3\,{}^4\boldsymbol{C} : \boldsymbol{a} : \boldsymbol{\sigma}_D^d + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} \\
\boldsymbol{s}_1 &= \boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a} \\
s_2 &= 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right)
\end{aligned}$$

3.4.3 Stiffness

To evaluate the iterative Updated Lagrange weighted residual equation not only the Cauchy stress $\boldsymbol{\sigma}$, but also the relation between the stress variation $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c$ has to be known, i.e. $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

The consistent stiffness tensor ${}^4\boldsymbol{M}$, eventually leads to the consistent stiffness matrix. It must be derived from the coupled nonlinear equations for $\boldsymbol{\sigma}$ and $\Delta\lambda$. Iterative changes (variations) of $\delta\boldsymbol{\sigma}$ and $\delta\lambda$ can be derived.

To simplify notation we omit again the upper index i, which indicates the iteration step number.

$$\left.\begin{aligned} &\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_D(t_n) - {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + \Delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a} = 0 \\ &3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n - \Delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right) = 0 \end{aligned}\right\}$$

$$\left.\begin{aligned}
&\delta\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_D(t_n) + {}^4\boldsymbol{C} : \delta\boldsymbol{\Lambda}_n - \delta\lambda\,{}^4\boldsymbol{C} : \boldsymbol{a} - \Delta\lambda\,{}^4\boldsymbol{C} : \delta\boldsymbol{a} = 0 \\
&3\delta\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{\Lambda}_n + 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \delta\boldsymbol{\Lambda}_n - \\
&\quad \delta\lambda \left(3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 2\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}}\right) - \\
&\quad \Delta\lambda \left(3\delta\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \boldsymbol{a} + 3\boldsymbol{\sigma}_D^d : {}^4\boldsymbol{C} : \delta\boldsymbol{a} + \right. \\
&\qquad\qquad \left. 2\delta\sigma_y H \sqrt{\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}} + 2\sigma_y H \tfrac{1}{2}[\tfrac{2}{3}\boldsymbol{a} : \boldsymbol{a}]^{-1/2} \tfrac{4}{3}\boldsymbol{a} : \delta\boldsymbol{a}\right) = 0
\end{aligned}\right\}$$

4 Linear viscoelastic material behavior

The modeling of linear viscoelastic material behavior is based on the principles of superposition and proportionality. Current stress and strain are given by a Boltzmann integral over the strain or stress history. Fourth-order relaxation (${}^4\boldsymbol{C}$) and creep (${}^4\boldsymbol{S}$) tensors relate stress to strain and vice versa.

Experiments show that long past history has less impact on the current stress than recent history. This fading memory property motivates the use of Prony series for ${}^4\boldsymbol{C}$ and ${}^4\boldsymbol{S}$. In the one-dimensional case they represent the behavior of discrete spring-dashpot models.

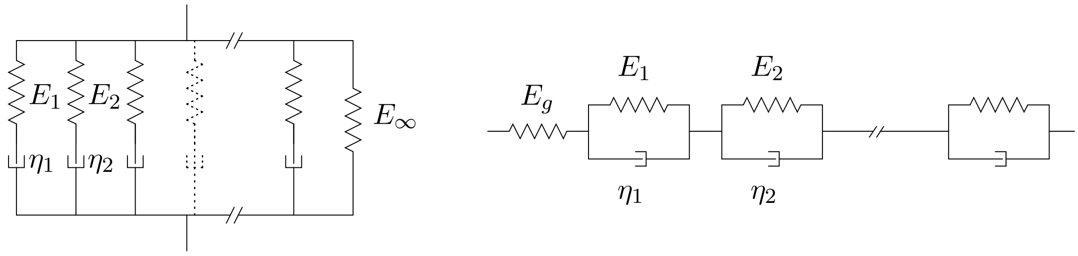


Fig. 4.21 : *Generalized Maxwell and Kelvin model*

$$\boldsymbol{\sigma}(t) = \int_{\tau=0}^{t} {}^4\boldsymbol{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \qquad ; \qquad \boldsymbol{\varepsilon}(t) = \int_{\tau=0}^{t} {}^4\boldsymbol{S}(t-\tau) : \dot{\boldsymbol{\sigma}}(\tau)\, d\tau$$

$${}^4\boldsymbol{C}(t) = {}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i e^{-\frac{t}{\tau_i}} \qquad ; \qquad {}^4\boldsymbol{S}(t) = {}^4\boldsymbol{S}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{S}_i \left\{1 - e^{-\frac{t}{\tau_i}}\right\}$$

4.1 Constitutive model

We now focus attention on the calculation of the current stress $\boldsymbol{\sigma}(t)$, because this is of importance in a numerical procedure like the finite element method. The hereditary integral is evaluated after substitution of the Prony series for ${}^4\boldsymbol{C}(t)$.

Using the Prony series expression for ${}^4\boldsymbol{C}(t)$ and assuming the initial strain to be zero ($\boldsymbol{\varepsilon}(\tau = 0) = \boldsymbol{O}$), an expression for $\boldsymbol{\sigma}(t)$ can be derived.

$$\left.\begin{aligned} \boldsymbol{\sigma}(t) &= \int_{\tau=0}^{t} {}^4\boldsymbol{C}(t-\tau) : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\ {}^4\boldsymbol{C}(t) &= {}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i\, e^{-\frac{t}{\tau_i}} \end{aligned}\right\} \quad \rightarrow$$

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \int_{\tau=0}^{t} \left[{}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i e^{-\frac{t-\tau}{\tau_i}} \right] : \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau = {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i : \int_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}}\, \dot{\boldsymbol{\varepsilon}}(\tau)\, d\tau \\ &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \end{aligned}$$

4.2 Incremental analysis

It is immediately clear that calculation of the stress involves the evaluation of a (large) number of integrals over the complete time history. For this reason the deformation time period is subdivided into a discrete number of time increments.

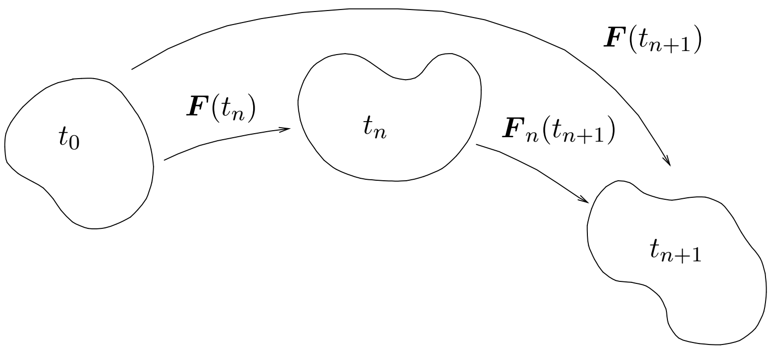


Fig. 4.22 : *Incremental deformation*

In the numerical analysis of the time dependent behavior, the total time interval $[0, t]$ is discretized :

$$[0, t] \quad \rightarrow \quad [t_1 = 0, t_2, t_3, .., t_n, t_{n+1} = t]$$

The timespan between two discrete moments in the time interval is a time increment. It is assumed that these increments are of equal length.

$$\Delta t = t_{i+1} - t_i \qquad ; \quad i = 1, ..., n$$

It is assumed that the strain is a linear function of time in each time increment.

$$\boldsymbol{\varepsilon}(\tau) = \boldsymbol{\varepsilon}(t_n) + (\tau - t_n)\,\frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \quad \rightarrow \quad \dot{\boldsymbol{\varepsilon}}(\tau) = \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t}$$

Stress update

The hereditary integral is split in an integral over $[0, t_n]$ and an integral over the last or current increment $[t_n, t_{n+1} = t]$. Here we consider only the i-th term of the series : $\boldsymbol{\sigma}_i(t)$.

$$\begin{aligned}
\boldsymbol{\sigma}_i(t) &= {}^4\boldsymbol{C}_i : \int_{\tau=0}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau = {}^4\boldsymbol{C}_i : \left[\int_{\tau=0}^{t_n} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau + \int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau\right] \\
&= {}^4\boldsymbol{C}_i : \left[e^{-\frac{\Delta t}{\tau_i}} \int_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau + \int_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}}\,\dot{\boldsymbol{\varepsilon}}(\tau)\,d\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\frac{\Delta t}{\tau_i}} \; {}^4\boldsymbol{C}_i : \int\limits_{\tau=0}^{t_n} e^{-\frac{t_n-\tau}{\tau_i}} \, \dot{\boldsymbol{\varepsilon}}(\tau) \, d\tau + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \, \dot{\boldsymbol{\varepsilon}}(\tau) \, d\tau \\
&= e^{-\frac{\Delta t}{\tau_i}} \, \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \, \dot{\boldsymbol{\varepsilon}}(\tau) \, d\tau
\end{aligned}
$$

The stress $\boldsymbol{\sigma}_i(t_n)$ is known from the previous increment. Calculation of $\Delta\boldsymbol{\sigma}_i(t)$ can be done analytically because it has been assumed that the strain is a linear function of time in each time increment.

$$
\begin{aligned}
\sigma_i(t) &= e^{-\frac{\Delta t}{\tau_i}} \, \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \, \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \, d\tau = e^{-\frac{\Delta t}{\tau_i}} \, \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \int\limits_{\tau=t_n}^{t} e^{-\frac{t-\tau}{\tau_i}} \, d\tau \, \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \\
&= e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t}
\end{aligned}
$$

Calculating the current stress does not mean that the Boltzmann integral has to be evaluated over the total deformation history. When results are stored properly we can easily update the stress $\boldsymbol{\sigma}(t)$.

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\
&= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta\boldsymbol{\varepsilon}}{\Delta t} \right]
\end{aligned}
$$

4.2.1 Stiffness

The variation of $\boldsymbol{\sigma}(t)$ results in the consistent material stiffness tensor.

$$
\begin{aligned}
\delta\boldsymbol{\sigma} &= \left[{}^4\boldsymbol{C}_\infty + \sum_{i=1}^{N} {}^4\boldsymbol{C}_i \, \frac{\tau_i}{\Delta t} \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \right] : \delta\boldsymbol{\varepsilon} \\
&= {}^4\boldsymbol{M} : \delta\boldsymbol{\varepsilon}
\end{aligned}
$$

4.3 Isotropic material

For an isotropic material the mechanical behavior is the same in each material direction and is characterized by two material parameters, the Lamé coefficients λ and μ. The elastic stiffness tensor ${}^4\boldsymbol{C}$ can then be written as :

$$
{}^4\boldsymbol{C} = \lambda \boldsymbol{II} + 2\mu \, {}^4\boldsymbol{I}^s
$$

where the fourth-order tensors $\boldsymbol{II}$ and ${}^4\boldsymbol{I}^s$ have the following index equivalents :

$$
\begin{aligned}
&\boldsymbol{II} \quad \rightarrow \quad \delta_{ij}\delta_{kl} \\
&2\,{}^4\boldsymbol{I}^s = {}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc} \quad \rightarrow \quad \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}
\end{aligned}
$$

Using the above expression for ${}^4\boldsymbol{C}$ the hydrostatic and deviatoric parts of the stress tensor can be decoupled and expressed in the hydrostatic and deviatoric strain tensor, respectively.

Instead of the Lamé coefficients other elastic material parameters are often used : Young's modulus E, Poisson's ratio ν, shear modulus G and bulk modulus K. These parameters are related as only two independent material parameters exist.

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \\
&= \left[\lambda \boldsymbol{II} + 2\mu\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} = \left[\lambda \boldsymbol{II} + \mu\left({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}\right)\right] : \boldsymbol{\varepsilon} = \lambda \boldsymbol{I}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + 2\mu\,\boldsymbol{\varepsilon} \\
&= (3\lambda + 2\mu)\,\tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon}^d = (3\lambda + 2\mu)\,\boldsymbol{\varepsilon}^h + 2\mu\,\boldsymbol{\varepsilon}^d = 3K\,\boldsymbol{\varepsilon}^h + 2G\,\boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
K = \tfrac{1}{3}\,(3\lambda + 2\mu) = \frac{E}{3(1-2\nu)} \quad ; \quad \mu = G = \frac{E}{2(1+\nu)} \quad ; \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}
$$

For a viscoelastic isotropic material the stress tensor is also split into an hydrostatic and a deviatoric part. In analogy with the elastic model, time dependent bulk and shear moduli are used, which are expressed in a Prony series.

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \boldsymbol{\sigma}^h(t) + \boldsymbol{\sigma}^d(t) \\
&= 3 \int\limits_{\tau=0}^{t} K(t-\tau)\,\frac{d}{d\tau}\left\{\boldsymbol{\varepsilon}^h(\tau)\right\}\,d\tau + 2 \int\limits_{\tau=0}^{t} G(t-\tau)\,\frac{d}{d\tau}\left\{\boldsymbol{\varepsilon}^d(\tau)\right\}\,d\tau \\
K(t) &= K_\infty + \sum_{i=1}^{n} K_i\,e^{-\frac{t}{\tau_i}} = \frac{1}{3(1-2\nu)}\left[E_\infty + \sum_{i=1}^{n} E_i\,e^{-\frac{t}{\tau_i}}\right] \\
G(t) &= G_\infty + \sum_{i=1}^{n} G_i\,e^{-\frac{t}{\tau_i}} = \frac{1}{2(1+\nu)}\left[E_\infty + \sum_{i=1}^{n} E_i\,e^{-\frac{t}{\tau_i}}\right]
\end{aligned}
$$

4.3.1 Stress update

Discretising the total time interval $[0, t]$ in equal time increments $\Delta t = t_{i+1} - t_i;\ i = 1..n$ allows an efficient calculation of the stress where an integral only has to be evaluated over the current increment, which moreover can be done rather straightforwardly when it is assumed that the incremental strain rate is constant (= linear incremental strain).

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \boldsymbol{\sigma}_i(t) \\
&= {}^4\boldsymbol{C}_\infty : \boldsymbol{\varepsilon}(t) + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}}\boldsymbol{\sigma}_i(t_n) + {}^4\boldsymbol{C}_i : \tau_i \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right) \frac{\Delta \boldsymbol{\varepsilon}}{\Delta t}\right] \\
&= 3K_\infty\,\Delta\boldsymbol{\varepsilon}^h + 2G_\infty\,\Delta\boldsymbol{\varepsilon}^d + \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}}\boldsymbol{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i\Delta\boldsymbol{\varepsilon}^h + 2G_i\Delta\boldsymbol{\varepsilon}^d\right\}\right]
\end{aligned}
$$

4.3.2 Stiffness

The relation between a small change in stress and a small change in strain is straightforwardly derived from the incremental stress relation.

$$\delta\boldsymbol{\sigma} = 3K_\infty \delta\boldsymbol{\varepsilon}^h + 2G_\infty \delta\boldsymbol{\varepsilon}^d + \sum_{i=1}^{N} \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i \delta\boldsymbol{\varepsilon}^h + 2G_i \delta\boldsymbol{\varepsilon}^d\right\}$$

Matrix/column notation

The relation between the incremental stress and strain tensor can be written in indices with relation to a vector basis. Components can then be stored in columns and matrices. For a two-dimensional deformation the following columns for stress and strain components are defined :

$$\underset{\approx}{\sigma}^T = \left[\begin{array}{ccccc} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{12} & \sigma_{21} \end{array}\right] \quad ; \quad \underset{\approx}{\varepsilon}^T = \left[\begin{array}{ccccc} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \varepsilon_{12} & \varepsilon_{21} \end{array}\right]$$

Hydrostatic and deviatoric stress/strain components can be related to total stress/strain components with the following matrices :

$$\underline{\underline{A}}^h = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{\underline{A}}^d = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

resulting in : $\quad \Delta\underset{\approx}{\varepsilon}^h = \underline{\underline{A}}^h \, \Delta\underset{\approx}{\varepsilon} \quad ; \quad \Delta\underset{\approx}{\varepsilon}^d = \underline{\underline{A}}^d \, \Delta\underset{\approx}{\varepsilon}$

The stress column can then be rewritten.

$$\begin{aligned} \underset{\approx}{\sigma}(t) = {} & \left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \Delta\underset{\approx}{\varepsilon} + \\ & \sum_{i=1}^{N} \left[e^{-\frac{\Delta t}{\tau_i}} \underset{\approx}{\sigma}_i(t_n) + \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right\}\right] \Delta\underset{\approx}{\varepsilon} \end{aligned}$$

$$\begin{aligned} \delta\underset{\approx}{\sigma}(t) = {} & \left[\left(3K_\infty \underline{\underline{A}}^h + 2G_\infty \underline{\underline{A}}^d\right) \delta\underset{\approx}{\varepsilon} + \right. \\ & \left. \sum_{i=1}^{N} \frac{\tau_i}{\Delta t}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left(3K_i \underline{\underline{A}}^h + 2G_i \underline{\underline{A}}^d\right)\right] \delta\underset{\approx}{\varepsilon} \end{aligned}$$

4.3.3 Initial stiffness formulation

Some implementations of the linear viscoelastic model (e.g. MARC) are formulated in such a way that the initial moduli K_0 and G_0 are required. The initial moduli are defined as

$$K_0 = K_\infty + \sum_{i=1}^{N} K_i \qquad ; \qquad G_0 = G_\infty + \sum_{i=1}^{N} G_i$$

The relation for stress increment and stress variation can be derived easily.

$$\Delta\boldsymbol{\sigma}(t) = 3K_0\Delta\boldsymbol{\varepsilon}^h + 2G_0\Delta\boldsymbol{\varepsilon}^d - \sum_{i=1}^{N}\left[1 - \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\Delta\boldsymbol{\varepsilon}^h + 2G_i\Delta\boldsymbol{\varepsilon}^d\right\} -$$

$$\sum_{i=1}^{N}\left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\left\{\boldsymbol{\sigma}_i^h(t_n) + \boldsymbol{\sigma}_i^d(t_n)\right\}$$

$$\delta\boldsymbol{\sigma} = 3K_0\delta\boldsymbol{\varepsilon}^h + 2G_0\delta\boldsymbol{\varepsilon}^d - \sum_{i=1}^{N}\left[1 - \left(1 - e^{-\frac{\Delta t}{\tau_i}}\right)\frac{\tau_i}{\Delta t}\right]\left\{3K_i\delta\boldsymbol{\varepsilon}^h + 2G_i\delta\boldsymbol{\varepsilon}^d\right\}$$

4.4 Example

An axial strain step with amplitude 0.01 is prescribed on an axisymmetric tensile bar with initial cross-sectional area $A_0 = 10$ mm^2. The stress response is calculated for a 12-mode generalized Maxwell model. The modal parameters are listed in the table.

	E [MPa]	τ [s]		E [MPa]	τ [s]
1	3.0e6	3.1e-8	2	1.4e6	3.0e-7
3	3.9e6	3.0e-6	4	5.4e6	2.9e-5
5	1.3e6	2.8e-4	6	2.3e5	2.7e-3
7	7.6e4	2.6e-2	8	3.7e4	2.5e-1
9	3.3e4	2.5e+0	10	1.7e4	2.4e+1
11	8.0e3	2.3e+2	12	1.2e4	2.2e+3

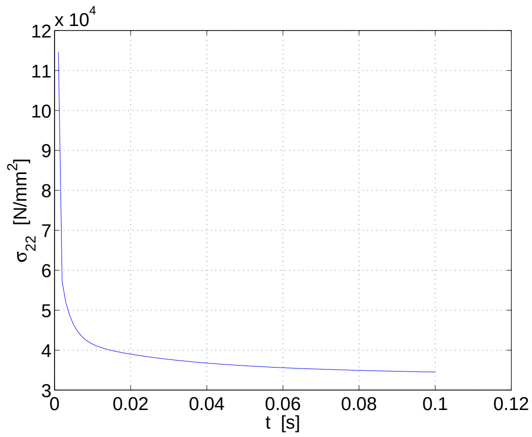


Fig. 4.23 : *Tensile stress versus time for axisymmetric element*

5 Viscoplastic material behavior

The one-dimensional mechanical representation of the elastoviscoplastic Perzyna model consists of a spring in series with a friction slider, a hardening spring and a linear viscous dashpot. The series-spring represents the elastic part of the material response. The viscoplastic response, represented by the hardening spring and the viscous dashpot, becomes manifest as soon as the friction slider "opens" when the stress σ exceeds a characteristic value, the yield stress σ_y.

After yielding, the total strain rate $\dot{\varepsilon}$ is the sum of the elastic strain rate $\dot{\varepsilon}_e$ and the viscoplastic strain rate $\dot{\varepsilon}_{vp}$. It is only for small strains that we can add strains.

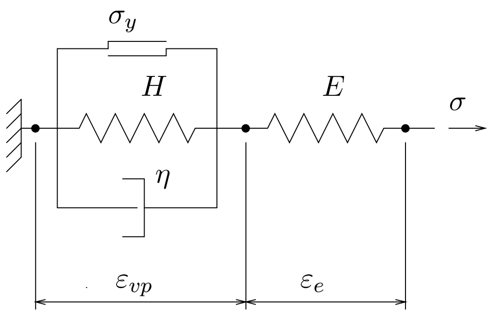


Fig. 5.24 : *Discrete model for viscoplastic material behavior*

5.1 Kinematics

Transformation from the undeformed configuration at time t_0 (position vector $\vec{x}_0$) to the current configuration at time t (position vector $\vec{x}$) is described by the deformation tensor $\boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c$, where $\vec{\nabla}_0$ is the gradient operator with respect to the undeformed state.

The right and left Cauchy-Green strain tensors, $\boldsymbol{C}$ and $\boldsymbol{B}$, are functions of $\boldsymbol{F}$ as is the Green-Lagrange strain tensor $\boldsymbol{E}$. Material velocity is taken into account by the deformation and rotation rate tensors $\boldsymbol{D}$ and $\boldsymbol{\Omega}$, the symmetric and skew-symmetric parts of the velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla} \vec{v})^c$, where $\vec{\nabla}$ is the gradient operator with respect to the current state and $\vec{v}$ is the velocity of the material volume.

In Perzyna's model the total deformation $\boldsymbol{F}$ is decomposed multiplicatively into an elastic and a viscoplastic contribution. Regarding the kinematics, this implies the introduction of elastic and viscoplastic (rate) tensors.

To make the decomposition unique it is commonly assumed that the viscoplastic rotation rate is zero, i.e. $\boldsymbol{\Omega}_p = \boldsymbol{O}$. Superimposed material rotations are thus fully represented in $\boldsymbol{F}_e$.

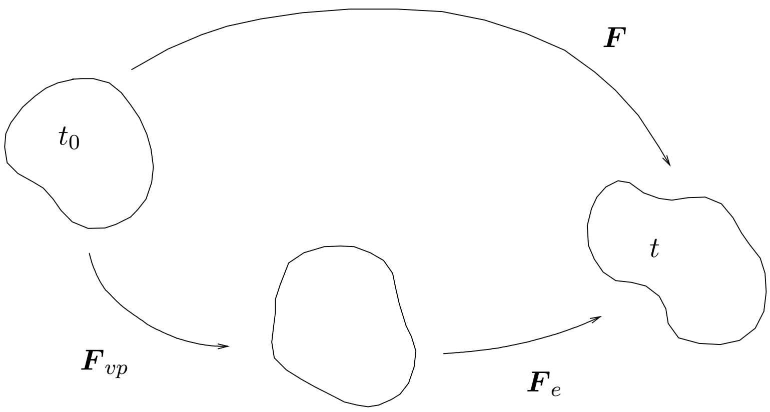


Fig. 5.25 : *Multiplicative decomposition of total deformation*

$$
\begin{aligned}
\boldsymbol{F} &= (\vec{\nabla}_0 \vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_{vp} \\
\boldsymbol{C} &= \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \qquad ; \qquad \boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \\
\boldsymbol{L} &= \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla} \vec{v})^c \\
&= \boldsymbol{L}_e + \boldsymbol{L}_{vp} = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_{vp} + \boldsymbol{\Omega}_{vp}) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_{vp}
\end{aligned}
$$

5.2 Constitutive relations

Elastic deformation

The stress is related to the elastic strain. Because we want to describe large elastic strains, the elastic behavior must be described with a hyper-elastic model. In that case it is assumed that an elastic strain energy function exists, which can be used to calculate the stress. The 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$ is related to the Green-Lagrange strain tensor $\boldsymbol{E}$. The current stress state is characterized by the Kirchhoff stress $\boldsymbol{\tau} = \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c$.

An elastic energy function is chosen, which characterizes isotropic, compressible material behavior. The fourth-order material tensor is completely determined by the volume ratio $J = \det(\boldsymbol{F})$ and by the constant Lamé coefficients λ and μ, which are related to Young's modulus and Poisson's ratio.

$$
\left.
\begin{aligned}
&\boldsymbol{P} = \frac{\partial W(\boldsymbol{E}_e)}{\partial \boldsymbol{E}_e} = 2\,\frac{\partial W}{\partial \boldsymbol{C}_e} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-c} \quad \rightarrow \quad \dot{\boldsymbol{P}} = 2\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \dot{\boldsymbol{C}} \\
&\\
&W(\lambda_1, \lambda_2, \lambda_3) = \tfrac{1}{2}\,\mu\left\{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\ln(J)\right\} + \tfrac{1}{2}\,\lambda\,\{\ln(J)\}^2 \\
&\text{with} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad ; \qquad \mu = \frac{E}{2(1+\nu)}
\end{aligned}
\right\}
$$

Yield criterion and hardening

A yield function F is used to evaluate the stress state and to check whether the deformation is purely elastic ($F < 0$) or viscoplastic ($F \geq 0$). The current stress state, represented by the equivalent or effective Kirchhoff stress $\bar{\tau}$, is compared to the current yield stress τ_y, which increases from its initial value τ_{y0} due to plastic deformation and is therefore related to the effective viscoplastic strain $\bar{\varepsilon}_{vp}$. The relation between τ_y and $\bar{\varepsilon}_{vp}$ is described by the hardening rule. To decide whether elastic or viscoplastic deformation occurs, the Kuhn-Tucker relations are used.

yield criterion	$F = \bar{\tau} - \tau_y(\bar{\varepsilon}_{vp})$		
effective viscoplastic strain	$\bar{\varepsilon}_{vp} = \int_{\tau=0}^{t} \dot{\bar{\varepsilon}}_{vp} \, d\tau$		
hardening law	$\tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp})$	with	$\dfrac{\partial \tau_y}{\partial \bar{\varepsilon}_p} = H(\bar{\varepsilon}_p)$
Kuhn-Tucker relations	$F < 0$	$\rightarrow$	elastic deformation
	$F \geq 0$	$\rightarrow$	viscoplastic deformation

5.2.1 Von Mises plasticity

For the Von Mises yield criterion, the yield surface is a circular cylinder in principal stress space. The equivalent Von Mises stress can be expressed in the deviatoric stress tensor $\boldsymbol{\tau}^d$. It is required that the disspated viscoplastic energy per unit of time is the product of the equivalent stress and the effective or equivalent plastic strain rate :

$$\boldsymbol{\tau} : \boldsymbol{D}_{vp} = \bar{\tau} \, \dot{\bar{\varepsilon}}_{vp}$$

which leads to the definition of the effective plastic strain rate as a function of the viscoplastic deformation rate tensor $\boldsymbol{D}_{vp}$.

$$\bar{\tau} = \sqrt{\tfrac{3}{2} \boldsymbol{\tau}^d : \boldsymbol{\tau}^d}$$

$$\dot{\bar{\varepsilon}}_{vp} = \sqrt{\tfrac{2}{3} \, \boldsymbol{D}_{vp} : \boldsymbol{D}_{vp}}$$

$$F = \sqrt{\tfrac{3}{2} \boldsymbol{\tau}^d : \boldsymbol{\tau}^d} - \tau_y(\bar{\varepsilon}_{vp})$$

Viscoplastic deformation

During viscoplastic deformation the *direction* of the viscoplastic strain rate is defined by the commonly used *normality* or *associative* flow rule : the viscoplastic strain rate is directed normal to the yield surface in stress space. The *length* of $\boldsymbol{D}_{vp}$ is characterized by the rate of the viscoplastic multiplier $\dot{\lambda}$. The normal to the yield surface can be expressed in the deviatoric stress $\boldsymbol{\tau}^d$. The time-derivative of $\boldsymbol{C}_{vp}$ can be related to $\boldsymbol{D}_{vp}$.

In contrast to elastoplastic models, stress states outside the yield surface can exist, which explains why these viscoplastic models are often called *over-stress models*.

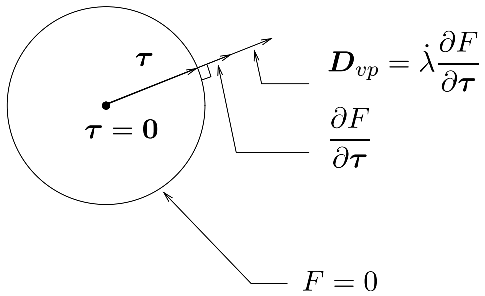


Fig. 5.26 : *Associative flow rule*

$$
\begin{aligned}
\boldsymbol{D}_{vp} &= \dot{\lambda}\,\frac{\partial F}{\partial \boldsymbol{\tau}} = \dot{\lambda}\boldsymbol{a} \quad \rightarrow \quad \dot{\boldsymbol{C}}_{vp} = 2\,\boldsymbol{F}^c \cdot \boldsymbol{D}_{vp} \cdot \boldsymbol{F} = 2\dot{\lambda}\,\boldsymbol{F}^c \cdot \boldsymbol{a} \cdot \boldsymbol{F} \\
\boldsymbol{a} &= \frac{\partial F}{\partial \boldsymbol{\tau}^d} : \frac{\partial \boldsymbol{\tau}^d}{\partial \boldsymbol{\tau}} = \left[\tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d : {}^4\boldsymbol{I}\right] : \left[\frac{\partial}{\partial \boldsymbol{\tau}}\left\{\boldsymbol{\tau} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\}\right] \\
&= \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) = \tfrac{3}{2}\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{-1/2} \boldsymbol{\tau}^d = \tfrac{3}{2}\,\frac{1}{\bar{\tau}}\,\boldsymbol{\tau}^d \\
\dot{\bar{\varepsilon}}_{vp} &= \dot{\lambda}\sqrt{\tfrac{2}{3}\,\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

5.3 Constitutive model

The material model can be summarized as a set of constitutive equations. During viscoplastic deformation the rate of the viscoplastic multiplier is related to an over-stress function $\phi(F)$ by a *fluidity parameter* γ and a *rate-sensitivity parameter* N, which has to satisfy $N \geq 1$ to make $\phi(F)$ convex.

The set of equations must be solved to determine the stress when (an approximation of) the deformation is known. From the same set of equations a relation between the variation of stress and deformation is derived.

$$
\begin{aligned}
F < 0 \quad &\rightarrow \quad \dot{\boldsymbol{C}} = \dot{\boldsymbol{C}}_e \\
&\dot{\boldsymbol{P}} = 2\,\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \dot{\boldsymbol{C}} \qquad ; \qquad \dot{\boldsymbol{C}}_{vp} = \boldsymbol{O} \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = 0
\end{aligned}
$$

$$
\begin{aligned}
F \geq 0 \quad &\rightarrow \quad \dot{\boldsymbol{C}} = \dot{\boldsymbol{C}}_e + \dot{\boldsymbol{C}}_{vp} \quad \rightarrow \\
&\left.\begin{aligned}
\dot{\boldsymbol{P}} &= 2\,\frac{\partial^2 W}{\partial \boldsymbol{C}^2} : \left(\dot{\boldsymbol{C}} - 2\,\boldsymbol{F}^c \cdot \dot{\lambda}\boldsymbol{a} \cdot \boldsymbol{F}\right) \\
\dot{\lambda} &= \gamma\phi(F) = \gamma\left(\frac{F}{\tau_{y0}}\right)^N
\end{aligned}\right\} \\
&\tau_y = \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}) \qquad ; \qquad \dot{\bar{\varepsilon}}_{vp} = \dot{\lambda}\sqrt{\tfrac{2}{3}\,\boldsymbol{a} : \boldsymbol{a}}
\end{aligned}
$$

5.4 Incremental analysis

The figure shows the relevant configurations in a large strain viscoplastic deformation process.

Starting from the undeformed configuration at time t_0 the external load is employed and the deformation leads to the current configuration at time t. During a numerical analysis of this deformation process the state of the material is determined at a finite number of discrete moments $t_i,\ i = 0, 1, .., n+1$. The period between two subsequent moments is an increment : $\Delta t_i = t_{i+1} - t_i$. The increments are assumed to be of equal length.

It is assumed that the analysis has brought us to $t = t_n$, the beginning of the current increment and that all relevant variables are known and satisfying all governing equations (balance laws, boundary conditions, constitutive relations). The state at the current time $t = t_{n+1}$, the end of the current increment has to be determined.

The transformation during the current increment is described by the deformation tensor $\boldsymbol{F}_n(\tau)$, where τ indicates a moment in time during the last (= current) increment : $t_n \leq \tau \leq t_{n+1}$.

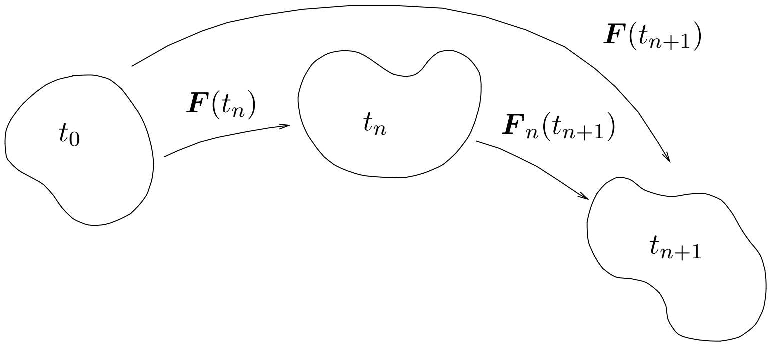


Fig. 5.27 : *Incremental deformation*

$$\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n)$$
$$\boldsymbol{F}_n = (\vec{\nabla}_n \vec{x})^c = \boldsymbol{R}_n \cdot \boldsymbol{U}_n \quad ; \quad J_n = \det(\boldsymbol{F}_n) \quad ; \quad \vec{\nabla} = \boldsymbol{F}_n^{-c} \cdot \vec{\nabla}_n$$

$$\boldsymbol{D} = \tfrac{1}{2}\left\{(\vec{\nabla}\vec{v})^c + (\vec{\nabla}\vec{v})\right\} = \tfrac{1}{2}\left(\dot{\boldsymbol{F}}_n \cdot \boldsymbol{F}_n^{-1} + \boldsymbol{F}_n^{-c} \cdot \dot{\boldsymbol{F}}_n^c\right)$$

5.4.1 Elastic stress predictor

The first step in evaluating the end-increment stress is the calculation of the elastic stress predictor. As a first assumption the current increment is taken to be purely elastic, so $\Delta\lambda = 0$. The elastic trial stress is used to evaluate the yield condition and to see if the assumption of elastic deformation holds. There are two possibilities :

1. the increment is indeed fully elastic,

2. the yield criterion is violated which implies that during the increment further elastoviscoplastic deformation has taken place.

elastic trial stress
$$\boldsymbol{P}_e = \boldsymbol{P}_n + 2\,\frac{\partial^2 W}{\partial \boldsymbol{G}^2} : (\boldsymbol{C} - \boldsymbol{C}(t_n)) \quad \rightarrow \quad \boldsymbol{\tau}_e = \boldsymbol{F} \cdot \boldsymbol{P}_e \cdot \boldsymbol{F}^c$$

yield criterion
$$F = \sqrt{\tfrac{3}{2}\,(\boldsymbol{\tau}_e)^d : (\boldsymbol{\tau}_e)^d} - \tau_y(\tau_{y0}, \bar{\varepsilon}_{vp}(t_n))$$

$$\begin{array}{lll} F < 0 & \rightarrow & \text{elastic increment} \\ F \geq 0 & \rightarrow & \text{elastoviscoplastic increment} \end{array}$$

matrix/column notation

$$\underset{\approx}{\tau}_e = \underset{\approx}{A} + \underline{\underline{H}}_c\, \underset{\approx}{e_n}$$

$$F = \sqrt{\tfrac{3}{2}\left(\underset{\approx}{\tau}_e\right)^T \left(\underset{\approx}{\tau}_e\right)_t} - \zeta(\kappa)$$

$$\text{with} \quad \left\{ \begin{array}{l} \underline{\underline{H}} = 2\,\{\mu - \lambda \ln(J)\}\,\underline{\underline{I}} + \lambda \underset{\approx}{I}\underset{\approx}{I}^T \\ \underline{e}_n = \tfrac{1}{2}\left(\underline{I} - \underline{F}_n^{-T}\underline{F}_n^{-1}\right) \rightarrow \underset{\approx}{e_n} \\ \underline{A} = \underline{F}_n\, \underline{\tau}(t_n)\underline{F}_n^T \rightarrow \underset{\approx}{A} \end{array} \right.$$

Elastic increment

When it is concluded that the current increment is purely elastic, the end-increment or current stress equals the calculated elastic trial stress. Viscoplastic strain does not need updating and is thus also known.

$$\begin{array}{lll} \boldsymbol{\tau}(t_{n+1}) = \boldsymbol{\tau}_e & ; & \Delta\lambda = 0 \\ \bar{\varepsilon}_{vp}(t_{n+1}) = \bar{\varepsilon}_{vp}(t_n) & ; & \tau_y(t_{n+1}) = \tau_y(t_n) \end{array}$$

5.4.2 Viscoplastic increment

During the increment $\Delta t = t_{n+1} - t_n$ the stress evolution equations are integrated using an implicit Euler integration scheme.

$$\left. \begin{array}{l} \dot{\boldsymbol{P}} = 2\,\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : (\dot{\boldsymbol{C}} - 2\,\boldsymbol{F}^c \cdot \dot{\lambda}\boldsymbol{a} \cdot \boldsymbol{F}) \\ \dot{\lambda} = \gamma\phi(F) = \gamma\left(\dfrac{F}{\tau_{y0}}\right)^N \end{array} \right\}$$

$$\left. \begin{array}{l} \boldsymbol{P} = \boldsymbol{P}(t_n) + 2\,\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \{\boldsymbol{C} - \boldsymbol{C}(t_n) - 2\,\boldsymbol{F}^c \cdot \Delta\lambda\,\boldsymbol{a} \cdot \boldsymbol{F}\} \\ \Delta\lambda = \Delta t \gamma \phi(F) \end{array} \right\}$$

The current or end-increment time $t = t_{n+1}$ is not indicated further. Constitutive equations are reformulated in the Kirchhoff stress tensor $\boldsymbol{\tau}$, using $\boldsymbol{\tau} = \boldsymbol{F} \cdot \boldsymbol{P} \cdot \boldsymbol{F}^c$, the incremental deformation tensor $\boldsymbol{F}_n$ and the Almansi strain tensor $\boldsymbol{e}_n$. A fourth-order elastic material tensor ${}^4\boldsymbol{H}$ is introduced and can be calculated for the Neo-Hookean elastic energy function W.

$$\left.\begin{array}{l} \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-c} = \boldsymbol{F}^{-1}(t_n) \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}^{-c}(t_n) + 2\dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \{\boldsymbol{C} - \boldsymbol{C}(t_n) - 2\,\boldsymbol{F}^c \cdot \Delta\lambda\,\boldsymbol{a} \cdot \boldsymbol{F}\} \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

$$\boldsymbol{F}_n = \boldsymbol{F} \cdot \boldsymbol{F}^{-1}(t_n) \quad \rightarrow \quad \boldsymbol{C} - \boldsymbol{C}(t_n) = \boldsymbol{F}^c \cdot (\boldsymbol{I} - \boldsymbol{F}_n^{-c} \cdot \boldsymbol{F}_n^{-1}) \cdot \boldsymbol{F} = 2\,\boldsymbol{F}^c \cdot \boldsymbol{e}_n \cdot \boldsymbol{F}$$

$$\left.\begin{array}{l} \boldsymbol{\tau} = \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + 4\boldsymbol{F} \cdot \dfrac{\partial^2 W}{\partial \boldsymbol{C}^2} : \boldsymbol{F}^c \cdot \left(\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}\right) \cdot \boldsymbol{F} \cdot \boldsymbol{F}^c \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

$${}^4\boldsymbol{H} = 4\,\boldsymbol{F} \cdot \left(\boldsymbol{F} \cdot \frac{\partial^2 W}{\partial \boldsymbol{C}^2} \cdot \boldsymbol{F}^c\right)^{lc,rc} \cdot \boldsymbol{F}^c = 2\{\mu - \lambda \ln(J)\}\,{}^4\boldsymbol{I}^{rc} + \lambda \boldsymbol{I}\boldsymbol{I}$$

$$\left.\begin{array}{l} \boldsymbol{\tau} = \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + {}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}) = \boldsymbol{\tau}_e - \Delta\lambda\,{}^4\boldsymbol{H} : \boldsymbol{a} \\ \Delta\lambda = \Delta t\,\gamma\phi(F) \end{array}\right\}$$

Iterative stress update

The coupled set of equations is solved iteratively following a Newton-Raphson procedure. In the stress update procedure it may be necessary to take into account the change in the elastic trial stress and deformation. This is the case in a plane stress situation. Both δJ and $\delta\boldsymbol{\tau}_{tr}$ can then be expressed in $\delta\boldsymbol{\tau}$ and $\delta\lambda$. New variables (J_1, $\boldsymbol{J}_2$, $\boldsymbol{M}_1$, ${}^4\boldsymbol{M}_2$) are introduced, which can be specified explicitly later.

From the coupled set of iterative equations $\delta\boldsymbol{\tau}$ and $\delta\lambda$ can be solved, whereupon new (better) values of $\boldsymbol{\tau}$ and $\Delta\lambda$ are determined. The iteration process is stopped when the residuals $\boldsymbol{s}_1$ and s_2 are small enough.

When the iteration process has converged, the current values of $\boldsymbol{\tau}$ and $\Delta\lambda$ are known. Then the Cauchy stress $\boldsymbol{\sigma}$ and the viscoplastic deformation rate $\boldsymbol{D}_{vp}$ can be determined. The latter is used to calculate the effective viscoplastic strain $\bar{\varepsilon}_{vp}$. Subsequently the yield stress is updated according to the hardening rule.

$$\left.\begin{array}{l} \delta\boldsymbol{\tau} - \delta\boldsymbol{\tau}_e + {}^4\boldsymbol{H} : \boldsymbol{a}\,\delta\lambda + \Delta\lambda\,\delta^4\boldsymbol{H} : \boldsymbol{a} + \Delta\lambda\,{}^4\boldsymbol{H} : \delta\boldsymbol{a} = -\boldsymbol{s}_1 \\ \delta\lambda - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\boldsymbol{a} : \delta\boldsymbol{\tau} - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right)\delta\lambda = -s_2 \end{array}\right\}$$

$$\text{with} \quad \begin{cases} \delta \boldsymbol{\tau}_e = \boldsymbol{M}_1 \delta\lambda + {}^4\boldsymbol{M}_2 : \delta\boldsymbol{\tau} \\ \delta^4 \boldsymbol{H} = \left(\dfrac{\partial\, {}^4\boldsymbol{H}}{\partial J} \right) \delta J = {}^4\boldsymbol{c}\, \delta J \\ \delta \boldsymbol{a} = \left(\dfrac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}} \right) : \delta\boldsymbol{\tau} = {}^4\boldsymbol{b} : \delta\boldsymbol{\tau} \\ \delta J = J_1 \delta\lambda + \boldsymbol{J}_2 : \delta\boldsymbol{\tau} \end{cases}$$

This can be rewritten with some abbreviations.

$$\left. \begin{aligned} {}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\, \delta\lambda &= -\boldsymbol{s}_1 \\ \boldsymbol{u} : \delta\boldsymbol{\tau} + v\, \delta\lambda &= -s_2 \end{aligned} \right\}$$

$$\begin{aligned} {}^4\boldsymbol{R} &= {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : {}^4\boldsymbol{b} \\ \boldsymbol{t} &= {}^4\boldsymbol{H} : \boldsymbol{a} \\ \boldsymbol{u} &= -\, \Delta t\, \gamma \left(\frac{\partial\phi}{\partial F} \right) \boldsymbol{a} \\ v &= 1 - \Delta t\, \gamma \left(\frac{\partial\phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \\ \boldsymbol{s}_1 &= \boldsymbol{\tau} - \boldsymbol{\tau}_e + \Delta\lambda\, {}^4\boldsymbol{H} : \boldsymbol{a} \\ s_2 &= \Delta\lambda - \Delta t\, \gamma\, \phi(F) \end{aligned}$$

Derivatives

The variations of various variables are determined by differentiation.

The hardening law relates the current yield stress to the equivalent viscoplastic strain. To describe the intrinsic softening followed by hardening, the relation between τ_y and $\bar{\varepsilon}_{vp}$ is taken to be a polynomial of 7th-order. Coefficients are fitted onto experimental data.

$$\tau_y = \tau_{y0} + h\bar{\varepsilon}_{vp} + a\bar{\varepsilon}_{vp}^2 + b\bar{\varepsilon}_{vp}^3 + c\bar{\varepsilon}_{vp}^4 + d\bar{\varepsilon}_{vp}^7$$

$$\begin{aligned} \frac{\partial\, {}^4\boldsymbol{H}}{\partial J} &= -2\lambda \frac{1}{J}\, {}^4\boldsymbol{I} = {}^4\boldsymbol{c} \quad \rightarrow \quad \underline{\underline{c}} = -2\lambda \frac{1}{J} \underline{\underline{I}} \\ \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) &= \frac{\partial}{\partial \bar{\varepsilon}_{vp}} \left(-\tau_y(\bar{\varepsilon}_{vp}) \right) = -h - 2a\bar{\varepsilon}_{vp} - 3b\bar{\varepsilon}_{vp}^2 - 4c\bar{\varepsilon}_{vp}^3 - 7d\bar{\varepsilon}_{vp}^6 \\ \left(\frac{\partial\phi}{\partial F} \right) &= \frac{\partial}{\partial F} \left\{ \left(\frac{F}{\tau_{y0}} \right)^N \right\} = \frac{N}{\tau_{y0}} \left(\frac{F(\underset{\sim}{\tau}, \bar{\varepsilon}_{vp})}{\tau_{y0}} \right)^{N-1} \\ \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}} &= \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}^d} : \frac{\partial \boldsymbol{\tau}^d}{\partial \boldsymbol{\tau}} \end{aligned}$$

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}^d} = \frac{\partial}{\partial \boldsymbol{\tau}^d} \left\{ \tfrac{3}{2} \bar{\tau}^{-1} \boldsymbol{\tau}^d \right\} = \tfrac{3}{2} \left(-\bar{\tau}^{-2} \frac{\partial\bar{\tau}}{\partial \boldsymbol{\tau}^d} \right) \boldsymbol{\tau}^d + \tfrac{3}{2} \bar{\tau}^{-1}\, {}^4\boldsymbol{I}$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{\tau}^d}{\partial \boldsymbol{\tau}} &= \frac{\partial}{\partial \boldsymbol{\tau}}\left\{\boldsymbol{\tau} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\} = {}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I} \\
\frac{\partial \bar{\tau}}{\partial \boldsymbol{\tau}^d} &= \frac{\partial}{\partial \boldsymbol{\tau}^d}\left\{\left(\tfrac{3}{2}\boldsymbol{\tau}^d : \boldsymbol{\tau}^d\right)^{1/2}\right\} = \tfrac{3}{2}\bar{\tau}^{-1}\boldsymbol{\tau}^d = \boldsymbol{a}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\bar{\tau}^{-1}\boldsymbol{a}\boldsymbol{a} + \tfrac{3}{2}\bar{\tau}^{-1}\,{}^4\boldsymbol{I}\right) : \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) \\
&= -\bar{\tau}^{-1}\boldsymbol{a}\boldsymbol{a} + \tfrac{3}{2}\bar{\tau}^{-1}\,{}^4\boldsymbol{I} - \tfrac{1}{2}\bar{\tau}^{-1}\,{}^4\boldsymbol{I} : \boldsymbol{I}\boldsymbol{I} = {}^4\boldsymbol{b} \quad \rightarrow
\end{aligned}
$$

$$
\underline{\underline{b}} = -\bar{\tau}^{-1}\underset{\approx}{a}\,\underset{\approx}{a}^T + \tfrac{3}{2}\bar{\tau}^{-1}\,\underline{\underline{I}} - \tfrac{1}{2}\bar{\tau}^{-1}\,\underset{\approx}{I}\,\underset{\approx}{I}^T = \bar{\tau}^{-1}\left(-\underset{\approx}{a}\,\underset{\approx}{a}^T + \tfrac{3}{2}\underline{\underline{I}} - \tfrac{1}{2}\underset{\approx}{I}\,\underset{\approx}{I}^T\right)
$$

5.4.3 Stiffness

To evaluate the iterative Updated Lagrange weighted residual equation, not only the Cauchy stress $\boldsymbol{\sigma}$, but also the relation between the stress variation $\delta\boldsymbol{\sigma}$ and $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c$ has to be known, i.e. $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

The consistent stiffness tensor ${}^4\boldsymbol{M}$, eventually leads to the consistent stiffness matrix. It must be derived from the coupled nonlinear equations for $\boldsymbol{\tau}$ and $\Delta\lambda$. Iterative changes (variations) of $\delta\boldsymbol{\tau}$ and $\delta\lambda$ can be derived. To simplify notation we omit again the upper index i, which indicates the iteration step number.

To arrive at a relation between $\delta\boldsymbol{\tau}$ and $\delta\boldsymbol{F}_n$ some new tensors are introduced which can be specified later, when a coordinate system is chosen.

$$
\left.
\begin{aligned}
&\boldsymbol{\tau} = \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + {}^4\boldsymbol{H} : \boldsymbol{e}_n - \Delta\lambda\,{}^4\boldsymbol{H} : \boldsymbol{a} \\
&\Delta\lambda = \Delta t\,\gamma\,\phi(F)
\end{aligned}
\right\} \quad \rightarrow
$$

$$
\left.
\begin{aligned}
\delta\boldsymbol{\tau} = {} & \delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c + \delta\,{}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}) + \\
& {}^4\boldsymbol{H} : \delta\boldsymbol{e}_n - {}^4\boldsymbol{H} : \boldsymbol{a}\,\delta\lambda - \Delta\lambda\,{}^4\boldsymbol{H} : \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}}\right) : \delta\boldsymbol{\tau} \\
\delta\lambda = {} & \left[\left\{\Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\right\} / \left\{1 - \Delta t\,\gamma\left(\frac{\partial\phi}{\partial F}\right)\left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}}\right)\right\}\right] \boldsymbol{a} : \delta\boldsymbol{\tau} = c_1\boldsymbol{a} : \delta\boldsymbol{\tau}
\end{aligned}
\right\} \quad \rightarrow
$$

$$
\begin{aligned}
&\left\{{}^4\boldsymbol{I} + \Delta\lambda\,{}^4\boldsymbol{H} : \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}}\right) + c_1\,{}^4\boldsymbol{H} : \boldsymbol{a}\boldsymbol{a}\right\} : \delta\boldsymbol{\tau} = \\
&\qquad \delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c + \delta\,{}^4\boldsymbol{H} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}) + {}^4\boldsymbol{H} : \delta\boldsymbol{e}_n \\
&{}^4\boldsymbol{V} : \delta\boldsymbol{\tau} = {}^4\boldsymbol{E} : \delta\boldsymbol{F}_n \quad \rightarrow \quad \delta\boldsymbol{\tau} = {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} : \delta\boldsymbol{F}_n
\end{aligned}
$$

The additional tensors are calculated below.

$$
\begin{aligned}
&\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c = {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n \\
&J = \det(\boldsymbol{F}_n) = \det(\boldsymbol{F}_n + \delta\boldsymbol{F}_n) = J(1 + \boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n) \quad \rightarrow \quad \delta J = J\,\boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n \\
&\delta\,{}^4\boldsymbol{H} = \left(\frac{\partial\,{}^4\boldsymbol{H}}{\partial J}\right)\delta J = \left(\frac{\partial\,{}^4\boldsymbol{H}}{\partial J}\right)\left(J\boldsymbol{F}_n^{-1} : \delta\boldsymbol{F}_n\right)
\end{aligned}
$$

$$
\left.
\begin{aligned}
&\delta\boldsymbol{e}_n = -\tfrac{1}{2}\delta\boldsymbol{F}_n^{-c} \cdot \boldsymbol{F}_n^{-1} - \tfrac{1}{2}\boldsymbol{F}_n^{-c} \cdot \delta\boldsymbol{F}_n^{-1} = -\,{}^4\boldsymbol{A}_1 : \delta\boldsymbol{F}_n^{-1} \\
&\delta\boldsymbol{F}_n^{-1} = -\boldsymbol{F}_n^{-1} \cdot \delta\boldsymbol{F}_n \cdot \boldsymbol{F}_n^{-1} = -\,{}^4\boldsymbol{A}_2 : \delta\boldsymbol{F}_n
\end{aligned}
\right\} \quad \rightarrow
$$

$$\delta \boldsymbol{e}_n = ({}^4\boldsymbol{A}_1 : {}^4\boldsymbol{A}_2) : \delta \boldsymbol{F}_n = {}^4\boldsymbol{P} : \delta \boldsymbol{F}_n$$

Using the definition $\boldsymbol{\tau} = J\boldsymbol{\sigma}$ a relation between $\delta\boldsymbol{\tau}$ and $\delta\boldsymbol{F}_n$ can be derived, which can be transformed to $\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u$.

$$\begin{aligned}
\boldsymbol{\tau} &= J\boldsymbol{\sigma} \quad \rightarrow \quad \boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{\tau} \quad \rightarrow \\
\delta\boldsymbol{\sigma} &= \frac{1}{J}(\delta\boldsymbol{\tau} - \boldsymbol{\sigma}\delta J) = \frac{1}{J}\left\{ {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma} J \boldsymbol{F}_n^{-1} \right\} : \delta\boldsymbol{F}_n = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n \\
&= {}^4\boldsymbol{C} : \left\{ \boldsymbol{F}^{-c}(t_n) \cdot \delta\boldsymbol{F}^c \right\}^c = {}^4\boldsymbol{C} : \left\{ \boldsymbol{F}^{-c}(t_n) \cdot \boldsymbol{F}^c \cdot \boldsymbol{L}_u^c \right\}^c \\
&= {}^4\boldsymbol{M} : \boldsymbol{L}_u
\end{aligned}$$

Matrix/column notation

The matrix/column notation for the consistent stiffness matrix is derived.

$$\begin{aligned}
\delta\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n && \rightarrow \quad \delta\underset{\sim}{\sigma} = \underline{\underline{C}}\,\delta\underset{\sim}{F_n}_t \\
\delta\boldsymbol{F}_n &= \left(\boldsymbol{F}^{-c}(t_n) \cdot \delta\boldsymbol{F}^c\right)^c && \rightarrow \quad \delta\underset{\sim}{F_n} = \left(\underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\sim}{F}_t\right)_t \rightarrow \delta\underset{\sim}{F_n}_t = \underline{\underline{F}}_t^{-1}(t_n)\delta\underset{\sim}{F}_t \\
\delta\boldsymbol{F}^c &= \boldsymbol{F}^c \cdot \boldsymbol{L}_u^c && \rightarrow \quad \delta\underset{\sim}{F}_t = \underline{\underline{F}}_t\underset{\sim}{L_u}_t
\end{aligned}$$

$$\delta\underset{\sim}{\sigma} = \left[\underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t\right]\underset{\sim}{L_u}_t = \underline{\underline{M}}\underset{\sim}{L_u}_t$$

$$\begin{aligned}
\underline{\underline{M}} &= \underline{\underline{C}}\,\underline{\underline{F}}_t^{-1}(t_n)\underline{\underline{F}}_t \\
\underline{\underline{C}} &= \frac{1}{J}\left(\underline{\underline{V}}^{-1}\,\underline{\underline{E}}_r - J\underset{\sim}{\sigma}\underset{\sim}{F_n}^{-T}\right) \\
\underline{\underline{V}} &= \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}_c\underline{\underline{b}} + c_1\underline{\underline{H}}_c\underset{\sim}{a}\underset{\sim}{a}^T \\
\underline{\underline{E}} &= \underline{\underline{T}} - 2\lambda\underline{\underline{I}}\left(\underset{\sim}{e_n} - \Delta\lambda\,\underset{\sim}{a}\right)\left(\underset{\sim}{F_n}^{-1}\right)^T + \underline{\underline{H}}_c\underline{\underline{P}}
\end{aligned}$$

5.4.4 Plane strain

For plane strain some terms in the stress update equations vanish. During viscoplastic deformation the volume will not change, so $\delta J = 0$. Also, the elastic trial stress will remain as it is, i.e. $\delta\boldsymbol{\tau}_{tr} = \boldsymbol{0}$.

$$\begin{aligned}
\delta J &= J_1\delta\lambda + \boldsymbol{J}_2 : \delta\boldsymbol{\tau} = 0 \\
\delta\boldsymbol{\tau}_{tr} &= \boldsymbol{M}_1\delta\lambda + {}^4\boldsymbol{M}_2 : \delta\boldsymbol{\tau} = \boldsymbol{O}
\end{aligned}$$

Iterative stress update

$$\left.\begin{aligned} &{}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\delta\lambda = -\boldsymbol{s}_1 \\ &\boldsymbol{u} : \delta\boldsymbol{\tau} + v\delta\lambda = -s_2 \end{aligned}\right\}$$

$$\begin{array}{lcl} {}^4\boldsymbol{R} = {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : {}^4\boldsymbol{b} & ; & \boldsymbol{t} = {}^4\boldsymbol{H} : \boldsymbol{a} \\ \boldsymbol{u} = -\Delta t\,\gamma \left(\dfrac{\partial\phi}{\partial F}\right)\boldsymbol{a} & ; & v = 1 - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\ \boldsymbol{s}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_{tr} + \Delta\lambda\, {}^4\boldsymbol{H} : \boldsymbol{a} & ; & s_2 = \Delta\lambda - \Delta t\,\gamma\,\phi(F) \end{array}$$

Matrix/column notation

It is assumed that there is no deformation in the x_3-direction ($u_3 = 0$), which results in the plane strain deformation in the (x_1x_2)-plane. The plane strain case can be derived rather straightforward from the three-dimensional formulation.

$$\begin{bmatrix} \underline{\underline{R}}_c & \underset{\approx}{t} \\ \underset{\approx}{u}_t^T & v \end{bmatrix} \begin{bmatrix} \delta\underset{\approx}{\tau} \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \underset{\approx}{s}_1 \\ s_2 \end{bmatrix}$$

$$\begin{array}{lcl} \underline{\underline{R}} = \underline{\underline{I}} + \Delta\lambda\,\underline{\underline{H}}\,\underline{\underline{b}}_t & ; & \underset{\approx}{t} = \underline{\underline{H}}\,\underset{\approx}{a}_t \\ \underset{\approx}{u} = -\Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\underset{\approx}{a} & ; & v = 1 - \Delta t\,\gamma\left(\dfrac{\partial\phi}{\partial F}\right)\left(\dfrac{\partial F}{\partial\bar{\varepsilon}_{vp}}\right) \\ \underset{\approx}{s}_1 = \underset{\approx}{\tau} - \underset{\approx}{\tau}_{tr} + \Delta\lambda\,\underline{\underline{H}}\,\underset{\approx}{a}_t & ; & s_2 = \Delta\lambda - \Delta t\,\gamma\phi(F) \end{array}$$

Stiffness

The plane strain stiffness in tensorial notation is analogous to the three-dimensional relation.

$$\delta\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n = \frac{1}{J}\left\{ {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma} J \boldsymbol{F}_n^{-1} \right\} : \delta\boldsymbol{F}_n$$

$$\begin{aligned} {}^4\boldsymbol{V} &= \left\{ {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : {}^4\boldsymbol{b} + c_1\, {}^4\boldsymbol{H} : \boldsymbol{a}\boldsymbol{a} \right\} \\ {}^4\boldsymbol{E} &= \left\{ {}^4\boldsymbol{T} + {}^4\boldsymbol{c} : (\boldsymbol{e}_n - \Delta\lambda\,\boldsymbol{a}) J \boldsymbol{F}_n^{-1} + {}^4\boldsymbol{H} : {}^4\boldsymbol{P} \right\} \\ \delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c &= {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n \\ \delta\boldsymbol{e}_n &= {}^4\boldsymbol{P} : \delta\boldsymbol{F}_n \end{aligned}$$

Matrix/column notation

Matrix/column notation of the consistent stiffness matrix for plain strain deformation.

$$\delta \underset{\sim}{\sigma} = \underline{\underline{C}} \left(\delta \underset{\sim}{F}_n \right)_t = \left[\frac{1}{J} \left\{ \underline{\underline{V}}^{-1} \underline{\underline{E}}_r - \underset{\sim}{\sigma} J \underset{\sim}{F}_n^{-T} \right\} \right] \left(\delta \underset{\sim}{F}_n \right)_t$$

$$\begin{aligned} \underline{\underline{V}} &= \underline{\underline{I}} + \Delta\lambda\, \underline{\underline{H}}_c \underline{\underline{b}} + c_1 \underline{\underline{H}}_c \underset{\sim}{a} \underset{\sim}{a}^T \\ \underline{\underline{E}} &= \underline{\underline{T}} + 2\lambda \underline{\underline{I}} \left(\underset{\sim}{e} - \Delta\lambda \underset{\sim}{a} \right) J \underset{\sim}{F}_n^{-T} + \underline{\underline{H}}_c \underline{\underline{P}} \end{aligned}$$

5.4.5 Plane stress

For plane stress we have to take into account the variation of the trial stress and the deformation.

Iterative stress update

Again the system of equations to be solved can be written with some abbreviations.

$$\left. \begin{aligned} & {}^4\boldsymbol{R} : \delta\boldsymbol{\tau} + \boldsymbol{t}\delta\lambda = -\boldsymbol{s}_1 \\ & \boldsymbol{u} : \delta\boldsymbol{\tau} + v\delta\lambda = -s_2 \end{aligned} \right\}$$

$$\begin{aligned} & {}^4\boldsymbol{R} = {}^4\boldsymbol{I} - {}^4\boldsymbol{M}_2 + \Delta\lambda\, {}^4\boldsymbol{C} : \boldsymbol{a}\boldsymbol{J}_2 + \Delta\lambda\, {}^4\boldsymbol{H} : {}^4\boldsymbol{b} \; ; && \boldsymbol{t} = -\boldsymbol{M}_1 + \Delta\lambda\, {}^4\boldsymbol{C} : \boldsymbol{a} J_1 + {}^4\boldsymbol{H} : \boldsymbol{a} \\ & \boldsymbol{u} = -\Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \boldsymbol{a} \qquad ; && v = 1 - \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \\ & \boldsymbol{s}_1 = \boldsymbol{\tau} - \boldsymbol{\tau}_{trial} + \Delta\lambda\, {}^4\boldsymbol{H} : \boldsymbol{a} \qquad ; && s_2 = \Delta\lambda - \Delta t\, \gamma\, \phi(F) \end{aligned}$$

Matrix/column notation

Introduction of a suitable (problem dependent !) coordinate system leads to the transformation of vectors and tensors into their components, which are stored in columns and matrices.

$$\begin{bmatrix} \underline{\underline{R}}_c & \underset{\sim}{t} \\ \underset{\sim}{u}_t^T & v \end{bmatrix} \begin{bmatrix} \delta \underset{\sim}{\tau} \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \underset{\sim}{s}_1 \\ s_2 \end{bmatrix}$$

$$\begin{aligned} & \underline{\underline{R}} = \underline{\underline{I}} - \underline{\underline{M}}_2 + \Delta\lambda \underline{\underline{C}}\, \underset{\sim}{a}_r \underset{\sim}{J}_2^T + \Delta\lambda \underline{\underline{H}}\, \underline{\underline{b}}_r \quad ; && \underset{\sim}{t} = -\underline{\underline{M}}_1 + \Delta\lambda \underline{\underline{C}}\, \underset{\sim}{a}_t J_1 + \underline{\underline{H}}\, \underset{\sim}{a}_t \\ & \underset{\sim}{u} = -\Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \underset{\sim}{a} \quad ; && v = 1 - \Delta t\, \gamma \left(\frac{\partial \phi}{\partial F} \right) \left(\frac{\partial F}{\partial \bar{\varepsilon}_{vp}} \right) \\ & \underset{\sim}{s}_1 = \underset{\sim}{\tau} - \underset{\sim}{\tau}_{tr} + \Delta\lambda \underline{\underline{H}}\, \underset{\sim}{a}_t \quad ; && s_2 = \Delta\lambda - \Delta t \gamma \phi(F) \end{aligned}$$

Stiffness

The plane stress stiffness in tensorial notation is analogous to the three-dimensional relation.

$$\delta\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \delta\boldsymbol{F}_n = \frac{1}{J}\left\{ {}^4\boldsymbol{V}^{-1} : {}^4\boldsymbol{E} - \boldsymbol{\sigma} J \boldsymbol{F}_n^{-1} \right\} : \delta\boldsymbol{F}_n$$

with

$$\begin{aligned}
{}^4\boldsymbol{V} &= \left\{ {}^4\boldsymbol{I} + \Delta\lambda\, {}^4\boldsymbol{H} : \left(\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\tau}}\right) + c_1\, {}^4\boldsymbol{H} : \boldsymbol{a}\boldsymbol{a} \right\} \\
{}^4\boldsymbol{E} &= \left\{ {}^4\boldsymbol{T} + \left(\frac{\partial\, {}^4\boldsymbol{H}}{\partial J}\right) : (\boldsymbol{e} - \Delta\lambda\, \boldsymbol{a}) J \boldsymbol{F}_n^{-1} + {}^4\boldsymbol{H} : {}^4\boldsymbol{P} \right\}
\end{aligned}$$

$$\delta\boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \boldsymbol{F}_n^c + \boldsymbol{F}_n \cdot \boldsymbol{\tau}(t_n) \cdot \delta\boldsymbol{F}_n^c = {}^4\boldsymbol{T} : \delta\boldsymbol{F}_n$$

Matrix-column notation

With the assumption that $\tau_{13} = \tau_{23} = \tau_{33} = 0$, the three-dimensional formulation reduces to that for two-dimensional plane stress deformation in the $(x_1 x_2)$-plane. Columns with relevant components of stress and deformation rate are :

$$\begin{aligned}
\underset{\sim}{\tau} &= \left[\begin{array}{cccc} \tau_{11} & \tau_{22} & \tau_{12} & \tau_{21} \end{array}\right]^T \\
\underset{\sim}{D} &= \left[\begin{array}{cccc} D_{11} & D_{22} & D_{12} & D_{21} \end{array}\right]^T
\end{aligned}$$

During the plane stress return mapping we have

$$\delta F_{11} = \delta F_{22} = \delta F_{12} = \delta F_{21} = 0 \qquad \text{and} \qquad \delta\underset{\sim}{\tau}^{trial} = \underset{\sim}{0}$$

As deformation in x_3-direction is allowed, δJ can be expressed in δF_{33} :

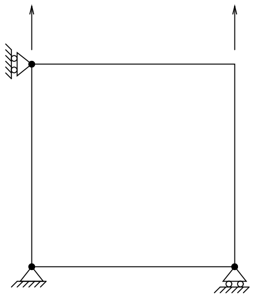
$$\delta J = (F_{11}F_{22} - F_{12}F_{21})\delta F_{33} = J_1 \delta\lambda + \underset{\sim}{J}_2^T \delta\underset{\sim}{\tau}$$

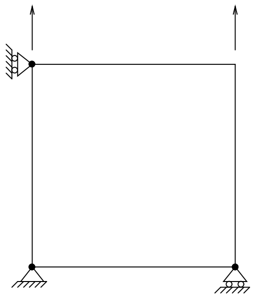
which results in the set of iterative equations for $\delta\underset{\sim}{\tau}$ and $\delta\lambda$.

5.5 Examples

5.5.1 Tensile test

A square plate or cylindrical bar is loaded uniaxially. Dimensions are listed in the table.





initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

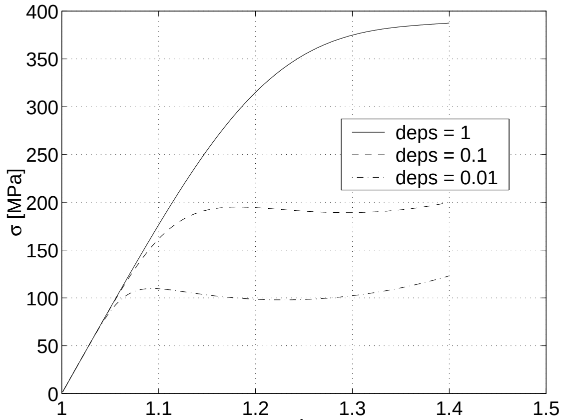
initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Tensile test at various strain rates

The Perzyna model parameter values for polycarbonate (PC) are used and listed in the table. The axial elongation is prescribed as a linear function of time with a constant elongation rate. The tensile bar is axisymmetric with initial cross-sectional area $A_0 = 10$ mm^2. The axial stress and force are shown in the figure as a function of the elongation.

E	1800	MPa	ν	0.37	-
σ_{y0}	37	MPa	H	-200	MPa
γ	0.001	1/s	N	3	-
a	500	MPa	b	700	MPa
c	800	MPa	d	30000	MPa

elongation rate $$\frac{\dot{\Delta l}}{h_0} = \{0.01, 0.1, 1\}\ \mathrm{s}^{-1}$$



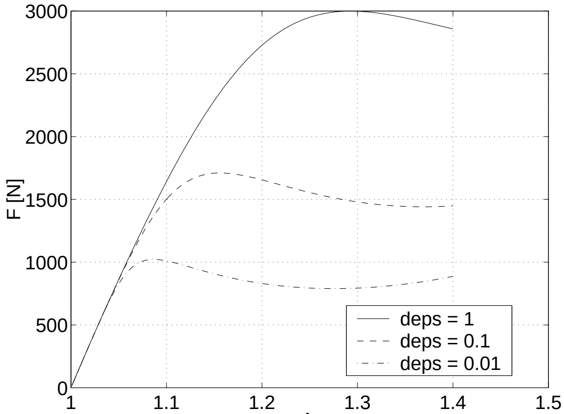
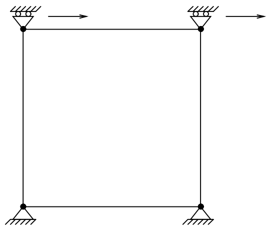


Fig. 5.28 : *Axial stress and force versus elongation for PC.*

5.5.2 Shear test

The simple shear test is analyzed with one element, where the horizontal displacement in the upper nodes is prescribed. Because there are no unknown degrees of freedom, the stiffness matrix is not used. The shear force is calculated for polycarbonate (PC). The prescribed strain rate is constant.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate $\qquad \dot{\gamma} = \dfrac{\dot{u}}{h_0} = 0.01 \text{ s}^{-1}$

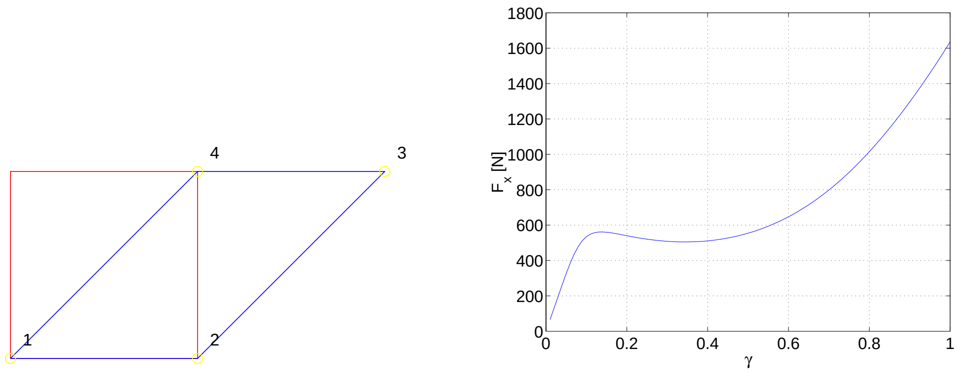


Fig. 5.29 : *Shear force versus shear strain for plane strain*

6 Nonlinear viscoelastic material behavior

The one-dimensional mechanical representation of the nonlinear viscoelastic (Leonov) model consists of a hardening spring in parallel with a Maxwell model, of which the viscosity is a nonlinear function of the stress.

For some materials the viscosity is decreased using a damage parameter, to describe intrinsic softening. Hardening at higher strains is described by the parallel spring.

In the model with hardening, the Cauchy stress σ is additively decomposed in an effective or driving stress s and a hardening stress w. This decomposition reflects the contribution of secondary interactions between polymer chains and that of the entangled polymer network.

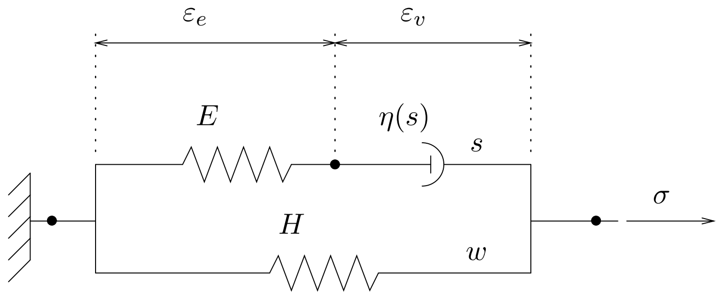


Fig. 6.30 : *Model for nonlinear viscoelastic behavior*

$$\sigma = s + w$$

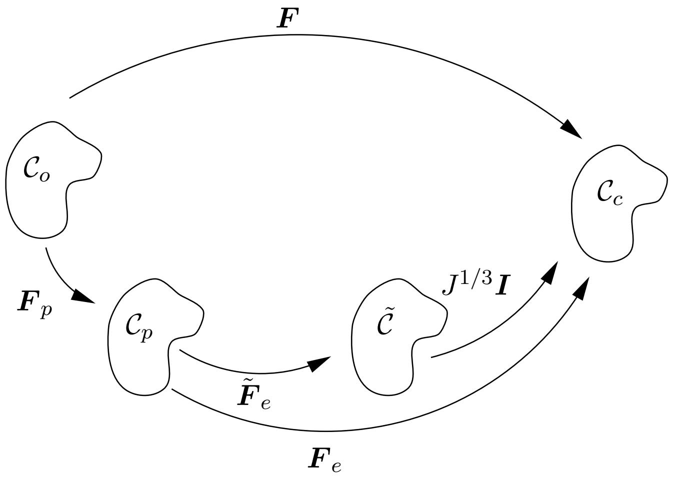
6.1 Kinematics

The deformation tensor $\boldsymbol{F}$ is multiplicatively decomposed into an elastic ($\boldsymbol{F}_e$) and a plastic ($\boldsymbol{F}_p$) contribution : $\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_p$. This decomposition follows from the postulate of a stress-free plastic intermediate configuration $\mathcal{C}_p$. As the decomposition is not unique with respect to rotational contributions, an extra assumption will later be needed regarding the rotations.

It is assumed that during plastic deformation the volume change is zero, i.e. $J_p = \det(\boldsymbol{F}_p) = 1$ and thus $J = \det(\boldsymbol{F}) = \det(\boldsymbol{F}_e)$. The elastic volume deformation is decoupled from the isochoric distortional deformation by the definition of the tensor $\tilde{\boldsymbol{F}}_e$ according to $\tilde{\boldsymbol{F}}_e = J^{-1/3}\boldsymbol{F}_e$.

The left Cauchy-Green strain tensor $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c$ is used as a strain measure. Its volume invariant elastic part is given by $\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c$. The velocity gradient tensor $\boldsymbol{L} = (\vec{\nabla}\vec{v})^c = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$ can be written as the sum of the symmetric deformation rate tensor $\boldsymbol{D}$ and the skew-symmetric spin tensor $\boldsymbol{\Omega}$: $\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{\Omega}$. Using the decomposition of $\boldsymbol{F}$, we can split $\boldsymbol{L}$ in an elastic and a plastic part. This leads to associated tensors $\boldsymbol{D}_e$, $\boldsymbol{D}_p$, $\boldsymbol{\Omega}_e$ and $\boldsymbol{\Omega}_p$.

To make the decomposition of $\boldsymbol{F}$ unique, $\boldsymbol{\Omega}_p$ is chosen equal to the null tensor. It has been shown by e.g. Boyce, that this specific choice regarding rotational contributions has no significant influence on the overall stress-strain behavior.



$$
\begin{aligned}
&\boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c = \boldsymbol{F}_e \cdot \boldsymbol{F}_p = J^{1/3} \boldsymbol{I} \cdot \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \\
&\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F} \qquad ; \qquad \boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^c \quad \rightarrow \quad \tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c \\
&\boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} = (\vec{\nabla} \vec{v})^c \\
&\quad = \boldsymbol{L}_e + \boldsymbol{L}_p = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + (\boldsymbol{D}_p + \boldsymbol{\Omega}_p) = (\boldsymbol{D}_e + \boldsymbol{\Omega}_e) + \boldsymbol{D}_p
\end{aligned}
$$

6.2 Constitutive relations

Stress decomposition

The deviatoric part of the driving stress, $\boldsymbol{s}^d$, is related to isochoric elastic deformation $\tilde{\boldsymbol{B}}_e^d$ through a generalized Hookean relation. The hydrostatic part $\boldsymbol{s}^h = -p\boldsymbol{I}$ (p = hydrostatic material pressure) is related to the volumetric deformation. Hardening is modeled according to Gaussian chain statistics as this model is applicable to a large number of thermoplastic polymers, both amorphous and semi-crystalline, up to very high extension ratios.

Material parameters are : the shear modulus G, the bulk modulus κ and the hardening modulus H. For the elastic part to be hyper-elastic, the shear modulus G should be replaced by $\frac{G}{J}$.

$$
\begin{aligned}
&\boldsymbol{\sigma} = \boldsymbol{s} + \boldsymbol{w} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w} \\
&\boldsymbol{s} = G\tilde{\boldsymbol{B}}_e^d + \kappa(J-1)\boldsymbol{I} \quad ; \quad \boldsymbol{w} = H\tilde{\boldsymbol{B}}^d
\end{aligned}
$$

Elastic deformation

As the model describes time- and history-dependent behavior, the elastic strain must be updated by integration of appropriate evolution equations for $\tilde{\boldsymbol{B}}_e$. The expression for $\dot{\tilde{\boldsymbol{B}}}_e$ can be derived starting from $\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c$ and using the decomposition $\tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p$ and the assumption $\boldsymbol{\Omega}_p = \boldsymbol{O}$.

$$
\begin{aligned}
&\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}_e \cdot \tilde{\boldsymbol{F}}_e^c \quad \rightarrow \quad \dot{\tilde{\boldsymbol{B}}}_e = \dot{\tilde{\boldsymbol{F}}}_e \cdot \tilde{\boldsymbol{F}}_e^c + \tilde{\boldsymbol{F}}_e \cdot \dot{\tilde{\boldsymbol{F}}}_e^c \\
&\qquad\qquad\qquad \tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \quad \rightarrow \quad \tilde{\boldsymbol{F}}_e = \tilde{\boldsymbol{F}} \cdot \boldsymbol{F}_p^{-1} \quad \rightarrow \quad \dot{\tilde{\boldsymbol{F}}}_e = \dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \\
&\dot{\tilde{\boldsymbol{B}}}_e = \left(\dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \right) \cdot \tilde{\boldsymbol{F}}_e^c + \tilde{\boldsymbol{F}}_e \cdot \left(\boldsymbol{F}_p^{-c} \cdot \dot{\tilde{\boldsymbol{F}}}^c + \dot{\boldsymbol{F}}_p^{-c} \cdot \tilde{\boldsymbol{F}}^c \right) \\
&\quad = \left(\dot{\tilde{\boldsymbol{F}}} \cdot \boldsymbol{F}_p^{-1} \cdot \tilde{\boldsymbol{F}}_e^{-1} + \tilde{\boldsymbol{F}} \cdot \dot{\boldsymbol{F}}_p^{-1} \cdot \tilde{\boldsymbol{F}}_e^{-1} \right) \cdot \tilde{\boldsymbol{B}}_e + \\
&\qquad\qquad \tilde{\boldsymbol{B}}_e \cdot \left(\tilde{\boldsymbol{F}}_e^{-c} \cdot \boldsymbol{F}_p^{-c} \cdot \dot{\tilde{\boldsymbol{F}}}^c + \tilde{\boldsymbol{F}}_e^{-c} \cdot \dot{\boldsymbol{F}}_p^{-c} \cdot \tilde{\boldsymbol{F}}^c \right) \\
&\quad = \left(\tilde{\boldsymbol{L}} + \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \cdot \dot{\boldsymbol{F}}_p^{-1} \tilde{\boldsymbol{F}}_e^{-1} \right) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot \left(\tilde{\boldsymbol{L}}^c + \tilde{\boldsymbol{F}}_e^{-c} \cdot \dot{\boldsymbol{F}}_p^{-c} \cdot \boldsymbol{F}_p^c \cdot \tilde{\boldsymbol{F}}_e^c \right) \\
&\qquad\qquad \boldsymbol{F}_p \cdot \boldsymbol{F}_p^{-1} = \boldsymbol{I} \quad \rightarrow \quad \boldsymbol{F}_p \cdot \dot{\boldsymbol{F}}_p^{-1} = -\dot{\boldsymbol{F}}_p \cdot \boldsymbol{F}_p^{-1} \quad \rightarrow \\
&\quad = (\tilde{\boldsymbol{L}} - \boldsymbol{D}_p) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot (\tilde{\boldsymbol{L}}^c - \boldsymbol{D}_p)
\end{aligned}
$$

Viscoplastic deformation

The viscoplastic deformation rate $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$, through the viscosity, which is a nonlinear function of the equivalent stress $\bar{s}$, the hydrostatic pressure p, the absolute temperature T and the damage parameter D.

For polymers the Eyring viscosity function is successfully used and for metal alloys the viscosity function of Bodner-Partom ([**?**],[**?**],[**?**]).

For the equivalent deviatoric stress $\bar{s}$ is the Von Mises definition is used.

$$
\boldsymbol{D}_p = \frac{1}{2\eta} \boldsymbol{s}^d
$$

$$
\begin{aligned}
&\eta = \eta(\bar{s}, p, T, D) \\
&\bar{s} = \sqrt{\tfrac{3}{2} \boldsymbol{s}^d : \boldsymbol{s}^d} \\
&p = \kappa (J - 1) \boldsymbol{I}
\end{aligned}
$$

Eyring viscosity

For polymer materials the plastic deformation rate tensor $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$ by an Eyring viscosity η. This is a function of the equivalent Von Mises stress $\bar{s}$, the hydrostatic stress p and the absolute temperature T. In the model, presented here, the viscosity is depending on an intrinsic softening quantity D, determined by an evolution equation, which has to be solved with the other constitutive relations.

Material parameters are :

A_0	time constant
ΔH	activation energy
R	universal gas constant
p	hydrostatic pressure
μ	parameter describing pressure dependence
V	shear activation volume
D_∞	saturation value of D

$$\eta = \frac{A\bar{s}}{\sqrt{3}\sinh\left(\dfrac{\bar{s}}{\sqrt{3}\tau_0}\right)}$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}$$

$$A = A_0 \exp\left[\frac{\Delta H}{RT} + \frac{\mu p}{\tau_0} - D\right]$$

$$\tau_0 = \frac{RT}{V} \qquad ; \qquad p = -\tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$$

$$\dot{D} = h\left(1 - \frac{D}{D_\infty}\right)\frac{\bar{s}}{\sqrt{6}\,\eta} \qquad ; \qquad D \in [0, D_\infty]$$

Bodner-Partom viscosity

To describe viscoplastic behavior of metals, the plastic deformation rate tensor $\boldsymbol{D}_p$ is related to the deviatoric stress $\boldsymbol{s}^d$ by a Bodner-Partom viscosity η. This is a function of the equivalent Von Mises stress $\bar{\sigma}$ and Z, the resistance to plastic flow. $\varGamma_0$ is a constant which reflects the smoothness of the transition from the elastic to the viscoplastic response and n characterizes the rate sensitivity of the viscoplastic response. The plastic flow resistance Z depends on the equivalent plastic strain $\bar{\varepsilon}_p$. Its lower and upper bounds are Z_0 and Z_1.

The Bodner-Partom model corresponds to isotropic hardening.

$$\eta = \frac{\bar{s}}{\sqrt{12\varGamma_0}} \exp\left[\tfrac{1}{2}\left(\frac{Z}{\bar{\sigma}}\right)^{2n}\right]$$

$$\bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}$$

$$Z = Z_1 + (Z_0 - Z_1)e^{-m\bar{\varepsilon}_p}$$

$$\dot{\bar{\varepsilon}}_p = \sqrt{\tfrac{2}{3}\boldsymbol{D}_p : \boldsymbol{D}_p} \qquad \rightarrow \qquad \bar{\varepsilon}_p$$

Plastic strain rate

The current value of $\tilde{\boldsymbol{B}}_e(t)$ can be determined by integration of $\dot{\tilde{\boldsymbol{B}}}_e$. However, the integrand $\dot{\tilde{\boldsymbol{B}}}_e$ is not objective, so that rigid body rotations will influence the results, which is of course not allowed. The problem of non-objectivity of $\dot{\tilde{\boldsymbol{B}}}_e$ can be circumvented by using an evolution equation for the Cauchy-Green plastic strain tensor $\boldsymbol{C}_p$, which is invariant.

Starting from $\tilde{\boldsymbol{F}}$ an expression for $\dot{\boldsymbol{C}}_p$ can be derived, containing $\dot{\tilde{\boldsymbol{B}}}_e$. With the earlier derived expression for $\dot{\tilde{\boldsymbol{B}}}_e$, $\dot{\boldsymbol{C}}_p$ can be expressed in $\tilde{\boldsymbol{B}}_e$ and $\boldsymbol{D}_p$. The relation between $\boldsymbol{D}_p$ and $\tilde{\boldsymbol{B}}_e^d$ allows $\dot{\boldsymbol{C}}_p$ to be related to $\tilde{\boldsymbol{B}}_e$ and $\boldsymbol{C}_p$. This equation states that the direction of the plastic strain rate is defined by the directional tensor $\boldsymbol{A}$, while the plastic strain rate magnitude is governed by the characteristic plastic deformation rate Γ.

The plastic strain rate is invariant for rigid body rotations. It is shown in literature that this formulation with a plastic predictor, can be used to apply an implicit, robustly stable and efficient time integration procedure.

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\tilde{\boldsymbol{F}} = \tilde{\boldsymbol{F}}_e \cdot \boldsymbol{F}_p \quad \rightarrow \quad \boldsymbol{C}_p = \boldsymbol{F}_p^c \cdot \boldsymbol{F}_p = \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}} \quad \rightarrow \\
\dot{\boldsymbol{C}}_p = \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \left[\tilde{\boldsymbol{B}}_e \cdot \tilde{\boldsymbol{L}}^c + \tilde{\boldsymbol{B}}_e \cdot \dot{\tilde{\boldsymbol{B}}}_e^{-1} \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{L}} \cdot \tilde{\boldsymbol{B}}_e \right] \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}}
\end{array}\right\} \\
\\
\left.\begin{array}{l}
\dot{\tilde{\boldsymbol{B}}}_e = (\tilde{\boldsymbol{L}} - \boldsymbol{D}_p) \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot (\tilde{\boldsymbol{L}}^c - \boldsymbol{D}_p) \quad \rightarrow \\
\tilde{\boldsymbol{B}}_e \cdot \dot{\tilde{\boldsymbol{B}}}_e^{-1} = -\tilde{\boldsymbol{L}} - \tilde{\boldsymbol{B}}_e \cdot \tilde{\boldsymbol{L}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} + \boldsymbol{D}_p + \tilde{\boldsymbol{B}}_e \cdot \boldsymbol{D}_p \cdot \tilde{\boldsymbol{B}}_e^{-1}
\end{array}\right\}
\end{array}\right\} \rightarrow$$

$$\begin{aligned}
\dot{\boldsymbol{C}}_p &= \tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \left[\boldsymbol{D}_p \cdot \tilde{\boldsymbol{B}}_e + \tilde{\boldsymbol{B}}_e \cdot \boldsymbol{D}_p \right] \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \tilde{\boldsymbol{F}} \\
&\qquad \text{with} \quad \boldsymbol{D}_p = \frac{1}{2\eta}\, \boldsymbol{s}^d = \frac{G}{2\eta}\, \tilde{\boldsymbol{B}}_e^d \quad \rightarrow \\
&= \frac{G}{\eta} \left(\tilde{\boldsymbol{C}} - \tfrac{1}{3}\, \mathrm{tr}(\tilde{\boldsymbol{B}}_e) \boldsymbol{C}_p \right) = \Gamma \left(\tilde{\boldsymbol{C}} - \frac{1}{\alpha}\, \boldsymbol{C}_p \right) = \Gamma\, \boldsymbol{A}
\end{aligned}$$

6.3 Constitutive model

The material model can be summarized as a set of constitutive equations. The differential equations must be integrated to determine the current elastic strain and stress. Also the variation of he stress must be derived from the constitutive model, representing the current stiffness.

$$\begin{array}{l}
J = \det(\boldsymbol{F}) \quad \rightarrow \quad \tilde{\boldsymbol{F}} = J^{-1/3} \boldsymbol{F} \quad \rightarrow \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}^c \quad \rightarrow \quad \boldsymbol{w} = H \tilde{\boldsymbol{B}}^d \\
p = \kappa(J - 1) \quad \rightarrow \quad \boldsymbol{s}^h = p\boldsymbol{I} \\
\left.\begin{array}{l}
\dot{\boldsymbol{C}}_p = \dfrac{G}{\eta} \left(\tilde{\boldsymbol{C}} - \tfrac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_e)\, \boldsymbol{C}_p \right) \\
\tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \tilde{\boldsymbol{F}}^c
\end{array}\right\} \quad \rightarrow \quad \boldsymbol{s}^d = G \tilde{\boldsymbol{B}}_e^d \quad \rightarrow \quad \bar{s} = \sqrt{\tfrac{3}{2} \boldsymbol{s}^d : \boldsymbol{s}^d}
\end{array}$$

$$\boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w}$$

6.4 Incremental analysis

The plastic strain $\boldsymbol{C}_p$ at the current time t must be determined by integration of the differential equation for $\dot{\boldsymbol{C}}_p(\tau)$. In an incremental procedure the total deformation period is subdivided into a number of sequential time increments : $\Delta t = t_{i+1} - t_i$; $i = 0 \cdots n$. A solution for the governing equations is determined for the discrete end-increment times, starting from the known state – with known values of all variables – at the begin-increment time. This implies that the differential equation for $\dot{\boldsymbol{C}}_p$ has to be solved for the last increment $t_n \rightarrow t_{n+1}$ assuming that $\boldsymbol{C}_p(t_n)$ is known. For simplicity we skip the indication of the current end-increment time t_{n+1}.

We now focus attention on the last increment $[t_n, t_{n+1}]$. It is assumed that at time t_n the configuration $\mathcal{C}_n$ is completely known and all equations are satisfied. The begin-increment state $\mathcal{C}_n$ at $\tau = t_n$ is taken as the reference configuration for deformation variables, which is known as the Updated Lagrange procedure.

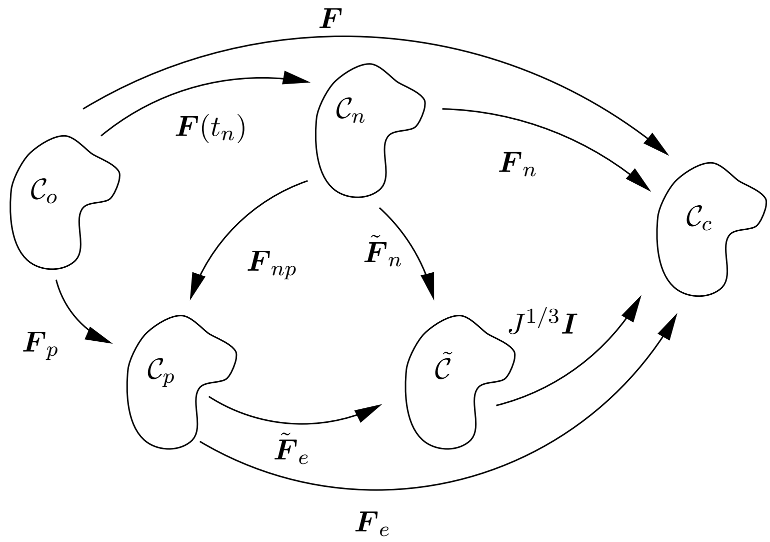


Fig. 6.31 : *Incremental deformation*

$$
\begin{aligned}
&\boldsymbol{F}(\tau) = \boldsymbol{F}_n(\tau) \cdot \boldsymbol{F}(t_n) \quad \rightarrow \quad \boldsymbol{F}_n(\tau) = \boldsymbol{F}(\tau) \cdot \boldsymbol{F}^{-1}(t_n) \\
&\tilde{\boldsymbol{F}}(\tau) = \tilde{\boldsymbol{F}}_n(\tau) \cdot \tilde{\boldsymbol{F}}(t_n) \\
&\boldsymbol{F}_n = \left(\vec{\nabla}_n \vec{x}\right)^c = \boldsymbol{R}_n \cdot \boldsymbol{U}_n
\end{aligned}
$$

Incremental plastic strain

Using the multiplicative decomposition, an expression for $\boldsymbol{C}_{p_n}(\tau)$ can be derived. It contains the tensor $\bar{\tilde{\boldsymbol{B}}}{}_{e_n}^{-1}$ which is the rotation neutralized version of $\tilde{\boldsymbol{B}}_e^{-1}$:

$$
\bar{\tilde{\boldsymbol{B}}}{}_{e_n}^{-1} = \boldsymbol{R}_n^c \cdot \tilde{\boldsymbol{B}}_e^{-1} \cdot \boldsymbol{R}_n
$$

where $\boldsymbol{R}_n$ is the incremental rotation tensor.

$$
\begin{aligned}
\boldsymbol{C}_p(\tau) &= \boldsymbol{F}_p^c(\tau) \cdot \boldsymbol{F}_p(\tau) = \tilde{\boldsymbol{F}}^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}(\tau) \\
&\qquad\qquad \text{with} \quad \tilde{\boldsymbol{F}}(\tau) = \tilde{\boldsymbol{F}}_n(\tau) \cdot \tilde{\boldsymbol{F}}(t_n) \quad \rightarrow \\
&= \tilde{\boldsymbol{F}}^c(t_n) \cdot \left[\tilde{\boldsymbol{F}}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}_n(\tau) \right] \cdot \tilde{\boldsymbol{F}}(t_n) \\
&= \tilde{\boldsymbol{F}}^c(t_n) \cdot \boldsymbol{C}_{p_n}(\tau) \cdot \tilde{\boldsymbol{F}}(t_n)
\end{aligned}
$$

incremental rotation neutralized plastic strain

$$
\begin{aligned}
\boldsymbol{C}_{p_n}(\tau) &= \tilde{\boldsymbol{F}}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \tilde{\boldsymbol{F}}_n(\tau) \\
&= \tilde{\boldsymbol{U}}_n(\tau) \cdot \left[\boldsymbol{R}_n^c(\tau) \cdot \tilde{\boldsymbol{B}}_e^{-1}(\tau) \cdot \boldsymbol{R}_n(\tau) \right] \cdot \tilde{\boldsymbol{U}}_n(\tau) \\
&= \tilde{\boldsymbol{U}}_n(\tau) \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1}(\tau) \cdot \tilde{\boldsymbol{U}}_n(\tau)
\end{aligned}
$$

Constitutive equations

With the incremental procedure the constitutive model is formulated in the incremental variables.

$$
J = \det(\boldsymbol{F}) \quad \rightarrow \quad \tilde{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F} \quad \rightarrow \quad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}^c \quad \rightarrow \quad \boldsymbol{w} = H\tilde{\boldsymbol{B}}^d
$$

$$
p = \kappa(J-1) \quad \rightarrow \quad \boldsymbol{s}^h = p\boldsymbol{I}
$$

$$
\left.
\begin{aligned}
&\dot{\boldsymbol{C}}_{p_n} = \frac{G}{\eta} \left(\tilde{\boldsymbol{C}}_n - \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n} \right) \boldsymbol{C}_{p_n} \right) \\
&\bar{\tilde{\boldsymbol{B}}}_{e_n} = \tilde{\boldsymbol{U}}_n \cdot \boldsymbol{C}_{p_n}^{-1} \cdot \tilde{\boldsymbol{U}}_n^c \rightarrow \tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}_n^c \\
&\dot{D} = h\left(1 - \frac{D}{D_\infty}\right) \frac{\bar{s}}{\sqrt{6}\,\eta} \\
&\eta = \eta(\bar{s}, p, T, D)
\end{aligned}
\right\} \rightarrow \boldsymbol{s}^d = G\tilde{\boldsymbol{B}}_e^d \rightarrow \bar{s} = \sqrt{\tfrac{3}{2}\boldsymbol{s}^d : \boldsymbol{s}^d}
$$

$$
\boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w}
$$

6.4.1 Stress update

The incremental plastic strain rate $\dot{\boldsymbol{C}}_{p_n}(\tau)$ can be integrated over the last increment $t_n \rightarrow t_{n+1}$ to determine $\boldsymbol{C}_{p_n}(t_{n+1})$. An implicit backward Euler integration scheme is used. With $\tilde{\boldsymbol{U}}_n(t_n) = \boldsymbol{I}$ we have $\boldsymbol{C}_{p_n}(t_n) = \bar{\tilde{\boldsymbol{B}}}_{e_n}^{-1}(t_n) = \tilde{\boldsymbol{B}}_{e_n}^{-1}(t_n)$.

The scalar λ is the so-called elasticity scalar, a state variable indicating the proportion of incremental elastic/plastic strains with respect to the incremental total strains ($\lambda = 1$, fully elastic increment, and $\lambda = 0$, fully plastic increment). This parameter depends on η and thus on $\boldsymbol{s}$ and $\boldsymbol{C}_p$. The isochoric elastic strain $\tilde{\boldsymbol{B}}_e$ can be calculated from $\boldsymbol{C}_{p_n}$ and $\tilde{\boldsymbol{F}}_n$.

$$
\begin{aligned}
&\dot{\boldsymbol{C}}_{p_n}(\tau) = \mathit{\Gamma}(\tau)\left[\tilde{\boldsymbol{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\boldsymbol{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n}\right) \\
&\frac{1}{\mathit{\Delta} t}\left[\boldsymbol{C}_{p_n} - \boldsymbol{C}_{p_n}(t_n)\right] = \mathit{\Gamma}\left[\tilde{\boldsymbol{C}}_n - \frac{1}{\bar{\alpha}_n}\boldsymbol{C}_{p_n}\right] \quad \rightarrow \\
&\boldsymbol{C}_{p_n} = \frac{\bar{\alpha}_n\,\mathit{\Delta} t\,\mathit{\Gamma}}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}}\tilde{\boldsymbol{C}}_n + \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}}\boldsymbol{C}_{p_n}(t_n) \quad \rightarrow \\
&\boldsymbol{C}_{p_n} = \bar{\alpha}_n(1-\lambda)\tilde{\boldsymbol{C}}_n + \lambda\boldsymbol{C}_{p_n}(t_n) \quad ; \quad \lambda = \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \mathit{\Delta} t\,\mathit{\Gamma}} = \text{elasticity parameter} \\
&\tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}_n^c = \tilde{\boldsymbol{F}}_n \cdot \boldsymbol{C}_{p_n}^{-1} \cdot \tilde{\boldsymbol{F}}_n^c
\end{aligned}
$$

Sub-incremental plastic strain update

The differential equation for the incremental plastic strain can be integrated more accurately by subdividing the current increment $\mathit{\Delta} t = t_{n+1} - t_n$ in a number (ns) of sub-increments $\delta t = \mathit{\Delta} t/ns$. The known iterative approximation for the end-increment deformation ($\tilde{\boldsymbol{F}}_n \rightarrow \tilde{\boldsymbol{C}}_n$) is also subdivided and subsequently values for $\boldsymbol{C}_{p_n}^j$ are determined with a backward Euler integration scheme.

The incremental rotation is not taken into account during this procedure but incorporated afterward at the end-increment time. It is also assumed that the principal strain directions do not change during the integration procedure.

The sub-incremental integration scheme results in a more accurate determination of $\boldsymbol{C}_{p_n}$ and thus $\boldsymbol{\sigma}$. It allows for larger incremental time steps.

Be aware that the final $\lambda^j = \lambda^{ns+1}$ is not the elasticity parameter λ introduced earlier, indicating the elastic part of the increment. This λ must be calculated without using sub-increments or just according to

$$
\lambda = \frac{\bar{\alpha}_n}{\bar{\alpha}_n + \mathit{\Delta} t\mathit{\Gamma}} = \frac{1}{1 + \mathit{\Delta} t\mathit{\Gamma}}
$$

where we assumed $\bar{\alpha}_n = 1$.

$$
\begin{aligned}
&\dot{\boldsymbol{C}}_{p_n}(\tau) = \mathit{\Gamma}(\tau)\left[\tilde{\boldsymbol{C}}_n(\tau) - \frac{1}{\bar{\alpha}_n(\tau)}\boldsymbol{C}_{p_n}(\tau)\right] \quad ; \quad \frac{1}{\bar{\alpha}_n} = \tfrac{1}{3}\mathrm{tr}\left(\bar{\tilde{\boldsymbol{B}}}_{e_n}\right) \\
&\left.\begin{array}{lll}
\text{sub-incremental deformation :} & j = 1 \cdots ns+1 & \\
j = 1 \quad : \quad \tau = t_n & ; \quad j = ns+1 \quad : & \tau = t_{n+1} \\
\delta t = \mathit{\Delta} t/ns \quad ; & \delta\tilde{\boldsymbol{C}}_n = \left\{\tilde{\boldsymbol{C}}_n\right\}^{1/ns} \quad ; & \tilde{\boldsymbol{C}}_n^j = \left\{\delta\tilde{\boldsymbol{C}}_n\right\}^j
\end{array}\right\} \\
&\frac{1}{\delta t}\left[\boldsymbol{C}_{p_n}^j - \boldsymbol{C}_{p_n}^{j-1}\right] = \mathit{\Gamma}^j\left[\tilde{\boldsymbol{C}}_n^j - \frac{1}{\bar{\alpha}_n^j}\boldsymbol{C}_{p_n}^j\right] \quad \rightarrow \\
&\boldsymbol{C}_{p_n}^j = \frac{\bar{\alpha}_n^j\,\delta t\,\mathit{\Gamma}^j}{\bar{\alpha}_n^j + \delta t\,\mathit{\Gamma}^j}\tilde{\boldsymbol{C}}_n^j + \frac{\bar{\alpha}_n^j}{\bar{\alpha}_n^j + \delta t\,\mathit{\Gamma}^j}\boldsymbol{C}_{p_n}^{j-1} \quad \rightarrow
\end{aligned}
$$

$$\boldsymbol{C}^j_{p_n} = \bar{\alpha}^j_n (1-\lambda^j)\tilde{\boldsymbol{C}}^j_n + \lambda^j \boldsymbol{C}^{j-1}_{p_n} \qquad ; \qquad \lambda^j = \frac{\bar{\alpha}^j_n}{\bar{\alpha}^j_n + \delta t\, \Gamma^j}$$

incremental plastic strain $\boldsymbol{C}_{p_n} = \boldsymbol{C}_{p_n}(t_{n+1}) = \boldsymbol{C}^{ns+1}_{p_n}$

total isochoric elastic strain $\tilde{\boldsymbol{B}}_e = \boldsymbol{R}_n \cdot \bar{\tilde{\boldsymbol{B}}}_{e_n} \cdot \boldsymbol{R}^c_n = \tilde{\boldsymbol{F}}_n \cdot \boldsymbol{C}^{-1}_{p_n} \cdot \tilde{\boldsymbol{F}}^c_n$

Iterative scalar variable update

The current plastic strain depends on two scalar variables : the elasticity parameter λ and the softening parameter D. These are a function of the stress $\boldsymbol{\sigma}$, which implies that the integration has to be carried out iteratively. A Newton-Raphson iterative procedure is employed and the resulting equation system involves partial derivatives of λ and D, which can be calculated rather straightforwardly.

After convergence of the iterative process the (sub)incremental plastic strain and stress is known, but beware that these are only approximations for the real end-increment values. The update procedure is part of the iterative procedure which has to be repeated until convergence is reached.

$$\begin{aligned}
&\lambda = 1/(1+\Delta t\Gamma) &&\rightarrow && f(\lambda, D) = \lambda(1+\Delta t\Gamma) = 1 \\
&\tfrac{1}{\Delta t}\left\{D - D(t_n)\right\} = \dot{D} &&\rightarrow && g(\lambda, D) = D - \Delta t\dot{D} = D(t_n)
\end{aligned}$$

Newton-Raphson iterative solution procedure

$$\begin{bmatrix} \dfrac{\partial f}{\partial \lambda} & \dfrac{\partial f}{\partial D} \\ \dfrac{\partial g}{\partial \lambda} & \dfrac{\partial g}{\partial D} \end{bmatrix} \begin{bmatrix} \delta\lambda \\ \delta D \end{bmatrix} = \begin{bmatrix} 1 - f^* \\ D(t_n) - g^* \end{bmatrix} = \begin{bmatrix} r^*_\lambda \\ r^*_D \end{bmatrix}$$

$$\begin{aligned}
\frac{\partial f}{\partial \lambda} &= 1 + \Delta t\Gamma + \lambda\Delta t\,\frac{\partial \Gamma}{\partial \lambda} = 1 + \Delta t\Gamma - \lambda\Delta t\frac{G}{\eta^2}\,\frac{\partial \eta}{\partial \bar{\sigma}}\,\frac{\partial \bar{\sigma}}{\partial \lambda} \\
&= 1 + \Delta t\Gamma - \lambda\Delta t\frac{G}{\eta^2}\left[\eta\left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0}\right)\right]\bar{\sigma} \\
\frac{\partial f}{\partial D} &= \lambda\Delta t\,\frac{\partial \Gamma}{\partial D} = -\lambda\Delta t\frac{G}{\eta^2}\,\frac{\partial \eta}{\partial D} = \lambda\Delta t\frac{G}{\eta^2}\,\eta = \lambda\Delta t\Gamma \\
\frac{\partial g}{\partial \lambda} &= -\Delta t\,\frac{\partial \dot{D}}{\partial \lambda} = -\Delta t\,\frac{\partial \dot{D}}{\partial \bar{\sigma}}\,\frac{\partial \bar{\sigma}}{\partial \lambda} = -\Delta t\,\left[\frac{\dot{D}}{\sqrt{3}\tau_0}\right]\bar{\sigma} \\
\frac{\partial g}{\partial D} &= 1 - \Delta t\,\frac{\partial \dot{D}}{\partial D} = 1 - \Delta t\,\left[\dot{D} - \frac{h\bar{\sigma}}{\sqrt{6}D_\infty \eta}\right]
\end{aligned}$$

Matrix/column notation

The tensors and vectors in the presented mathematics can be written in components w.r.t. a vector basis. The components are stored in columns and matrices and the tensor formulations are transferred into matrix/column formulations which can be implemented rather straightforwardly in a computer code.

$$J = \det(\underline{F}) \quad \rightarrow \quad \underline{\tilde{F}} = J^{-1/3}\underline{F} \quad \rightarrow \quad \underline{\tilde{B}} = \underline{\tilde{F}}\,\underline{\tilde{F}}^T \quad \rightarrow \quad \underline{w} = H\underline{\tilde{B}}^d$$

$$p = \kappa(J-1) \quad \rightarrow \quad \underline{s}^h = p\underline{I}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \lambda = 1/(1+\Delta t\Gamma) \\[2ex] \frac{1}{\Delta t}\left\{D - D(t_n)\right\} = \dot{D} \end{array}\right\} \quad \rightarrow \quad \lambda, D \\[2ex] \underline{C}_{p_n} = (1-\lambda)\underline{\tilde{C}}_n + \lambda\underline{C}_{p_n}(t_n) \\[2ex] \underline{\bar{\tilde{B}}}_{e_n} = \underline{\tilde{U}}_n\,\underline{C}_{p_n}^{-1}\,\underline{\tilde{U}}_n^T \\[2ex] \underline{\bar{s}}^d = G\underline{\bar{\tilde{B}}}_{e_n} \quad \rightarrow \quad \bar{s} = \sqrt{\frac{3}{2}\mathrm{tr}(\underline{\bar{s}}^d\,\underline{\bar{s}}^d)} \\[2ex] \eta = \eta(\bar{s}, p, T, D) \end{array}\right\} \rightarrow \underline{\bar{\tilde{B}}}_{e_n} \rightarrow \left\{\begin{array}{l} \underline{\tilde{B}}_e = \underline{R}_n\,\underline{\bar{\tilde{B}}}_{e_n}\,\underline{R}_n^T \rightarrow \\[2ex] \underline{s}^d = G\underline{\tilde{B}}_e^d \end{array}\right.$$

$$\underline{\sigma} = \underline{s}^d + \underline{s}^h + \underline{w}$$

6.4.2 Stiffness

The stress is related to the elastic isochoric strain $\tilde{\boldsymbol{B}}_e$, the volume change J and the total isochoric strain $\tilde{\boldsymbol{B}}$. Each of the three quantities will be considered separately and relations between their variations and $\delta\boldsymbol{F}$ will be derived.

The consistent material stiffness tensor relates the iterative change of the Cauchy stress tensor $\delta\boldsymbol{\sigma}$ to the iterative displacement $\delta\vec{u}$. In the derivation of this relation it is assumed that approximate end-increment values of all relevant variables are known. ${}^4\boldsymbol{S}_d$, ${}^4\boldsymbol{S}_h$ and ${}^4\boldsymbol{H}$ are properly defined fourth-order tensors.

$$\left.\begin{array}{l} \boldsymbol{\sigma} = \boldsymbol{s}^d + \boldsymbol{s}^h + \boldsymbol{w} = G\tilde{\boldsymbol{B}}_e^d + \kappa\boldsymbol{I}(J-1) + H\tilde{\boldsymbol{B}}^d \\[2ex] \quad \left.\begin{array}{l} \tilde{\boldsymbol{B}}_e = \tilde{\boldsymbol{F}}\cdot\boldsymbol{C}_p^{-1}\cdot\tilde{\boldsymbol{F}}^c \\ \boldsymbol{C}_p = (1-\lambda)\tilde{\boldsymbol{C}} + \lambda\boldsymbol{C}_p(t_n) \\ \tilde{\boldsymbol{F}} = J^{-1/3}\boldsymbol{F} \end{array}\right\} \end{array}\right\}$$

$$
\begin{aligned}
\delta \boldsymbol{\sigma} &= \delta \boldsymbol{s}^d + \delta \boldsymbol{s}^h + \delta \boldsymbol{w} \\
&= G\,\delta \tilde{\boldsymbol{B}}_e^d + \kappa\,\boldsymbol{I}\delta J + H\,\delta \tilde{\boldsymbol{B}}^d = \left({}^4\boldsymbol{S}_d + {}^4\boldsymbol{S}_h + {}^4\boldsymbol{H}\right) : \delta \boldsymbol{F} \\
&= {}^4\boldsymbol{S} : \delta \boldsymbol{F} = {}^4\boldsymbol{S}^{rc} : \delta \boldsymbol{F}^c \quad \text{with} \qquad \delta \boldsymbol{F}^c = \vec{\nabla}_0 \vec{u} = \boldsymbol{F}^c \cdot \vec{\nabla} \vec{u} = \boldsymbol{F}^c \cdot \boldsymbol{L}_u^c \\
&= {}^4\boldsymbol{S}^{rc} : (\boldsymbol{F}^c \cdot \boldsymbol{L}_u^c) \\
&= {}^4\boldsymbol{M} : \boldsymbol{L}_u^c
\end{aligned}
$$

Elastic strain variation

The elastic strain $\tilde{\boldsymbol{B}}_e$ must be calculated from the total deformation $\tilde{\boldsymbol{F}}$ and the plastic strain $\boldsymbol{C}_p$. Its variation is related to $\delta\tilde{\boldsymbol{F}}$ and $\delta\boldsymbol{C}_p$ which will be considered separately.

$$
\begin{aligned}
\tilde{\boldsymbol{B}}_e &= \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \tilde{\boldsymbol{F}}^c \\
\delta \tilde{\boldsymbol{B}}_e &= \delta \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \tilde{\boldsymbol{F}}^c - \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \delta \boldsymbol{C}_p \cdot \boldsymbol{C}_p^{-1} \cdot \tilde{\boldsymbol{F}}^c + \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \delta \tilde{\boldsymbol{F}}^c \\
&= \left(\tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-c} \cdot \delta \tilde{\boldsymbol{F}}^c\right)^c - \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \left(\tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-c} \cdot \delta \boldsymbol{C}_p^c\right)^c + \tilde{\boldsymbol{F}} \cdot \boldsymbol{C}_p^{-1} \cdot \delta \tilde{\boldsymbol{F}}^c \\
&= \left(\boldsymbol{M}^{(1)} \cdot \delta \tilde{\boldsymbol{F}}^c\right)^c - \boldsymbol{M}^{(2)} \cdot \left(\boldsymbol{M}^{(1)} \cdot \delta \boldsymbol{C}_p^c\right)^c + \boldsymbol{M}^{(2)} \cdot \delta \tilde{\boldsymbol{F}}^c
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\boldsymbol{B}}_e^d &= \tilde{\boldsymbol{B}}_e - \tfrac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{B}}_e)\boldsymbol{I} = \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \tilde{\boldsymbol{B}}_e \\
\delta \tilde{\boldsymbol{B}}_e^d &= \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \delta \tilde{\boldsymbol{B}}_e
\end{aligned}
$$

Plastic strain variation

The variation of the plastic strain $\boldsymbol{C}_p$ is related to $\delta\tilde{\boldsymbol{F}}$ (via $\delta\tilde{\boldsymbol{C}}$) and $\delta\lambda$. These variations will be considered separately.

$$
\begin{aligned}
\boldsymbol{C}_p &= (1-\lambda)\tilde{\boldsymbol{C}} + \lambda \boldsymbol{C}_p(t_n) \\
\delta \boldsymbol{C}_p &= (1-\lambda)\delta\tilde{\boldsymbol{C}} + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda \\
&= (1-\lambda)\left(\delta\tilde{\boldsymbol{F}}^c \cdot \tilde{\boldsymbol{F}} + \tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right) + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda \\
&= (1-\lambda)\left[\left(\tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right)^c + \tilde{\boldsymbol{F}}^c \cdot \delta\tilde{\boldsymbol{F}}\right] + \left(\boldsymbol{C}_p(t_n) - \tilde{\boldsymbol{C}}\right)\delta\lambda
\end{aligned}
$$

Deformation tensor variation

The variation of the isochoric deformation tensor $\tilde{\boldsymbol{F}}$ can be expressed in the variation of the total deformation tensor $\boldsymbol{F}$. The volume ratio J is assumed to be constant in this variation.

$$
\begin{aligned}
\tilde{\boldsymbol{F}} &= J^{-1/3}\boldsymbol{F} \\
\delta\tilde{\boldsymbol{F}} &= -\tfrac{1}{6}J^{-1/3}\boldsymbol{F}\boldsymbol{I} : \left(\delta\boldsymbol{F} \cdot \boldsymbol{F}^{-1} + \boldsymbol{F}^{-c} \cdot \delta\boldsymbol{F}^c\right) + J^{-1/3}\delta\boldsymbol{F} \\
&= -\tfrac{1}{3}J^{-1/3}\boldsymbol{F}\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) + J^{-1/3}\delta\boldsymbol{F}
\end{aligned}
$$

Elasticity scalar variation

The variation $\delta\lambda$ of the elasticity parameter λ can be expressed in $\delta\tilde{\boldsymbol{B}}_e$ and $\delta\boldsymbol{F}$, starting from

$$\lambda = \frac{1}{1+\Delta t\Gamma} = \frac{\eta}{\eta + G\Delta t} \quad \rightarrow \quad \delta\lambda = \frac{\lambda\Delta t\Gamma}{G\Delta t+\eta}\,\delta\eta$$

The variation $\delta\eta$ can be written as :

$$\begin{aligned}
\delta\eta &= \frac{\partial\eta}{\partial\bar{\sigma}}\,\delta\bar{\sigma} + \frac{\partial\eta}{\partial p}\,\delta p + \frac{\partial\eta}{\partial D}\,\frac{\partial D}{\partial\bar{\sigma}}\,\delta\bar{\sigma} + \frac{\partial\eta}{\partial D}\,\frac{\partial D}{\partial p}\,\delta p \\
&\qquad \delta\bar{\sigma} = \frac{3G^2}{2\bar{\sigma}}\,\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e^d = \frac{3G^2}{2\bar{\sigma}}\,\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e \\
&\qquad \delta p = -\kappa J \mathrm{tr}(\delta\boldsymbol{F}) = -\kappa J\boldsymbol{I} : \delta\boldsymbol{F} \\
&= \frac{3G^2}{2\bar{\sigma}}\left(\frac{\partial\eta}{\partial\bar{\sigma}} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial\bar{\sigma}}\right)\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e - \kappa J\left(\frac{\partial\eta}{\partial p} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial p}\right)\boldsymbol{I} : \delta\boldsymbol{F} \\
&= h_1\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e + h_2\boldsymbol{I} : \delta\boldsymbol{F}
\end{aligned}$$

A number of partial derivatives must be calculated to determine h_1 and h_2.

$$\delta\lambda = \frac{\lambda\Delta t\Gamma}{G\Delta t+\eta}\,\delta\eta = l_1\tilde{\boldsymbol{B}}_e^d : \delta\tilde{\boldsymbol{B}}_e + l_2\boldsymbol{I} : \delta\boldsymbol{F}$$

$$\begin{aligned}
& l_1 = \frac{\lambda\Delta t\,\Gamma\,h_1}{\Delta t\,G+\eta} \quad ; \quad l_2 = \frac{l_1 h_2}{h_1} \\
& h_1 = \frac{3G^2}{2\bar{\sigma}}\left(\frac{\partial\eta}{\partial\bar{\sigma}} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial\bar{\sigma}}\right) \quad ; \quad h_2 = -\kappa J\left(\frac{\partial\eta}{\partial p} + \frac{\partial\eta}{\partial D}\frac{\partial D}{\partial p}\right) \\
& \frac{\partial\eta}{\partial\bar{\sigma}} = \eta\left(\frac{1}{\bar{\sigma}} - \frac{1}{\sqrt{3}\tau_0}\right) \quad ; \quad \frac{\partial\eta}{\partial p} = \frac{\eta\mu}{\tau_0} \quad ; \quad \frac{\partial\eta}{\partial D} = -\eta \\
& \frac{\partial D}{\partial\bar{\sigma}} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial\bar{\sigma}}}{1-\Delta t\,\frac{\partial\dot{D}}{\partial D}} \quad ; \quad \frac{\partial D}{\partial p} = \frac{\Delta t\,\frac{\partial\dot{D}}{\partial p}}{1-\Delta t\,\frac{\partial\dot{D}}{\partial D}} \\
& \frac{\partial\dot{D}}{\partial\bar{\sigma}} = \frac{\dot{D}}{\sqrt{3}\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial p} = -\frac{\dot{D}\mu}{\tau_0} \quad ; \quad \frac{\partial\dot{D}}{\partial D} = \dot{D} - \frac{h\bar{\sigma}}{\sqrt{6}\,D_\infty\,\eta} \\
& \text{with} \qquad \dot{D} = h\left(1-\frac{D}{D_\infty}\right)\frac{\bar{\sigma}}{\sqrt{6}\,\eta}
\end{aligned}$$

Deviatoric stress variation

The variation of the deviatoric stress tensor is related to $\delta\tilde{\boldsymbol{B}}_e^d$ and subsequently to $\delta\boldsymbol{F}$:

$$\delta\boldsymbol{s}^d = G\,\delta\tilde{\boldsymbol{B}}_e^d = {}^4\boldsymbol{S}_d : \delta\boldsymbol{F}$$

Hydrostatic stress variation

The variation of the hydrostatic stress $\boldsymbol{s}^h$ is related to the variation of the volume factor J. The latter can be related to the variation of $\boldsymbol{F}$, resulting in a relation between $\delta\boldsymbol{s}^h$ and $\delta\boldsymbol{F}$.

$$\delta\boldsymbol{s}^h = \kappa\,\boldsymbol{I}\delta J = {}^4\boldsymbol{S}_h : \delta\boldsymbol{F}$$

$$\begin{aligned}
\dot{J} &= J\,\mathrm{tr}(\boldsymbol{D}) = J\,\tfrac{1}{2}\,\mathrm{tr}\left\{\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + \left(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\right)^c\right\} \quad \rightarrow \\
\delta J &= \tfrac{1}{2}J\,\mathrm{tr}\left(\delta\boldsymbol{F}\cdot\boldsymbol{F}^{-1} + \boldsymbol{F}^{-c}\cdot\delta\boldsymbol{F}^c\right) = \tfrac{1}{2}J\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) + \tfrac{1}{2}J\left(\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c\right) \\
&= J\,\boldsymbol{F}^{-c} : \delta\boldsymbol{F}^c = J\,\boldsymbol{F}^{-1} : \delta\boldsymbol{F}
\end{aligned}$$

Hardening stress variation

The hardening stress $\boldsymbol{w}$ is related to the deviatoric total volume invariant strain $\tilde{\boldsymbol{B}}^d$. The variation $\delta\boldsymbol{w}$ can be related to $\delta\boldsymbol{F}$.

$$\delta\boldsymbol{w} = H\,\delta\tilde{\boldsymbol{B}}^d = {}^4\boldsymbol{H} : \delta\boldsymbol{F}$$

$$\begin{aligned}
\tilde{\boldsymbol{B}} &= \tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}^c \\
\delta\tilde{\boldsymbol{B}} &= \delta\tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}^c + \tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c
\end{aligned}$$

$$\begin{aligned}
\tilde{\boldsymbol{B}}^d &= \tilde{\boldsymbol{B}} - \tfrac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{B}})\boldsymbol{I} = \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \tilde{\boldsymbol{B}} \\
\delta\tilde{\boldsymbol{B}}^d &= \left({}^4\boldsymbol{I} - \tfrac{1}{3}\boldsymbol{I}\boldsymbol{I}\right) : \left\{\left(\tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c\right)^c + \tilde{\boldsymbol{F}}\cdot\delta\tilde{\boldsymbol{F}}^c\right\}
\end{aligned}$$

6.4.3 Consistent material stiffness tensor

The variation of the Cauchy stress $\delta\boldsymbol{\sigma}$ is related to the variation of the deformation tensor $\delta\boldsymbol{F}$. In the iterative weighted residual equation $\delta\boldsymbol{\sigma}$ must be related to the gradient of the iterative displacement $\boldsymbol{L}_u = (\vec{\nabla}\vec{u})^c = (\vec{\nabla}\delta\vec{x})^c$. The resulting fourth-order tensor ${}^4\boldsymbol{M}$ is the consistent material stiffness tensor.

The components of $\delta\boldsymbol{\sigma}$ (in column $\delta\underset{\approx}{\sigma}$) and $\boldsymbol{L}_u$ (in column $\underset{\approx}{L}_u$) are related by the consistent stiffness matrix $\underline{\underline{M}}$.

$$\begin{aligned}
\delta\boldsymbol{\sigma} &= \delta\boldsymbol{s}^d + \delta\boldsymbol{s}^h + \delta\boldsymbol{w} \\
&= \left({}^4\boldsymbol{S}_d + {}^4\boldsymbol{S}_h + {}^4\boldsymbol{H}\right) : \delta\boldsymbol{F} = {}^4\boldsymbol{S} : \delta\boldsymbol{F} = {}^4\boldsymbol{S}^{rc} : \delta\boldsymbol{F}^c \\
&\qquad\text{with}\quad \delta\boldsymbol{F}^c = \vec{\nabla}_0\vec{u} = \boldsymbol{F}^c\cdot\vec{\nabla}\vec{u} = \boldsymbol{F}^c\cdot\boldsymbol{L}_u^c \quad \rightarrow \\
&= {}^4\boldsymbol{S}^{rc} : (\boldsymbol{F}^c\cdot\boldsymbol{L}_u^c) = {}^4\boldsymbol{M} : \boldsymbol{L}_u^c
\end{aligned}$$

6.4.4 Matrix/column notation

$$\delta\underset{\approx}{\tilde{B}}_e = \left(\underline{\underline{M}}^{(1)}_{cr} + \underline{\underline{M}}^{(2)}_{c}\right)\delta\underset{\approx}{\tilde{F}} - \underline{\underline{M}}^{(2)}_{c}\underline{\underline{M}}^{(1)}_{c}\delta\underset{\approx}{C}_p \quad ; \quad \underline{\underline{M}}^{(1)} = \underline{\underline{\tilde{F}}}\,\underline{\underline{C}}_p^{-T} \ ; \ \underline{\underline{M}}^{(2)} = \underline{\underline{\tilde{F}}}\,\underline{\underline{C}}_p^{-1}$$
$$= \underline{\underline{A}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\delta\underset{\approx}{C}_p$$

$$\delta\underset{\approx}{\tilde{B}}{}^d_e = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}^T_t\right)\delta\underset{\approx}{\tilde{B}}_e$$

$$\delta\underset{\approx}{C}_p = \left[(1-\lambda)\left(\underline{\underline{\tilde{F}}}_{tr} + \underline{\underline{\tilde{F}}}_t\right)\right]\delta\underset{\approx}{\tilde{F}} + \left(\underset{\approx}{C}_p(t_n) - \underset{\approx}{\tilde{C}}\right)\delta\lambda = \underline{\underline{C}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underset{\approx}{C}^{(2)}\delta\lambda$$

$$\delta\underset{\approx}{\tilde{F}} = \left[-\tfrac{1}{3}J^{-1/3}\underset{\approx}{F}\left(\underset{\approx}{F}^{-1}\right)^T_t + J^{-1/3}\underline{\underline{I}}\right]\delta\underset{\approx}{F} = \underline{\underline{F}}\,\delta\underset{\approx}{F}$$

$$\delta\lambda = = l_1\left(\underset{\approx}{\tilde{B}}{}^d_e\right)^T_t\delta\underset{\approx}{\tilde{B}}_e + l_2\underset{\approx}{I}^T_t\delta\underset{\approx}{F}$$

6.4.5 Matrix/column notation

$$\delta\underset{\approx}{\tilde{B}}_e = \underline{\underline{A}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\delta\underset{\approx}{C}_p = \left(\underline{\underline{A}}^{(1)} + \underline{\underline{A}}^{(2)}\underline{\underline{C}}^{(1)}\right)\delta\underset{\approx}{\tilde{F}} + \underline{\underline{A}}^{(2)}\underset{\approx}{C}^{(2)}\delta\lambda = \underline{\underline{B}}^{(1)}\delta\underset{\approx}{\tilde{F}} + \underset{\approx}{B}^{(2)}\delta\lambda$$
$$= \underline{\underline{B}}^{(1)}\underline{\underline{F}}\delta\underset{\approx}{F} + l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}^d_e\right)^T_t\delta\underset{\approx}{\tilde{B}}_e + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}^T_t\delta\underset{\approx}{F}$$

$$\delta\underset{\approx}{\tilde{B}}_e = \left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}^d_e\right)^T_t\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}^T_t\right]\delta\underset{\approx}{F}$$

$$\delta\underset{\approx}{\tilde{B}}{}^d_e = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}^T_t\right)\delta\underset{\approx}{\tilde{B}}_e = \left(\underline{\underline{I}} - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}^T_t\right)\left[\underline{\underline{I}} - l_1\underset{\approx}{B}^{(2)}\left(\underset{\approx}{\tilde{B}}{}^d_e\right)^T_t\right]^{-1}\left[\underline{\underline{B}}^{(1)}\underline{\underline{F}} + l_2\underset{\approx}{B}^{(2)}\underset{\approx}{I}^T_t\right]\delta\underset{\approx}{F} = \underline{\underline{B}}^{(3)}\delta\underset{\approx}{F}$$

$$\delta\underset{\approx}{\tilde{B}}{}^d = \left(\underline{\underline{I}}_c - \tfrac{1}{3}\underset{\approx}{I}\underset{\approx}{I}^T_t\right)\left(\underline{\underline{\tilde{F}}}_{cr} + \underline{\underline{\tilde{F}}}_c\right)\delta\underset{\approx}{F}$$
$$= \left(\underline{\underline{\tilde{F}}}_{cr} + \underline{\underline{\tilde{F}}}_c - \tfrac{2}{3}\underset{\approx}{I}\underset{\approx}{I}^T_t\underline{\underline{\tilde{F}}}_c\right)\underline{\underline{F}}\delta\underset{\approx}{F} = \underline{\underline{B}}^{(4)}\delta\underset{\approx}{F}$$

6.4.6 Matrix/column notation

The components of the deviatoric, hydrostatic and hardening stress tensors are now stored in columns.

$$\delta\underset{\approx}{s}^d = G\delta\underset{\approx}{\tilde{B}}{}^d_e = G\underline{\underline{B}}^{(3)}\delta\underset{\approx}{F} = \underline{\underline{S}}_d\delta\underset{\approx}{F}$$

$$\delta\underset{\approx}{s}^h = \kappa\,\underset{\approx}{I}\delta J = \kappa J\,\underset{\approx}{I}\left(\underset{\approx}{F}^{-1}\right)^T_t\delta\underset{\approx}{F} = \underline{\underline{S}}_h\delta\underset{\approx}{F}$$

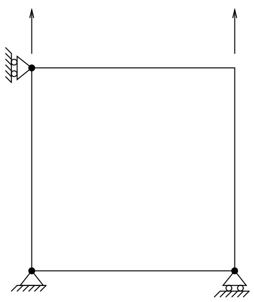
$$\delta\underset{\approx}{w} = H\,\delta\tilde{\underset{\approx}{B}}^d = H\,\underline{\underline{B}}^{(4)}\delta\underset{\approx}{F} = \underline{\underline{H}}\delta\underset{\approx}{F}$$

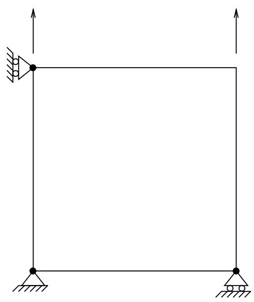
$$\begin{aligned}
\delta\underset{\approx}{\sigma} &= \delta\underset{\approx}{s}^d + \delta\underset{\approx}{s}^h + \delta\underset{\approx}{w} \\
&= \left(\underline{\underline{S}}_d + \underline{\underline{S}}_h + \underline{\underline{H}}\right)\delta\underset{\approx}{F} = \underline{\underline{S}}\delta\underset{\approx}{F} = \underline{\underline{S}}_c\,\delta\underset{\approx}{F}_t \quad \text{with} \quad \delta\underset{\approx}{F}_t = \underline{\underline{F}}_t\left(\underset{\approx}{L}_u\right)_t \\
&= \underline{\underline{S}}_c\underline{\underline{F}}_t\left(\underset{\approx}{L}_u\right)_t \\
&= \underline{\underline{M}}\left(\underset{\approx}{L}_u\right)_t
\end{aligned}$$

6.5 Examples

6.5.1 Tensile test

A square plate or cylindrical bar is loaded uniaxially. Dimensions are listed in the table.





initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

initial radius	r_0	$\sqrt{(10/\pi)}$	mm
initial height	h_0	100	mm

Viscoelastic model in tensile test

The axial elongation is prescribed with a constant elongation rate. The axial stress and force are calculated for polycarbonate (PC). Parameter values are listed in the table. The deformation is assumed to be plane strain.

E	2305	MPa	ν	0.37	-
H	29	MPa	h	270	-
D_∞	19	-	A_0	9.7573E-27	s
ΔH	2.9E5	J/mol	μ	0.06984	-
τ_0	0.72	MPa			

elongation rate $$\frac{\dot{\Delta l}}{h_0} = 0.01 \text{ s}^{-1}$$

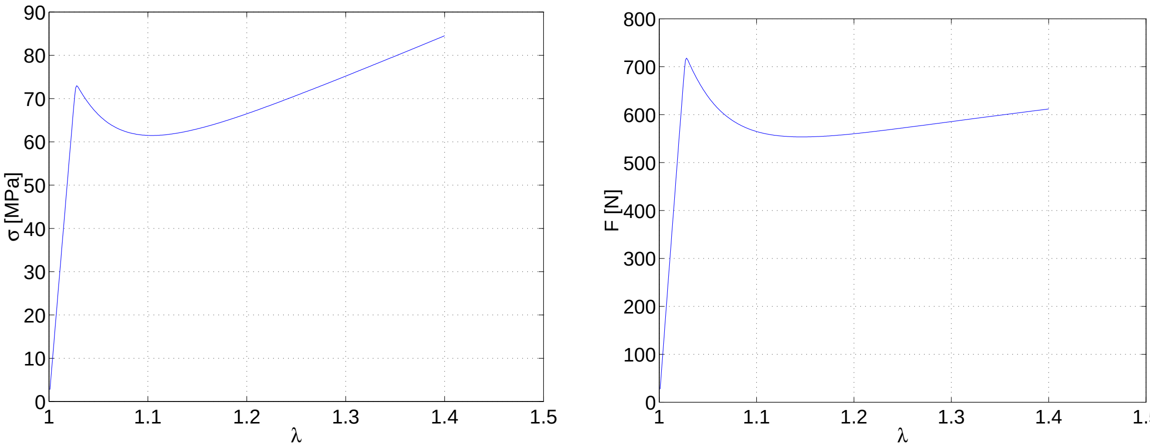
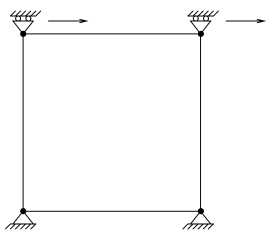


Fig. 6.32 : *Axial stress and force versus elongation*

6.5.2 Shear test

The simple shear test is analyzed with one element, where the horizontal displacement in the upper nodes is prescribed. Because there are no unknown degrees of freedom, the stiffness matrix is not used. Only strains, stresses and reaction forces are calculated. Material parameters for polycarbonate (PC) are used. The prescribed strain rate is constant.



initial width	w_0	100	mm
initial height	h_0	100	mm
initial thickness	d_0	0.1	mm

strain rate $$\dot{\gamma} = \frac{\dot{u}}{h_0} = 0.01 \text{ s}^{-1}$$

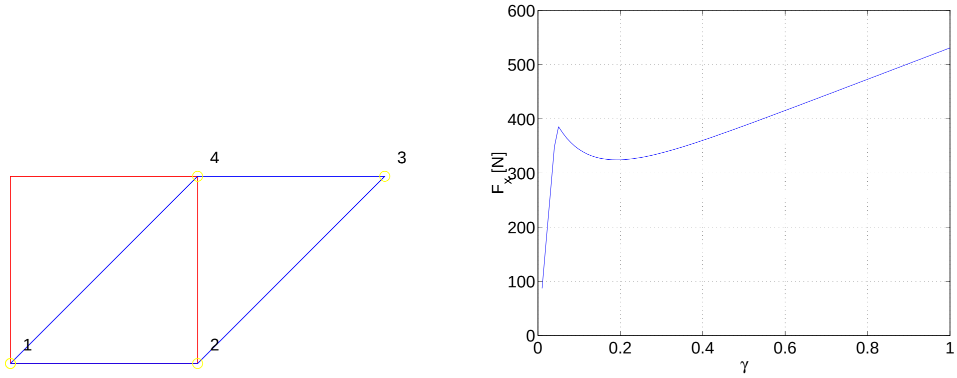


Fig. 6.33 : *Shear force versus shear strain for plane strain*