ANALYTICAL SOLUTIONS

1 Analytical solutions

In the following sections we present various problems, which have an analytical solution. The equations are presented and the solution is given without extensive derivations. Many problems involve the calculation of integration constants from boundary conditions. For such problems these integration constants can be found in appendix B. Examples with numerical values for parameters, are presented. More examples can be found in the above-mentioned appendix.

1.1 Cartesian, planar

For planar problems in a Cartesian coordinate system, two partial differential equations for the displacement components u_x and u_y, the so-called Navier equations, have to be solved, using specific boundary conditions. Only for very simply cases, this can be done analytically. For practical problems, approximate solutions have to be determined with numerical solution procedures. The Navier equations have been derived in section **??** and are repeated below for the static case, where no material acceleration is considered. The material parameters A_p, B_p, Q_p and K have to be specified for plane stress or plane strain and for the material model concerned (see section **??** and appendix **??**.

$$A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = 0$$

$$K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = 0$$

1.1.1 Tensile test

When a square plate (length a) of homogeneous material is loaded uniaxially by a uniform tensile edge load p, this load constitutes an equilibrium system, i.e. the stresses satisfy the equilibrium equations : $\sigma_{xx} = p$ and $\sigma_{yy} = \sigma_{xy} = \sigma_{zz} = 0$. The deformation can be calculated directly from Hooke's law.

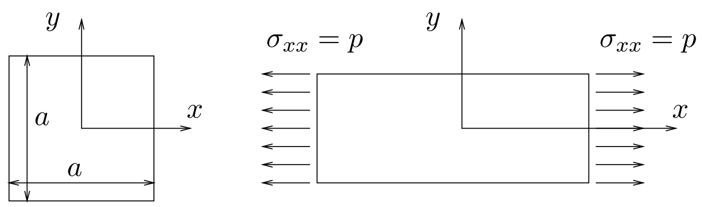


Fig. 1.1 : *Uniaxial tensile test*

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx} = \frac{p}{E} \quad \rightarrow \quad u_x = \frac{p}{E}x + c \quad ; \quad u_x(x=0) = 0 \rightarrow c = 0$$

$$u_x = \frac{p}{E} x \quad \rightarrow \quad u_x(x = a) = \frac{p}{E} a$$

$$\varepsilon_{yy} = -\nu \varepsilon_{xx} = -\nu \frac{p}{E} \quad \rightarrow \quad u_y = -\nu \frac{p}{E} y + c \quad ; \quad u_y(y = 0) = 0 \quad \rightarrow \quad c = 0$$

$$u_y = -\nu \frac{p}{E} y \quad ; \quad u_y(y = a/2) = -\nu \frac{p}{E} \frac{a}{2}$$

1.1.2 Orthotropic plate

A square plate is loaded in its plane so that a plane stress state can be assumed with $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. The plate material is a "matrix" in which long fibers are embedded, which have all the same orientation along the direction indicated as 1 in the "material" $1,2$-coordinate system. Both matrix and fibers are linearly elastic. The volume fraction of the fibers is V. The angle between the 1-direction and the x-axis is α. In the $1,2$-coordinate system the material behavior for plane stress is given by the orthotropic material law, which is found in appendix **??**.

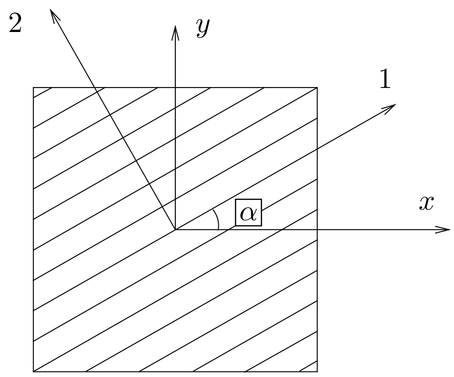


Fig. 1.2 : *Orthotropic plate*

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{1 - \nu_{12}\nu_{21}} \begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1 - \nu_{12}\nu_{21})G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{\sigma}^* = \underline{\underline{C}}^* \underset{\sim}{\varepsilon}^*$$

In appendix **??** the transformation of matrix components due to a rotation of the coordinate axes is described for the three-dimensional case. For planar deformation, the anticlockwise rotation is only about the global z-axis. For stress and strain components, stored in columns $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$, respectively, the transformation is described by transformation matrices $\underline{T}_\sigma$ for stress and $\underline{T}_\varepsilon$ for strain. The components of these matrices are cosine (c) and sine (s) functions of the rotation angle α, which is positive for an anti-clockwise rotation about the z-axis.

The properties in the material coordinate system are known. The stress-strain relations in the global coordinate system can than be calculated.

$$\underset{\sim}{\sigma}^* = \left[\begin{array}{ccc} \sigma_{11} & \sigma_{22} & \sigma_{12} \end{array}\right]^T \qquad \underset{\sim}{\varepsilon}^* = \left[\begin{array}{ccc} \varepsilon_{11} & \varepsilon_{22} & \gamma_{12} \end{array}\right]^T$$

$$\underset{\sim}{\sigma} = \left[\begin{array}{ccc} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{array}\right]^T \qquad \underset{\sim}{\varepsilon} = \left[\begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{yy} & \gamma_{xy} \end{array}\right]^T$$

$$\underline{T}_\sigma = \left[\begin{array}{ccc} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{array}\right] \qquad \underline{T}_\sigma^{-1} = \left[\begin{array}{ccc} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{array}\right]$$

$$\underline{T}_\varepsilon = \left[\begin{array}{ccc} c^2 & s^2 & cs \\ s^2 & c^2 & -cs \\ -2cs & 2cs & c^2 - s^2 \end{array}\right] \qquad \underline{T}_\varepsilon^{-1} = \left[\begin{array}{ccc} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{array}\right]$$

$$\underset{\sim}{\sigma}^* = \underline{T}_\sigma \, \underset{\sim}{\sigma} \qquad \underset{\sim}{\varepsilon}^* = \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon}$$

$$\begin{array}{lllll} \underset{\sim}{\sigma}^* = \underline{\underline{C}}^* \, \underset{\sim}{\varepsilon}^* & \rightarrow & \underline{T}_\sigma \underset{\sim}{\sigma} = \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} & \rightarrow & \underset{\sim}{\sigma} = \underline{T}_\sigma^{-1} \, \underline{\underline{C}}^* \, \underline{T}_\varepsilon \, \underset{\sim}{\varepsilon} = \underline{\underline{C}} \, \underset{\sim}{\varepsilon} \\ \underset{\sim}{\varepsilon}^* = \underline{\underline{S}}^* \, \underset{\sim}{\sigma}^* & \rightarrow & \underline{T}_\varepsilon \underset{\sim}{\varepsilon} = \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} & \rightarrow & \underset{\sim}{\varepsilon} = \underline{T}_\varepsilon^{-1} \, \underline{\underline{S}}^* \, \underline{T}_\sigma \, \underset{\sim}{\sigma} = \underline{\underline{S}} \, \underset{\sim}{\sigma} \end{array}$$

Example : stiffness of an orthotropic plate

The material parameters in the material 1, 2-coordinate system are known from experiments :

$$E_1 = 100 \text{ N/mm}^2 \ ; \ E_2 = 20 \text{ N/mm}^2 \ ; \ G_{12} = 50 \text{ N/mm}^2 \ ; \ \nu_{12} = 0.4$$

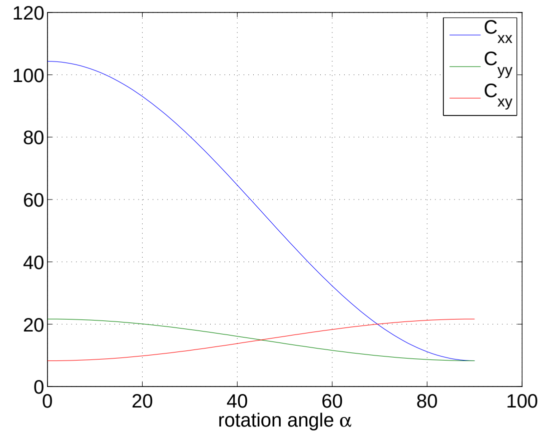
Due to symmety of the stiffness matrix (and of course the compliance matrix), the second Poisson ratio can be calculated :

$$\nu_{12} E_2 = \nu_{21} E_1 \quad \rightarrow \quad \nu_{21} = \nu_{12} \frac{E_2}{E_1} = (0.4) * (20/100) = 0.08$$

The stiffness matrix in the material coordinate system can than be calculated. When the material coordinate system is rotated over $\alpha = 20^o$ anti-clockwise w.r.t. the global x-axis, the transformation matrices can be generated and used to calculate the stiffness matrix w.r.t. the global axes.

$$\underline{\underline{C}}^* = \left[\begin{array}{ccc} 103.3058 & 8.2645 & 0 \\ 8.2645 & 20.6612 & 0 \\ 0 & 0 & 50.0000 \end{array}\right] \quad ; \quad \underline{\underline{C}} = \left[\begin{array}{ccc} 93.0711 & 9.8316 & -31.8180 \\ 19.4992 & 20.0940 & 31.8180 \\ 29.9029 & -3.3414 & 39.0683 \end{array}\right]$$

We can also concentrate on components of $\underline{\underline{C}}$ and investigate how they changes, when the rotation angle α varies within a certain range. The next plot shows C_{xx}, C_{yy} and C_{xy} as a function of α.



1.2 Axi-symmetric, planar, $u_t = 0$

The differential equation for the radial displacement u_r is derived in chapter **??** by substitution of the stress-strain relation (material law) and the strain-displacement relation in the equilibrium equation w.r.t. the radial direction. It is repeated here for orthotropic material behavior with isotropic thermal expansion. Material parameters A_p and Q_p have to be specified for plane stress and plane strain and can be found in appendix **??**.

$$u_{r,rr} + \frac{1}{r}\,u_{r,r} - \zeta^2\,\frac{1}{r^2}\,u_r = f(r)$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r}$$

A general solution for the differential equation can be determined as the addition of the homogeneous solution $\hat{u}_r$ and the particulate solution $\bar{u}_r$, which depends on the specific loading $f(r)$. From the general solution the radial and tangential strains can be calculated according to their definitions.

$$\hat{u}_r = r^\lambda \quad \rightarrow \quad \hat{u}_{r,r} = \lambda\,r^{\lambda-1} \quad \rightarrow \quad \hat{u}_{r,rr} = \lambda(\lambda-1)\,r^{\lambda-2} \quad \rightarrow$$
$$\left[\lambda(\lambda-1) + \lambda - \zeta^2\right] r^{\lambda-2} = 0 \quad \rightarrow$$
$$\lambda^2 = \zeta^2 \quad \rightarrow \quad \lambda = \pm\zeta \quad \rightarrow \quad \hat{u}_r = c_1\,r^\zeta + c_2 r^{-\zeta}$$

$$u_r = c_1\,r^\zeta + c_2 r^{-\zeta} + \bar{u}_r$$

For the isotropic case, this can be done also as follows, where the result includes an expression for the part $\bar{u}_r$.

$$u_{r,rr} + \frac{1}{r}\, u_{r,r} - \frac{1}{r^2}\, u_r = f(r)$$

$$\text{with} \qquad f(r) = \frac{\rho}{A_p}\, (\ddot{u}_r - q_r)$$

$$\begin{aligned}
&\hat{u}_r = r^\lambda \quad \rightarrow \quad \hat{u}_{r,r} = \lambda\, r^{\lambda-1} \quad \rightarrow \quad \hat{u}_{r,rr} = \lambda(\lambda-1)\, r^{\lambda-2} \quad \rightarrow \\
&[\lambda(\lambda-1) + \lambda - 1]\, r^{\lambda-2} = 0 \quad \rightarrow \\
&\lambda^2 = 1 \quad \rightarrow \quad \lambda = \pm 1 \quad \rightarrow \quad \hat{u}_r = c_1\, r + \frac{c_2}{r}
\end{aligned}$$

$$u_r = c_1\, r + \frac{c_2}{r} + \bar{u}_r$$

Orthotropic material

The general solution for radial displacement, strains and stresses is presented here.

$$\text{general solution} \qquad u_r = c_1 r^\zeta + c_2 r^{-\zeta} + \bar{u}_r$$

$$\begin{aligned}
\varepsilon_{rr} &= c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} + \bar{u}_{r,r} \\
\varepsilon_{tt} &= c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}
\end{aligned}$$

$$\begin{aligned}
\sigma_{rr} &= (A_p \zeta + Q_p) c_1 r^{\zeta-1} - (A_p \zeta - Q_p) c_2 r^{-\zeta-1} + A_p \bar{u}_{r,r} + Q_p \frac{\bar{u}_r}{r} - (\Theta_{p1}) \alpha \Delta T \\
\sigma_{tt} &= (Q_p \zeta + B_p) c_1 r^{\zeta-1} - (Q_p \zeta - B_p) c_2 r^{-\zeta-1} + Q_p \bar{u}_{r,r} + B_p \frac{\bar{u}_r}{r} - (\Theta_{p2}) \alpha \Delta T
\end{aligned}$$

For isotropic material the relations can be simplified, as in that case we have $A_p = B_p$ and thus $\zeta = 1$. When there is no right-hand loading term $f(r)$ in the differential equation, the particulate part $\bar{u}_r$ will be zero. Then, for isotropic material, the radial and tangential strains are uniform, i.e. no function of the radius r. For a state of plane stress, the axial strain is calculated as a weighted summation of the in-plane strains, so also ε_{zz} will be uniform (see section **??**). The thicknesss of the axi-symmetric object will remain uniform. For non-isotropic material behavior this is not the case, however.

$$u_r = c_1 r + \frac{c_2}{r} + \bar{u}_r$$

$$\begin{aligned}
\varepsilon_{rr} &= u_{r,r} = c_1 - \frac{c_2}{r^2} + \bar{u}_{r,r} \\
\varepsilon_{tt} &= \frac{u_r}{r} = c_1 + \frac{c_2}{r^2} + \frac{\bar{u}_r}{r}
\end{aligned}$$

$$\begin{aligned}
\sigma_{rr} &= A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} \\
&= (A_p + Q_p) c_1 - (A_p - Q_p) \frac{c_2}{r^2} + A_p \bar{u}_{r,r} + Q_p \frac{\bar{u}_r}{r} \\
\sigma_{tt} &= Q_p \varepsilon_{rr} + A_p \varepsilon_{tt} \\
&= (Q_p + A_p) c_1 - (Q_p - A_p) \frac{c_2}{r^2} + Q_p \bar{u}_{r,r} + A_p \frac{\bar{u}_r}{r}
\end{aligned}$$

When there is no right-hand loading term $f(r)$ in the differential equation, the particulate part $\bar{u}_r$ will be zero. Then, for isotropic material, the radial and tangential strains are uniform, i.e. no function of the radius r. For a state of plane stress, the axial strain is calculated as a weighted summation of the in-plane strains, so also ε_{zz} will be uniform (see section **??**). The thicknesss of the axi-symmetric object will remain uniform. For non-isotropic material behavior this is not the case, however.

Loading and boundary conditions

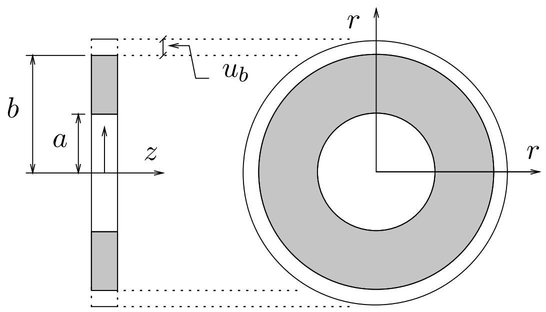
In the following subsections, different geometries and loading conditions will be considered. The external load determines the right-hand side $f(r)$ of the differential equation and as a consequence the particulate part $\bar{u}_r$ of the general solution. Boundary conditions must be used subsequently to determine the integration constants c_1 and c_2. Finally the parameters A_p, B_p and Q_p must be chosen in accordance with the material behavior and specified for plane stress or plane strain (see appendix **??**).

The algebra, which is involved with these calculations, is not very difficult, but rather cumbersome. In appendix B a number of examples is presented. When numerical values are provided, displacements, strains and stresses can be calculated and plotted with a Matlab program, which is available on the website of this course. Based on the input, it selects the proper formulas for the calculation. Instructions for its use can be found in the program source file. The figures in the next subsections are made with this program.

1.2.1 Prescribed edge displacement

The outer edge of a disc with a central hole is given a prescribed displacement $u(r = b) = u_b$. The inner edge is stress-free. With these boundary conditions, the integration constants in the general solution can be determined. They can be found in appendix B.

For the parameter values listed below, the radial displacement u_r and the stresses are calculated and plotted as a function of the radius r.



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r = b) = u_b$$
$$\sigma_{rr}(r = a) = 0$$
plane stress
c_1, c_2 : App. B

Fig. 1.3 : *Edge displacement of circular disc*

$$|\, u_b = 0.01 \text{ m} \,|\, a = 0.25 \text{ m} \,|\, b = 0.5 \text{ m} \,|\, h = 0.05 \text{ m} \,|\, E = 250 \text{ GPa} \,|\, \nu = 0.33 \,|$$

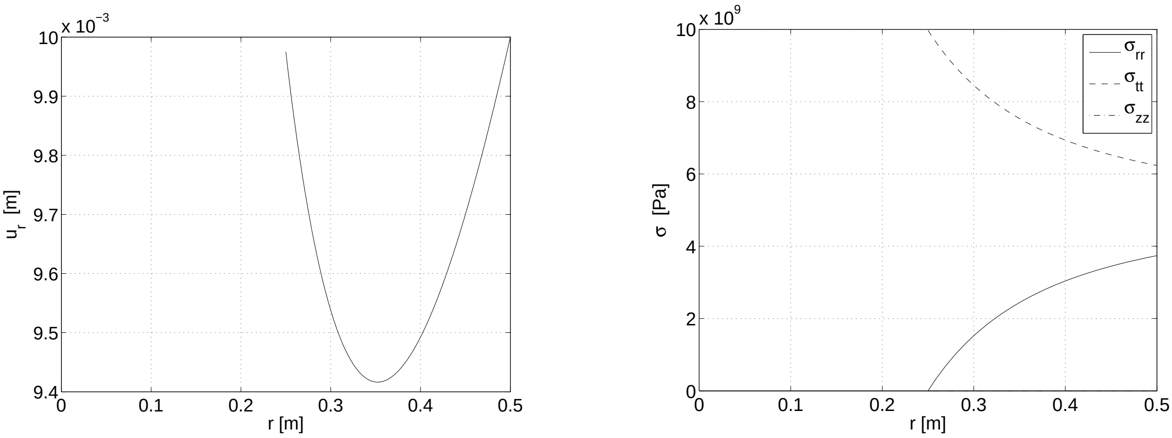


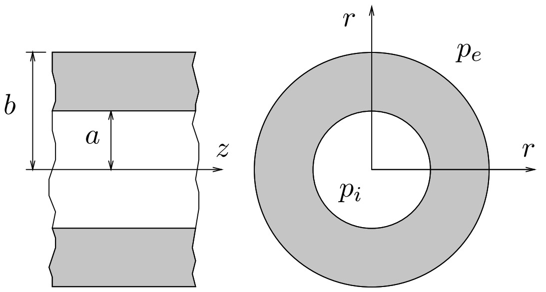
Fig. 1.4 : *Displacement and stresses for plane stress ($\sigma_{zz} = 0$)*

1.2.2 Edge load

A cylinder has inner radius $r = a$ and outer radius $r = b$. It is loaded with an internal (p_i) and/or an external (p_e) pressure.

The general solution to the equilibrium equation has two integration constants, which have to be determined from boundary conditions. In appendix B they are determined for the case that an open cylinder is subjected to an internal pressure p_i and an external pressure p_e. For plane stress ($\sigma_{zz} = 0$) the cylinder is free to deform in axial direction. The solution was first derived by Lamé in 1833 and therefore this solution is referred to as Lamé's equations. When these integration constants for isotropic material are substituted in the stress solution, it appears that the stresses are independent of the material parameters. This implies that radial and tangential stresses are the same for plane stress and plane strain. For the plane strain case, the axial stress σ_{zz} can be calculated directly from the radial and tangential stresses.

The Tresca and Von Mises limit criteria for a pressurized cylinder can be calculated according to their definitions (see chapter **??**).



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$\sigma_{rr}(r = a) = -p_i$$
$$\sigma_{rr}(r = b) = -p_e$$
$$c_1, c_2 \ : \ \text{App. B}$$

Fig. 1.5 : *Cross-section of a thick-walled circular cylinder*

$$\sigma_{rr} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} - \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2} \quad ; \quad \sigma_{tt} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} + \frac{a^2 b^2 (p_i - p_e)}{b^2 - a^2} \frac{1}{r^2}$$

$$\sigma_{TR} = 2\tau_{max} = \max \left[\, |\sigma_{rr} - \sigma_{tt}|, |\sigma_{tt} - \sigma_{zz}|, |\sigma_{zz} - \sigma_{rr}| \,\right]$$
$$\sigma_{VM} = \sqrt{\tfrac{1}{2} \left\{ (\sigma_{rr} - \sigma_{tt})^2 + (\sigma_{tt} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 \right\}}$$

An open cylinder is analyzed with the parameters from the table below. Stresses are plotted as a function of the radius.

$p_i = 100$ MPa	$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$

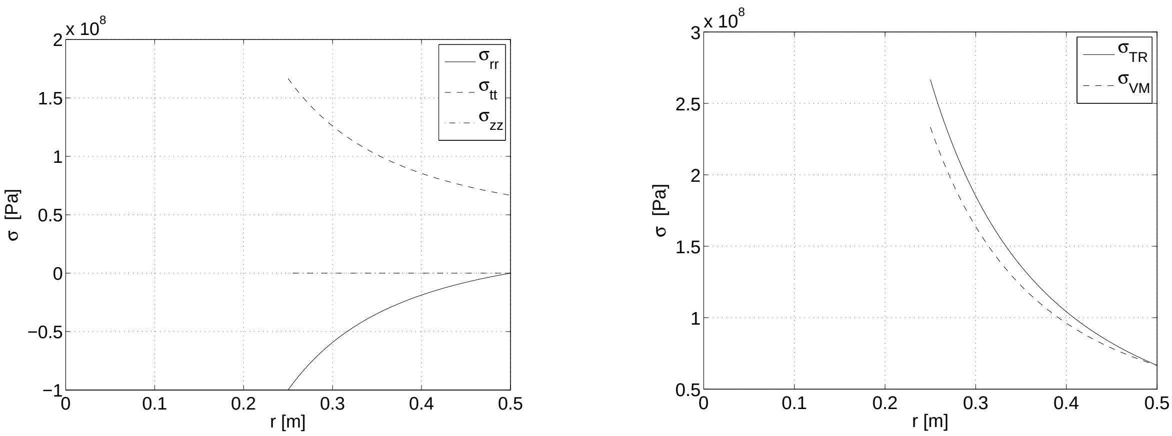


Fig. 1.6 : *Stresses in a thick-walled pressurized cylinder for plane stress* ($\sigma_{zz} = 0$)

That the inner material is under much higher tangential stress than the outer material, can be derived by reasoning, when we only consider an internal pressure. This pressure will result in enlargement of the diameter for each value of r, but it will also compress the material and result in reduction of the wall thickness. The inner diameter will thus increase more than the outer diameter – which is also calculated and plotted in the figure below – and the tangential stress will be much higher at the inner edge.

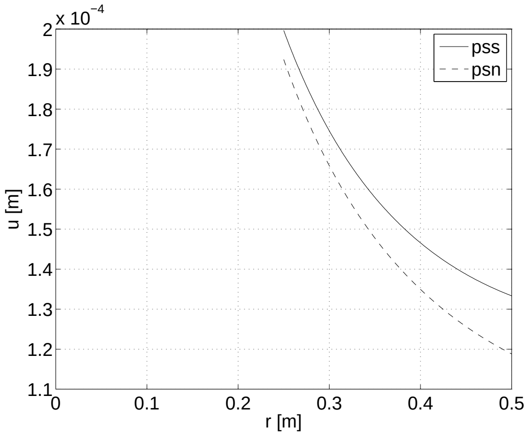


Fig. 1.7 : *Radial displacement in a thick-walled pressurized cylinder for plane stress (pss) and plane strain (psn)*

Closed cylinder

A closed cylinder is loaded in axial direction by the internal and the external pressure. This load leads to an axial stress σ_{zz}, which is uniform over the wall thickness. It can be determined from axial equilibrium and can be considered as an The radial and tangential stress are not influenced by this axial load.

The resulting radial displacement due to the contraction caused by the axial load, u_{ra}, can be calculated from Hooke's law.

$$\text{axial equilibrium} \qquad \sigma_{zz} = \frac{p_i a^2 - p_e b^2}{b^2 - a^2} \qquad \rightarrow \qquad u_{ra} = \varepsilon_{tta} r = -\frac{\nu}{E}\sigma_{zz} r$$

1.2.3 Shrink-fit compound pressurized cylinder

A compound cylinder is assembled of two individual cylinders. Before assembling the outer radius of the inner cylinder b_i is larger than the inner radius of the outer cylinder a_o. The difference between the two radii is the *shrinking allowance* $b_i - a_o$. By applying a pressure at the outer surface of the inner cylinder and at the inner surface of the outer cylinder, a clearance between the two cylinders is created and the cylinders can be assembled. After assembly the pressure is released, the clearance is eliminated and the two cylinders are fitted together.

The cylinders can also be assembled by heating up the outer cylinder to ΔT, due to which it will expand. The radial displacement of the inner radius has to be larger than the shrinking allowance. With α being the coefficient of thermal expansion, this means :

$$\varepsilon_{tt_{\Delta T}}(r = a_o) = u_{r_o}(r = a_o)/a_o = \alpha\Delta T \rightarrow u_{r_o}(r = a_o) = a_o\alpha\Delta T > b_i - a_o$$

After assembly the outer cylinder is cooled down again and the two cylinders are fitted together. Residual stresses will remain in both cylinders. At the interface between the two cylinders the radial stress is the contact pressure, indicated as p_c. The stresses in both cylinders, loaded with this contact pressure, can be calculated with the Lame's equations.

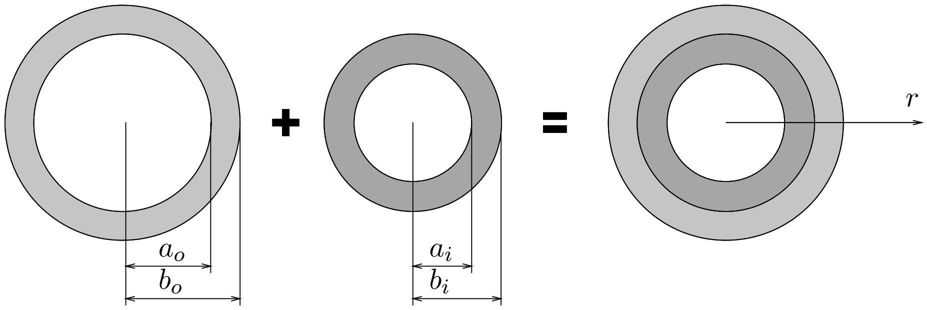


Fig. 1.8 : *Shrink-fit assemblage of circular cylinders*

$$\sigma_{rr_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} + \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)r^2} \quad ; \quad \sigma_{tt_i} = \frac{-p_c b_i^2}{b_i^2 - a_i^2} - \frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)r^2}$$

$$\sigma_{rr_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} - \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)r^2} \quad ; \quad \sigma_{tt_o} = \frac{p_c a_o^2}{b_o^2 - a_o^2} + \frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)r^2}$$

The radial displacement can also be calculated for both cylinders. For plane stress, these relations are shown below. They can be derived with subsequential reference to the pages a6 and **??**. The inner and outer radius of the compound cylinder is then known. The *radial interference* δ is the difference the displacemnts at the interface, which is located at radius r_c.

The location of the contact interface is indicated as r_c. The contact pressure p_c can be solved from the relation for r_c.

$$u_{r_i}(r = a_i) = -\frac{2}{E}\frac{p_c a_i b_i^2}{b_i^2 - a_i^2} \quad ; \quad u_{r_o}(r = b_o) = \frac{2}{E}\frac{p_c a_o^2 b_o}{b_o^2 - a_o^2}$$

$$u_{r_o}(r = a_o) = \frac{1-\nu}{E}\frac{p_c a_o^2}{b_o^2 - a_o^2}a_o + \frac{1+\nu}{E}\frac{p_c a_o^2 b_o^2}{(b_o^2 - a_o^2)}\frac{1}{a_o}$$

$$u_{r_i}(r = b_i) = -\frac{1-\nu}{E}\frac{p_c b_i^2}{b_i^2 - a_i^2}b_i - \frac{1+\nu}{E}\frac{p_c a_i^2 b_i^2}{(b_i^2 - a_i^2)}\frac{1}{b_i}$$

$$r_i = a_i + u_i(r = a_i) \quad ; \quad r_o = b_o + u_o(r = b_o)$$

$$r_c = b_i + u_{r_i}(r = b_i) = a_o + u_{r_o}(r = a_o) \quad \rightarrow$$

$$p_c = \frac{E(b_i - a_o)(b_o^2 - a_o^2)(b_i^2 - a_i^2)}{a_o(b_i^2 - a_i^2)\{(b_0^2 + a_o^2) + \nu(b_o^2 - a_o^2)\} + b_i(b_o^2 - a_o^2)\{(b_i^2 + a_i^2) - \nu(b_i^2 - a_i^2)\}}$$

For the parameter values listed below, the radial and tangential stresses are calculated and plotted as a function of the radius r.

$$|\, a_i = 0.4 \text{ m} \,|\, b_i = 0.7 \text{ m} \,|\, a_o = 0.699 \text{ m} \,|\, b_o = 1 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$

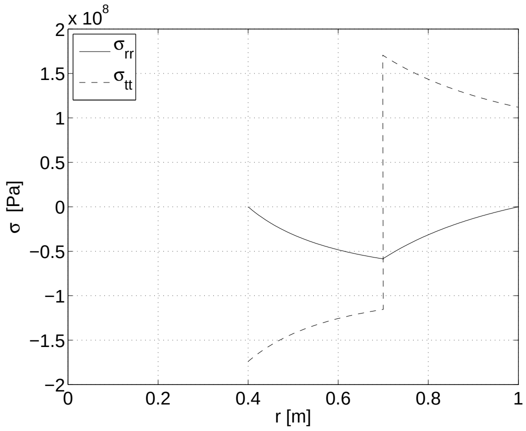


Fig. 1.9 : *Residual stresses in shrink-fit assemblage of two cylinders*

Shrink-fit and service state

After assembly, the compound cylinder is loaded with an internal pressure p_i. The residual stresses from the shrink-fit stage and the stresses due to the internal pressure can be added, based on the superposition principle. The resulting stresses are lower than the stresses due to internal pressure. Compound cylinders can carry large pressures more efficiently.

1.2.4 Circular hole in infinite medium

When a circular hole is located in an infinite medium, we can derive the stresses from Lame's equations by taking $b \rightarrow \infty$. For the general case this leads to limit values of the integration constants c_1 and c_2. These can then be substituted in the general solutions for displacement and stresses.

Pressurized hole in infinite medium

For a pressurized hole in an infinite medium the external pressure p_e is zero. In that case the absolute values of radial and tangential stresses are equal. The radial displacement can also be calculated.

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius. Note that we take a large but finite value of for b.

$$b \to \infty \quad ; \quad p_i = p \quad ; \quad p_e = 0 \qquad \to \qquad \sigma_{rr} = -\frac{pa^2}{r^2} \quad ; \quad \sigma_{tt} = \frac{pa^2}{r^2}$$

$$|\, p_i = 100 \text{ MPa} \,|\, a = 0.2 \text{ m} \,|\, b = 20 \text{ m} \,|\, h = 0.5 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$

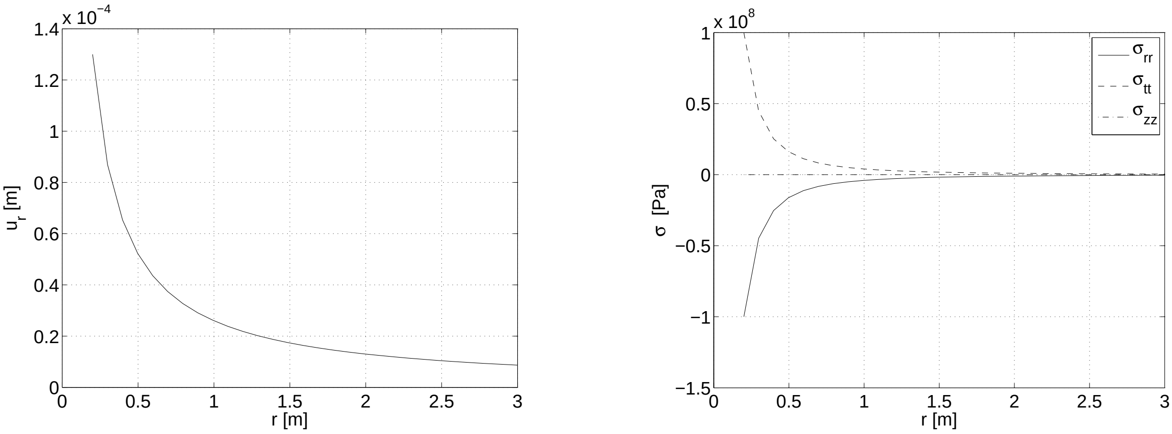


Fig. 1.10 : *Displacement and stresses in a pressurized circular hole in an infinite medium*

Stress-free hole in bi-axially loaded infinite medium

We consider the case of a stress-free hole of radius a in an infinite medium, which is bi-axially loaded at infinity by a uniform load T, equal in x- and y-direction. Because the load is applied at boundaries which are at infinite distance from the hole center, the bi-axial load is equivalent to an externally applied radial edge load $p_e = -T$.

Radial and tangential stresses are different in this case. The tangential stress is maximum for $r = a$ and equals $2T$.

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius.

$$b \to \infty \; ; \; p_i = 0 \; ; \; p_e = -T \quad \to \quad \sigma_{rr} = T\left(1 - \frac{a^2}{r^2}\right) \; ; \; \sigma_{tt} = T\left(1 + \frac{a^2}{r^2}\right)$$

stress concentration factor $$K_t = \frac{\sigma_{max}}{T} = \frac{\sigma_{tt}(r = a)}{T} = \frac{2T}{T} = 2$$

$$|\, p_e = -100 \text{ MPa} \,|\, a = 0.2 \text{ m} \,|\, b = 20 \text{ m} \,|\, h = 0.5 \text{ m} \,|\, E = 200 \text{ GPa} \,|\, \nu = 0.3 \,|$$

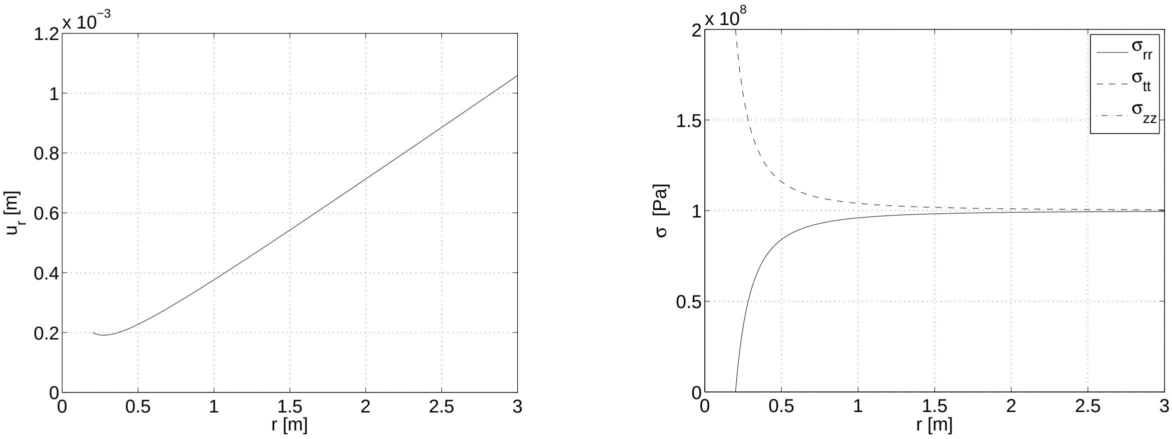


Fig. 1.11 : *Displacement and stresses in a pressurized circular hole in an infinite medium*

1.2.5 Rotating disc

A circular disc, made of isotropic material, rotates with angular velocity ω [rad/s]. The outer radius of the disc is taken to be b. The disc may have a central circular hole with radius a. When $a = 0$ there is no hole and the disc is called "solid". Boundary conditions for a disc with a central hole are rather different than those for a "solid" disc, which results in different solutions for radial displacement and stresses. The external load $f(r)$ is the result of the radial acceleration of the material (see appendix **??**).

$$\ddot{u}_r = -\omega^2 r \quad \rightarrow \quad f(r) = \frac{\rho}{A_p}\, \ddot{u}_r = -\,\frac{\rho}{A_p}\, \omega^2 r$$

The particulate solution $\bar{u}_r$ can be calculated to be

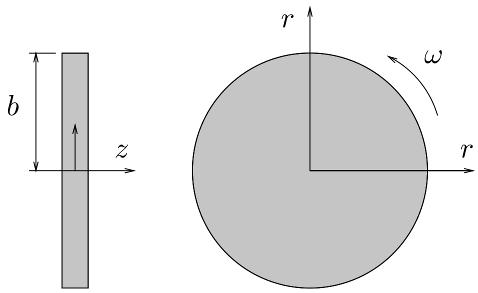
$$\bar{u}_r = -\,\frac{1}{8}\,\frac{1}{A_p}\, \rho\omega^2 r^3$$

For orthotropic and isotropic material, the general solution for the radial displacement and the radial and tangential stresses can be calculated.

Solid disc

In a disc without a central hole (solid disc) there are material points at radius $r = 0$. To prevent infinite displacements for $r \rightarrow 0$ the second integration constant c_2 must be zero. At the outer edge the radial stress σ_{rr} must be zero, because this edge is unloaded. With these boundary conditions the integration constants in the general solution can be calculated (see appendix B).

For a plane stress state and with the listed parameter values, the stresses are calculated and plotted as a function of the radius.



$$u_r(r = 0) \neq \infty$$
$$\sigma_{rr}(r = b) = 0$$
$$c_1, c_2 \ : \ \text{App. B}$$

Fig. 1.12 : *A rotating solid disc*

$$| \, \omega = 6 \text{ c/s} \quad | \, a = 0 \text{ m} \, | \, b = 0.5 \text{ m} \, | \, t = 0.05 \text{ m} \, | \, \rho = 7500 \text{ kg/m}^3 \, |$$
$$| \, E = 200 \text{ GPa} \, | \, \nu = 0.3 \ \ |$$

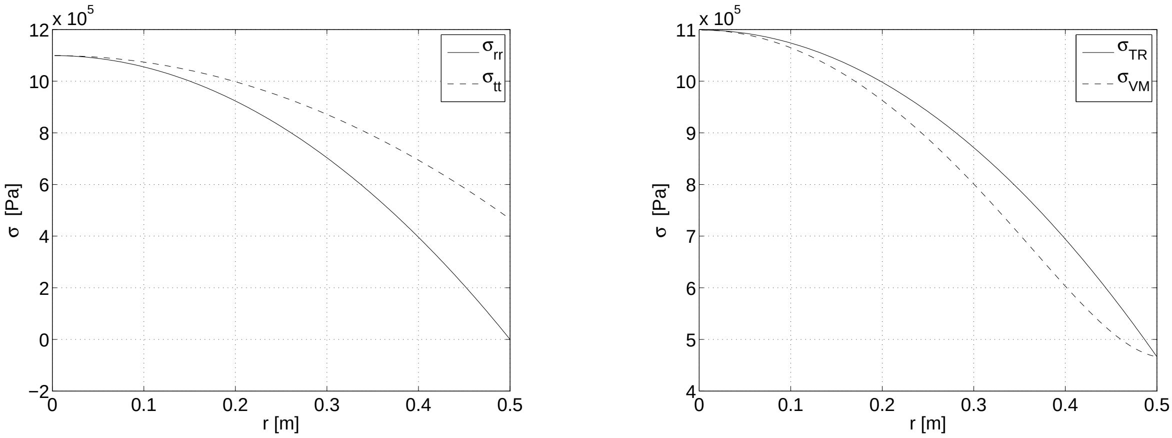


Fig. 1.13 : *Stresses in a rotating solid disc in plane stress*

In a rotating solid disc the radial and tangential stresses are equal in the center of the disc. They both decrease with increasing radius, where of course the radial stress reduces to zero at the outer radius. The equivalent Tresca and Von Mises stresses are not very different and also decrease with increasing radius. In the example the disc is assumed to be in a state of plane stress.

When the same disc is fixed between two rigid plates, a plane strain state must be modelled. In that case the axial stress is not zero. As can be seen in the plots below, the axial stress influences the Tresca and Von Mises equivalent stresses.

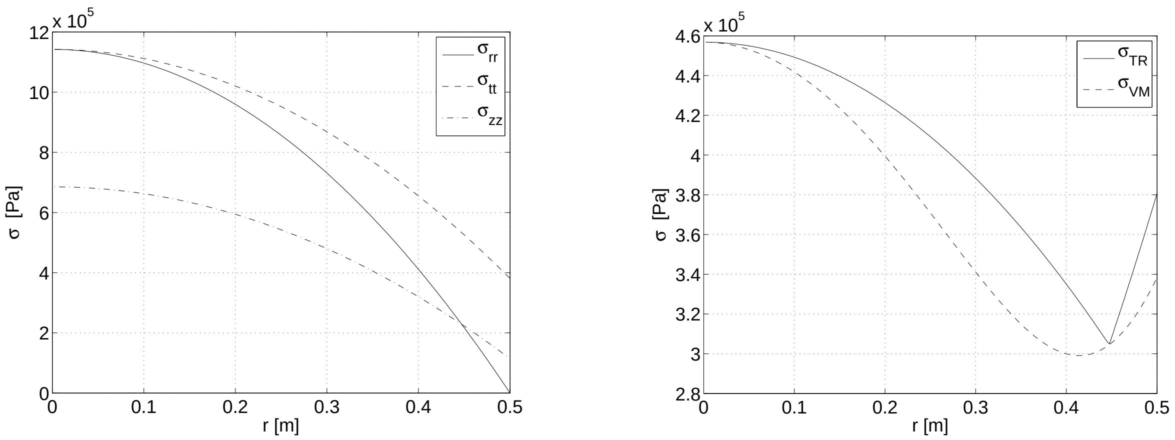
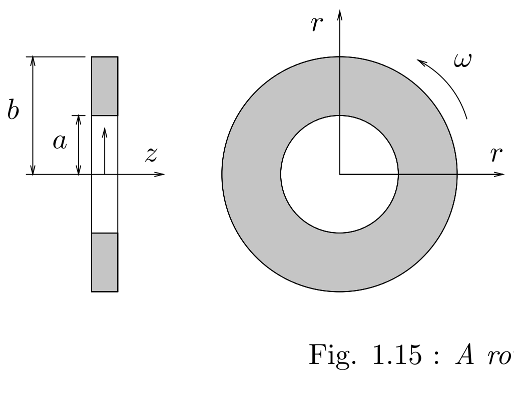


Fig. 1.14 : *Stresses in a rotating solid disc in plane strain*

Disc with central hole

When the disc has a central circular hole, the radial stress at the inner edge and at the outer edge must both be zero, which provides two equations to solve the two integration constants.

For a plane stress state and with parameter values listed below, the radial displacement and the stresses are calculated and plotted as a function of the radius.



$\sigma_{rr}(r = a) = 0$
$\sigma_{rr}(r = b) = 0$
c_1, c_2 : App. B

Fig. 1.15 : *A rotating disc with central hole*

$\omega = 6$ c/s	$a = 0.2$ m	$b = 0.5$ m	$t = 0.05$ m	$\rho = 7500$ kg/m^3
$E = 200$ GPa	$\nu = 0.3$			

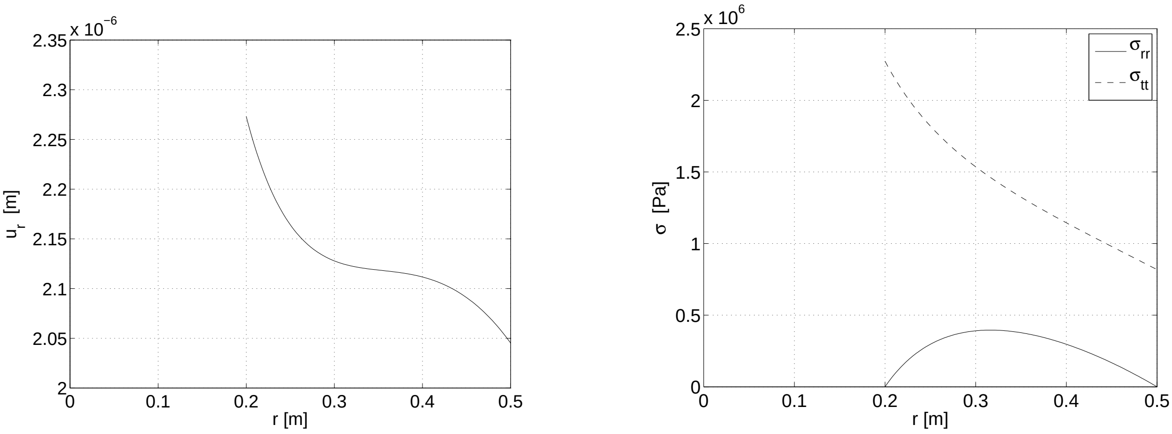
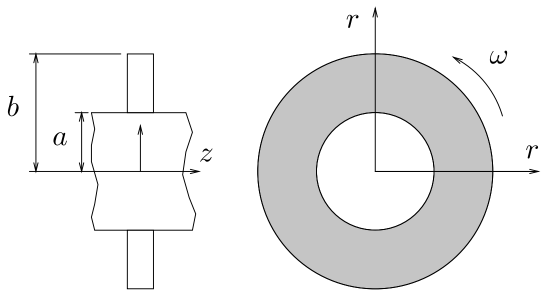


Fig. 1.16 : *Displacement and stresses in a rotating disc with a central hole*

Disc fixed on rigid axis

When the disc is fixed on an axis and the axis is assumed to be rigid, the displacement of the inner edge is suppressed. The radial stress at the outer edge is obviously zero.

For a plane stress state and with parameter values listed below the stresses and tha radial displacement is calculated and plotted.



$u_r(r = a) = 0$

$\sigma_{rr}(r = b) = 0$

c_1, c_2 : App. B

Fig. 1.17 : *Disc fixed on rigid axis*

$\omega = 6$ c/s	$a = 0.2$ m	$b = 0.5$ m	$t = 0.05$ m	$\rho = 7500$ kg/m^3
$E = 200$ GPa	$\nu = 0.3$			

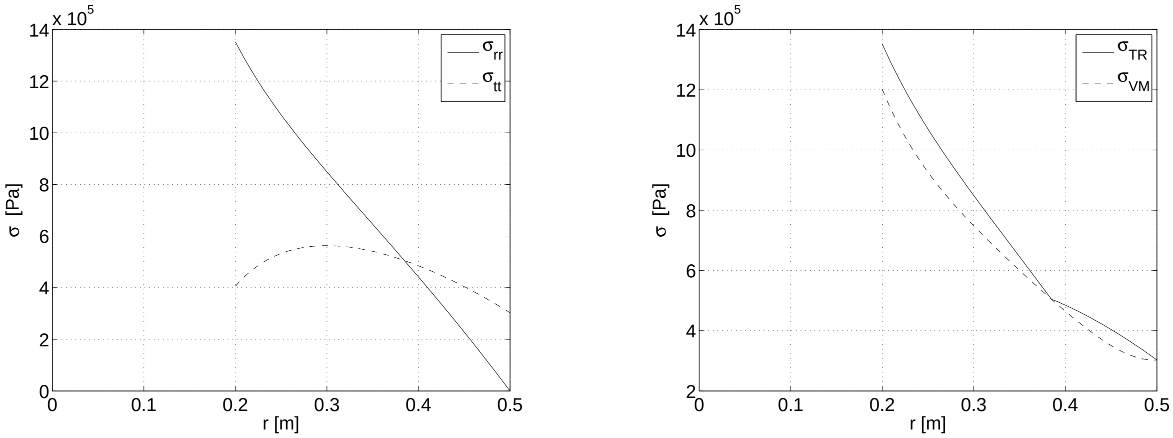


Fig. 1.18 : *Stresses in a rotating disc, fixed on an axis*

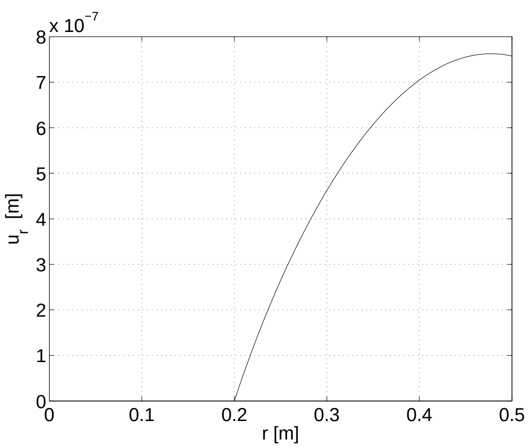


Fig. 1.19 : *Displacement in a rotating disc*

1.2.6 Rotating disc with variable thickness

For a rotating disc with variable thickness $t(r)$ the equation of motion in radial direction can be derived. For a disc with inner and outer radius a and b, respectively, and a thickness distribution $t(r) = \frac{t_a}{2}\frac{a}{r}$, a general solution for the stresses can be derived. The integration constants can be determined from the boundary conditions, e.g. $\sigma_{rr}(r = a) = \sigma_{rr}(r = b) = 0$.

equilibrium $$\frac{\partial(t(r)r\sigma_{rr})}{\partial r} - t(r)\sigma_{tt} = -\rho\omega^2 t(r)r^2 \quad \text{with } t(r) = \frac{t_a}{2}\frac{a}{r}$$

stresses
$$\sigma_{rr} = \frac{2c_1}{at_a}\, r^{d_1} + \frac{2c_2}{at_a}\, r^{d_2} - \frac{3+\nu}{5+\nu}\rho\omega^2 r^2 \quad ; \quad \sigma_{tt} = \frac{2c_1}{at_a}\, d_1 r^{d_1} + \frac{2c_2}{at_a}\, d_2 r^{d_2} - \frac{1+3\nu}{5+\nu}\rho\omega^2 r^2$$

with $$d_1 = -\tfrac{1}{2} + \sqrt{\tfrac{5}{4} + \nu} \quad ; \quad d_2 = -\tfrac{1}{2} - \sqrt{\tfrac{5}{4} + \nu}$$

boundary conditions $\quad \sigma_{rr}(r=a) = \sigma_{rr}(r=b) = 0 \quad \rightarrow$

$$\frac{2c_1}{at_a} = \frac{3+\nu}{5+\nu}\,\rho\omega^2\, a^{-d_1}\left[a^2 - a^{d_2}\left(\frac{b^2 - a^{-d_1}b^{d_1}a^2}{b^{d_2} - a^{d_2}a^{-d_1}b^{d_1}}\right)\right]$$
$$\frac{2c_2}{at_a} = \frac{3+\nu}{5+\nu}\,\rho\omega^2\left(\frac{b^2 - a^{-d_1}b^{d_1}a^2}{b^{d_2} - a^{d_2}a^{-d_1}b^{d_1}}\right)$$

A disc with a central hole and a variable thickness rotates with an angular velocity of 6 cycles per second. The stresses are plotted as a function of the radius.

isotropic	plane stress	$\omega = 6$ c/s			
$a = 0.2$ m	$b = 0.5$ m	$t_a = 0.05$ m	$\rho = 7500$ kg/m^3	$E = 200$ GPa	$\nu = 0.3$

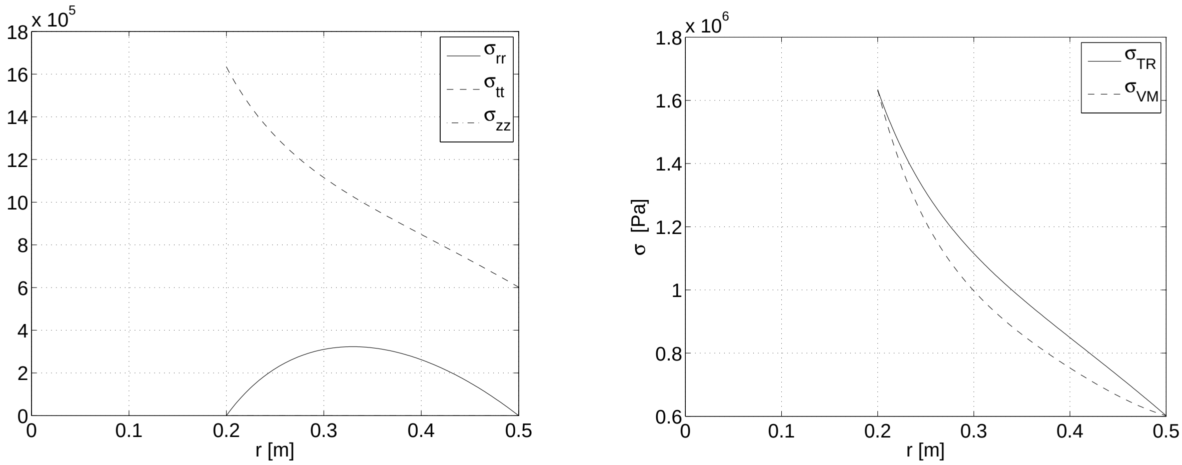


Fig. 1.20 : *Stresses in a rotating disc with variable thickness*

1.2.7 Thermal load

For a disc loaded with a distributed temperature $\Delta T(r)$ the external load $f(r)$ is due to thermal expansion. The coefficient of thermal expansion is α and the general material parameters for planar deformation are A_p and Q_p.

The part $\bar{u}_r$ has to be determined for a specific radial temperature loading. It is assumed here that the temperature is a third order function of the radius r.

diff.eq. $\quad u_{r,rr} + \frac{1}{r}\,u_{r,r} - \frac{1}{r^2}\,u_r = f(r)$

load $\quad f(r) = \frac{\Theta_{p1}}{A_p}\,\alpha(\Delta T)_{,r}$

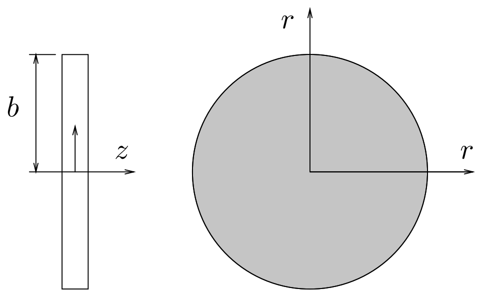
$$\Delta T(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 \quad \rightarrow \quad f(r) = \frac{\Theta_{p1}}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right)$$

diff.eq.	$u_{r,rr} + \frac{1}{r}\,u_{r,r} - \frac{1}{r^2}\,u_r = \frac{\Theta_{p1}}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right)$
hom.sol.	$\hat{u}_r = c_1\,r + \frac{c_2}{r}$
part.sol.	$\bar{u}_r(r) = \frac{\Theta_{p1}}{A_p}\,\alpha\left(\frac{1}{3}a_1 r^2 + \frac{1}{4}a_2 r^3 + \frac{1}{5}a_3 r^4\right)$
gen.sol.	$u_r = c_1 r + \frac{c_2}{r} + \frac{\Theta_{p1}}{A_p}\,\alpha\left(\frac{1}{3}a_1 r^2 + \frac{1}{4}a_2 r^3 + \frac{1}{5}a_3 r^4\right)$
stresses	$\sigma_{rr} = \cdots \quad ; \quad \sigma_{tt} = \cdots$
BC's	$\rightarrow \quad c_1 \quad \text{and} \quad c_2$

Solid disc, free outer edge

As is always the case for a solid disc, the constant c_2 has to be zero to prevent the displacement to become infinitely large for $r = 0$. The constant c_1 must be calculated from the other boundary condition. It can be found in appendix B.

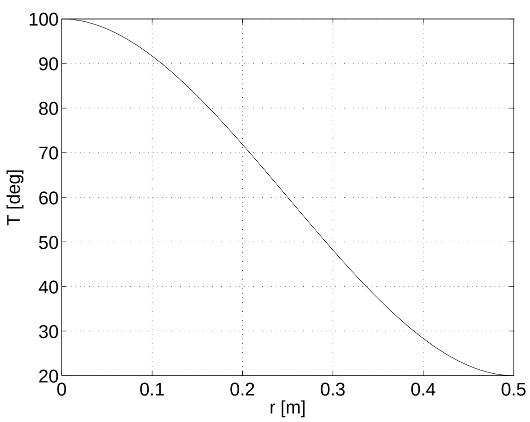
An isotropic solid disc is subjected to the radial temperature profile, shown in the figure below. The temperature gradient is zero at the center and at the outer edge. For a plane stress state and with parameter values listed below the stresses are calculated and plotted as a function of the radius.



$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$
$$c_1, c_2 \;:\; \text{App. B}$$

Fig. 1.21 : *Solid disc with a radial temperature gradient*

$$|\; \underset{\sim}{a}^T = [100\ 20\ 0\ 0] \;|\; a = 0\ \text{m} \;|\; b = 0.5\ \text{m} \;|\; E = 200\ \text{GPa} \;|\; \nu = 0.3 \;|\; \alpha = 10^{-6}\ 1/^o\text{C} \;|$$



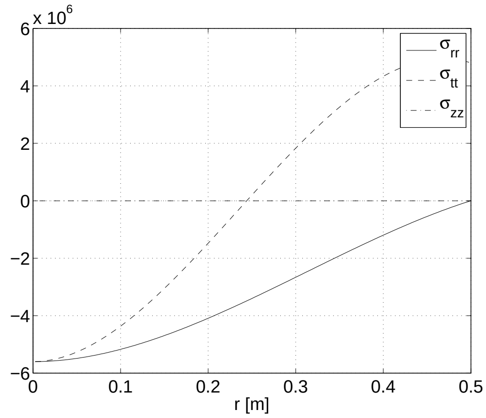


Fig. 1.22 : *Radial temperature profile and stresses in a solid disc in plane stress*

When the temperature field is uniform, $\Delta T(r) = a_0$, we have $\bar{u}_r = 0$, and for the solid disc $c_2 = 0$ to assure $u_r(r = 0) \neq \infty$. From the general solution for the stresses – see page a10 – it follows that both the radial and the tangential stresses are uniform. Because the outer edge is stress-free, they have to be zero. The integration constant c_1 can be determined to be $c_1 = \alpha a_0$, which gives for the radial displacement :

$$u_r = \alpha a_0 r$$

When the outer edge is clamped, the condition $u_r(r = b) = 0$ leads to $c_1 = 0$, so the radial displacement is uniformly zero. Radial and tangential stresses are uniform and equal :

$$\sigma_{rr} = \sigma_{tt} = -\alpha(A_p + Q_p)a_0$$

Different results for plane stress and plane strain emerge after substitution of the appropriate values for A_p and Q_p, see section **??** and appendix **??**. For plane stress, the thickness strain can be calculated (see section **??**) :

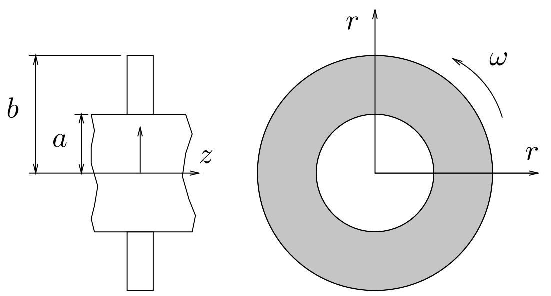
$$\varepsilon_{zz} = r\sigma_{11} + s\varepsilon_{22} + \alpha\Delta T$$

For plane strain, the axial stress can be calculated :

$$\sigma_{zz} = -\frac{r}{c}\sigma_{rr} - \frac{s}{c}\sigma_{tt} - \frac{\alpha}{c}\Delta T$$

Disc on a rigid axis

When the disc is mounted on a rigid axis, the radial displacement at the inner radius of the central hole is zero. The radial stress at the outer edge is again zero.



$u_r(r = a) = 0$
$\sigma_{rr}(r = b) = 0$
c_1, c_2 : App. B

Fig. 1.23 : *Disc on a rigid axis*

isotropic	plane stress	$\underset{\sim}{a}^T = [100\ 20\ 0\ 0]$		
$a = 0.2$ m	$b = 0.5$ m	$E = 200$ GPa	$\nu = 0.3$	$\alpha = 10^{-6}$ 1/°C

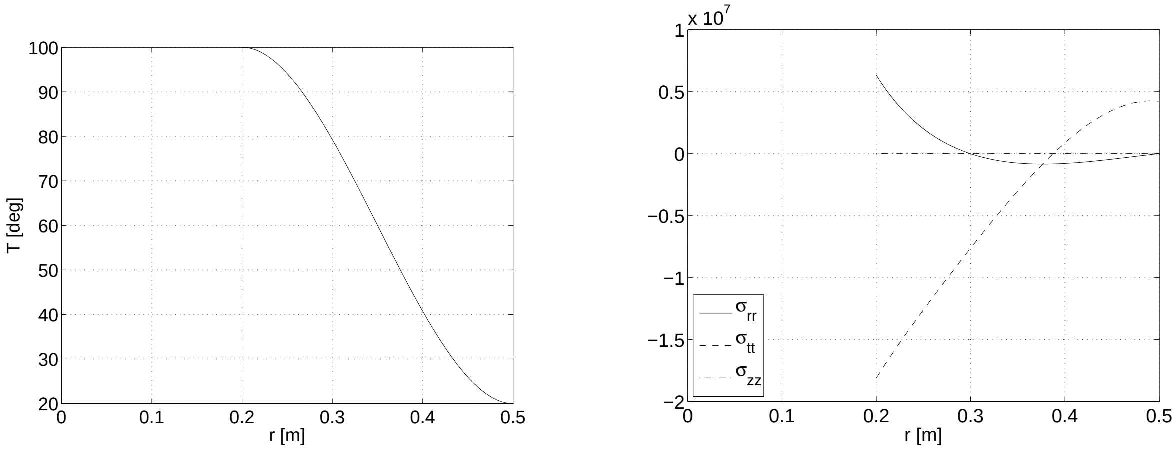


Fig. 1.24 : *Radial temperature profile and stresses in a disc which is fixed on a rigid axis*

1.2.8 Large thin plate with central hole

A large rectangular plate is loaded with a uniform stress $\sigma_{xx} = \sigma$. In the center of the plate is a hole with radius a, much smaller than the dimensions of the plate.

The stresses around the hole can be determined, using an Airy stress function approach. The relevant stresses are expressed as components in a cylindrical coordinate system, with coordinates r, measured from the center of the hole, and θ in the circumferential direction.

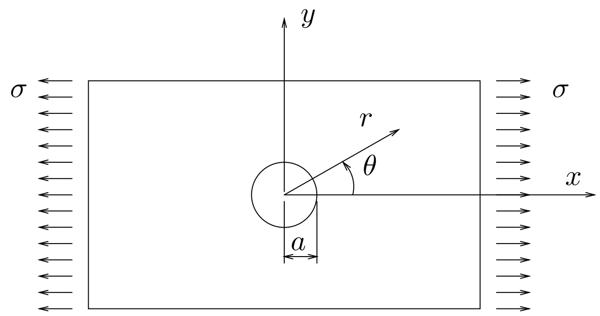


Fig. 1.25 : *Large thin plate with a central hole*

$$\sigma_{rr} = \frac{\sigma}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + 3\,\frac{a^4}{r^4} - 4\,\frac{a^2}{r^2}\right)\cos(2\theta)\right]$$

$$\sigma_{tt} = \frac{\sigma}{2}\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + 3\,\frac{a^4}{r^4}\right)\cos(2\theta)\right]$$

$$\sigma_{rt} = -\,\frac{\sigma}{2}\left[1 - 3\,\frac{a^4}{r^4} + 2\,\frac{a^2}{r^2}\right]\sin(2\theta)$$

At the inner edge of the hole, the tangential stress reaches a maximum value of 3σ for $\theta = 90^o$. For $\theta = 0^o$ a compressive tangential stress occurs. The stress concentration factor K_t is independent of material parameters and the hole diameter.

At a large distance from the hole, so for $r \gg a$, the stress components are a function of the angle θ only.

stress concentrations

$$\sigma_{tt}(r = a, \theta = \tfrac{\pi}{2}) = 3\sigma \quad ; \quad \sigma_{tt}(r = a, \theta = 0) = -\sigma$$

$$\text{stress concentration factor} \qquad K_t = \frac{\sigma_{max}}{\sigma} = 3$$

stress at larger r

$$\sigma_{rr} = \frac{\sigma}{2}\left[1 + \cos(2\theta)\right] = \sigma\, \cos^2(\theta)$$

$$\sigma_{tt} = \frac{\sigma}{2}\left[1 - \cos(2\theta)\right] = \sigma\left[1 - \cos^2(\theta)\right] = \sigma \sin^2(\theta)$$

$$\sigma_{rt} = -\frac{\sigma}{2}\, \sin(2\theta) = -\sigma\, \sin(\theta)\cos(\theta)$$

For parameters values listed below stress components are calculated and plotted for $\theta = 0$ and for $\theta = \frac{\pi}{2}$ as a function of the radial distance r.

$$|\, a = 0.05 \text{ m} \,|\, \sigma = 1000 \text{ Pa} \,|\, E = 250 \text{ GPa} \,|\, \nu = 0.3 \,|$$

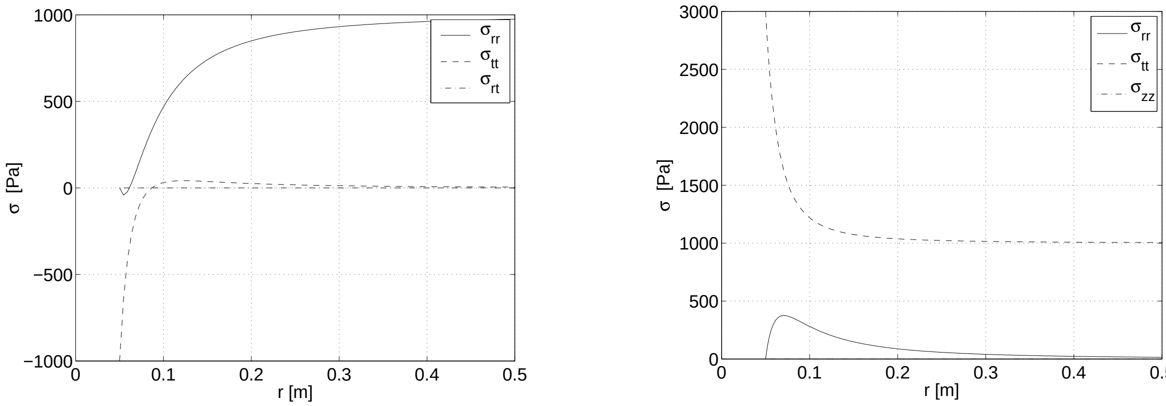


Fig. 1.26 : *Stresses in plate for $\theta = 0$ and $\theta = \frac{\pi}{2}$*

1.2.9 Optimized thickness distribution

An optimized shape can be found for the requirement $\sigma_{rr} = \sigma_{tt} = \sigma$. The equilibrium equation results in a differential equation for the thickness and can be solved.

$$\frac{dt}{dr} + \frac{\rho\omega^2}{\sigma} tr = 0 \quad \rightarrow \quad t(r) = t_0 \, e^{-\frac{\rho\omega^2}{2\sigma} r^2} \qquad \text{with} \qquad t_0 = t(r = r_i)$$

APPENDICES

A Radial temperature field

An axi-symmetric material body may be subjected to a radial temperature field, resulting in non-homogeneous strains and stresses. The considered temperature is a cubic function of the radial coordinate. Its constants are a related to the temperatures and temperature gradients at the inner and the outer edges of the body.

Cubic temperature function

The radial temperature field is a third-order function of the radius.

$$\begin{aligned} T(r) &= a_0 + a_1 r + a_2 r^2 + a_3 r^3 \\ \frac{dT}{dr} &= a_1 + 2a_2 r + 3a_3 r^2 \end{aligned}$$

Boundary values

The coefficients in the temperature function are expressed in the boundary values of temperature T and its derivative $\frac{dT}{dr} = T_{,r}$. The boundaries are the inner and outer edge of the disc, with radius r_1 and r_2, respectively.

$$\begin{aligned} T(r = r_1) &= f_1 = a_0 + a_1 r_1 + a_2 r_1^2 + a_3 r_1^3 \\ T(r = r_2) &= f_2 = a_0 + a_1 r_2 + a_2 r_2^2 + a_3 r_2^3 \\ T_{,r}(r = r_1) &= f_3 = a_1 + 2a_2 r_1 + 3a_3 r_1^2 \\ T_{,r}(r = r_2) &= f_4 = a_1 + 2a_2 r_2 + 3a_3 r_2^2 \end{aligned}$$

in matrix notation

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 1 & r_2 & r_2^2 & r_2^3 \\ 0 & 1 & 2r_1 & 3r_1^2 \\ 0 & 1 & 2r_2 & 3r_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \rightarrow \quad \underset{\sim}{f} = \underline{M}\, \underset{\sim}{a}$$

Coefficients

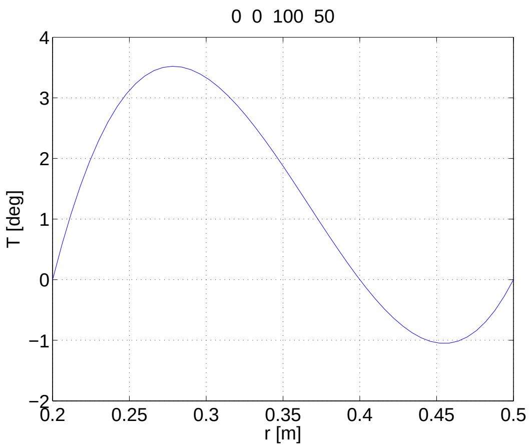
The coefficients can be expressed in the boundary values of temperature and temperature derivative. This can be done numerically by inversion of $\underset{\sim}{f} = \underline{M}\,\underset{\sim}{a}$. Here, also the analytical expressions are presented.

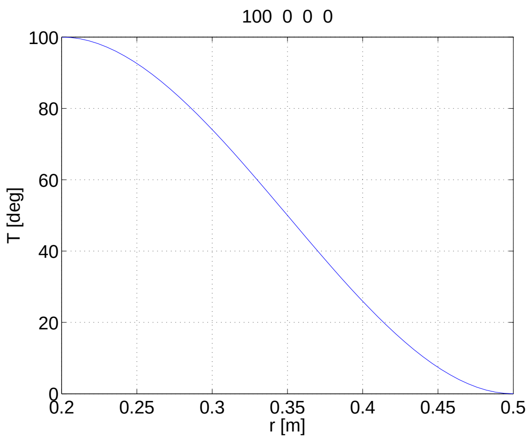
$$
\begin{aligned}
a_3 &= \frac{F_4 X_{12} - F_3 X_{22}}{X_{13} X_{22} - X_{23} X_{12}} \\
a_2 &= -\frac{F_3}{X_{12}} - \frac{X_{13}}{X_{12}} a_3 \\
a_1 &= \frac{f_2 - f_1}{r_2 - r_1} - (r_2 + r_1) a_2 - \frac{r_2^3 - r_1^3}{r_2 - r_1} a_3 \\
a_0 &= f_1 - r_1 a_1 - r_1^2 a_2 - r_1^3 a_3
\end{aligned}
$$

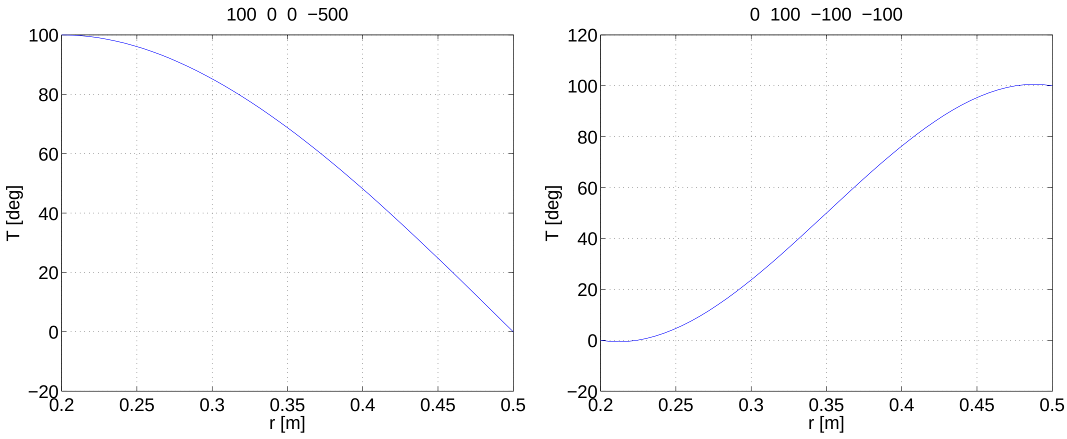
$$
\begin{aligned}
F_3 &= f_3(r_2 - r_1) - (f_2 - f_1) \\
F_4 &= f_4(r_2 - r_1) - (f_2 - f_1) \\
X_{12} &= r_2^2 - r_1^2 - 2r_1(r_2 - r_1) \\
X_{13} &= r_2^3 - r_1^3 - 3r_1^2(r_2 - r_1) \\
X_{22} &= r_2^2 - r_1^2 - 2r_2(r_2 - r_1) \\
X_{23} &= r_2^3 - r_1^3 - 3r_2^2(r_2 - r_1)
\end{aligned}
$$

Temperature fields

As an example, some temperature fields are plotted. The radius ranges between 0.2 and 0.5 m. The title of the plots gives the values of $T(r_1)$, $T(r_2)$, $T_{,r}(r1)$ and $T_{,r}(r2)$, respectively.







Radial temparature fields

B Examples

In this appendix a number of examples is shown. The general relations for the analytical solutions can be found in chapter 1 and are specified here. The integration constants are calculated for the specific boundary conditions and loading, and are given here.

B.1 Governing equations and general solution

$$\boxed{u_{r,rr} + \frac{1}{r}\,u_{r,r} - \zeta^2\,\frac{1}{r^2}\,u_r = f(r)}$$

with $\quad \zeta = \sqrt{\dfrac{B_p}{A_p}}$

and $\quad f(r) = \dfrac{\rho}{A_p}\,(\ddot{u}_r - q_r) + \dfrac{A_p + Q_p}{A_p}\,\alpha(\varDelta T)_{,r} + \dfrac{A_p - B_p}{A_p}\dfrac{1}{r}\,\alpha\varDelta T$

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$$\varepsilon_{rr} = c_1\zeta r^{\zeta-1} - c_2\zeta r^{-\zeta-1} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p\bar{u}_{r,r} + Q_p\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\varDelta T$$

$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p\bar{u}_{r,r} + B_p\frac{\bar{u}_r}{r} - (B_p + Q_p)\alpha\varDelta T$$

isotropic material :

general solution $\quad u_r = c_1 r + \dfrac{c_2}{r} + \bar{u}_r$

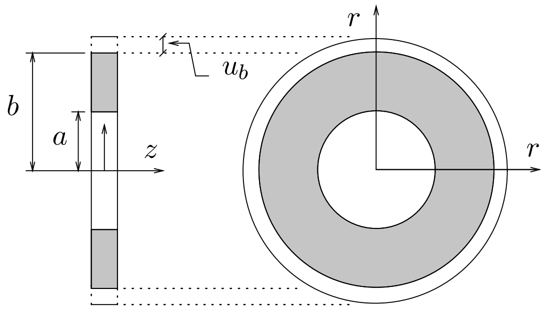
$$\varepsilon_{rr} = c_1 - c_2 r^{-2} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2} + \frac{\bar{u}_r}{r}$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} + A_p\bar{u}_{r,r} + Q_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\varDelta T$$

$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2} + Q_p\bar{u}_{r,r} + A_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\varDelta T$$

For plane strain and plane stress the material parameters can be found in appendix **??**.

B.2 Disc, edge displacement



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$u_r(r = b) = u_b$$
$$\sigma_{rr}(r = a) = 0;$$

Edge displacement of circular disc

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta}$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1}$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1}$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1}$$

$$c_1 = \frac{(A_p\zeta - Q_p)b^{\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}} \quad ; \quad c_2 = \frac{(A_p\zeta + Q_p)b^{\zeta} a^{2\zeta}\, u_b}{(A_p\zeta + Q_p)a^{2\zeta} + (A_p\zeta - Q_p)b^{2\zeta}}$$

isotropic material :

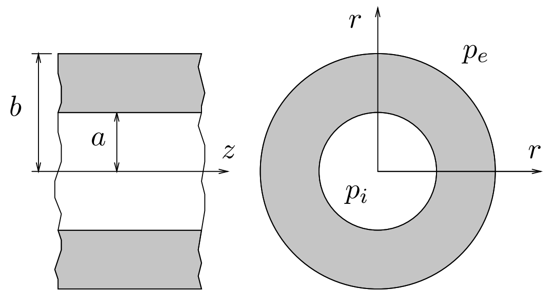
general solution $\quad u_r = c_1 r + \frac{c_2}{r}$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2}$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2}$$
$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2}$$

$$c_1 = \frac{(A_p - Q_p)b}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b \quad ; \quad c_2 = \frac{(A_p + Q_p)ba^2}{(A_p + Q_p)a^2 + (A_p - Q_p)b^2}\, u_b$$

B.3 Disc/cylinder, edge load



$$f(r) = 0 \quad \rightarrow \quad \bar{u}_r = 0$$
$$\sigma_{rr}(r = a) = -p_i$$
$$\sigma_{rr}(r = b) = -p_e$$

Cross-section of a thick-walled circular cylinder

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta}$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1}$$

$$\sigma_{rr} = (A_p \zeta + Q_p) c_1 r^{\zeta-1} - (A_p \zeta - Q_p) c_2 r^{-\zeta-1}$$
$$\sigma_{tt} = (Q_p \zeta + B_p) c_1 r^{\zeta-1} - (Q_p \zeta - B_p) c_2 r^{-\zeta-1}$$

$$c_1 = \frac{1}{A_p \zeta + Q_p} \frac{a^{\zeta+1} p_i - b^{\zeta+1} p_e}{b^{2\zeta} - a^{2\zeta}} \quad ; \quad c_2 = \frac{1}{A_p \zeta - Q_p} \frac{a^{\zeta+1} b^{2\zeta} p_i - b^{\zeta+1} a^{2\zeta} p_e}{b^{2\zeta} - a^{2\zeta}}$$

isotropic material :

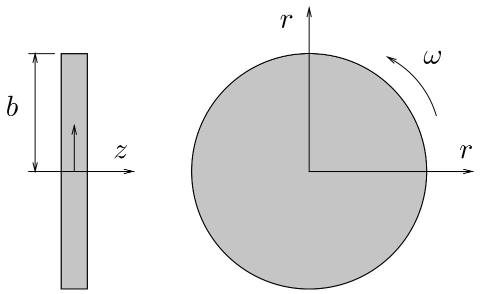
general solution $\quad u_r = c_1 r + \frac{c_2}{r}$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2}$$

$$\sigma_{rr} = (A_p + Q_p) c_1 - (A_p - Q_p) \frac{c_2}{r^2}$$
$$\sigma_{tt} = (Q_p + A_p) c_1 - (Q_p - A_p) \frac{c_2}{r^2}$$

$$c_1 = \frac{1}{A_p + Q_p} \frac{1}{b^2 - a^2} (p_i a^2 - p_e b^2) \quad ; \quad c_2 = \frac{1}{A_p - Q_p} \frac{a^2 b^2}{b^2 - a^2} (p_i - p_e)$$

B.4 Rotating solid disc



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$
$$\beta = \frac{1}{1-\zeta}\,\rho\omega^2 \quad ; \quad \zeta = \sqrt{\frac{B_p}{A_p}}$$

A rotating solid disc

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_2 = 0 \quad ; \quad c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\,\beta b^{-\zeta+3}$$

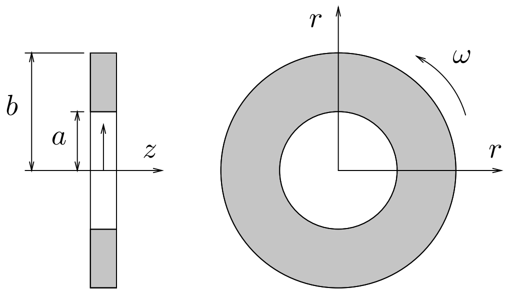
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_2 = 0 \quad ; \quad c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}\,\beta b^2$$

B.5 Rotating disc with central hole



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$\sigma_{rr}(r=a) = 0$$
$$\sigma_{rr}(r=b) = 0$$
$$\beta = \frac{1}{1-\zeta}\,\rho\omega^2 \quad ; \quad \zeta = \sqrt{\frac{B_p}{A_p}}$$

A rotating disc with central hole

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_1 = \frac{3A_p + Q_p}{A_p(A_p\zeta + Q_p)}\left(\frac{b^{\zeta+3} - a^{\zeta+3}}{b^{2\zeta} - a^{2\zeta}}\right)\beta$$
$$c_2 = \frac{3A_p + Q_p}{A_p(A_p\zeta - Q_p)}\left(\frac{a^{2\zeta-2}b^{\zeta+1} - a^{\zeta+1}b^{2\zeta-2}}{b^{2\zeta} - a^{2\zeta}}\right)(a^2b^2)\beta$$

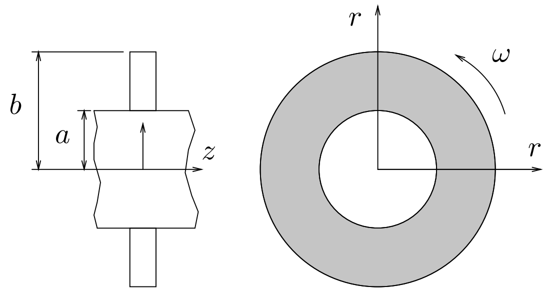
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_1 = \frac{(3A_p + Q_p)}{A_p(A_p + Q_p)}\,(a^2 + b^2)\beta \quad ; \quad c_2 = \frac{(3A_p + Q_p)}{A_p(A_p - Q_p)}\,(a^2b^2)\beta$$

B.6 Rotating disc fixed on rigid axis



$$f(r) = -\frac{\rho}{A_p}\,\omega^2 r$$
$$u_r(r=a) = 0$$
$$\sigma_{rr}(r=b) = 0$$
$$\beta = \frac{1}{1-\zeta}\,\rho\omega^2 \quad ; \quad \zeta = \sqrt{\frac{B_p}{A_p}}$$

Disc fixed on rigid axis

orthotropic material :

$$u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} - \frac{1}{A_p}\beta r^3 \quad \text{with} \quad \beta = \frac{1}{9-\zeta}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} - \frac{3A_p + Q_p}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} - \frac{3Q_p + B_p}{A_p}\,\beta r^2$$

$$c_1 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{\frac{3A_p + Q_p}{A_p}\,b^4 a^{-\zeta+1} + \frac{A_p\zeta - Q_p}{A_p}\,b^{-\zeta+1}a^4\right\}$$
$$c_2 = \frac{\beta}{(A_p\zeta + Q_p)b^{\zeta+1}a^{-\zeta+1} + (A_p\zeta - Q_p)b^{-\zeta+1}a^{\zeta+1}} \left\{\frac{A_p\zeta + Q_p}{A_p}\,b^{\zeta+1}a^4 - \frac{3A_p + Q_p}{A_p}\,b^4 a^{\zeta+1}\right\}$$

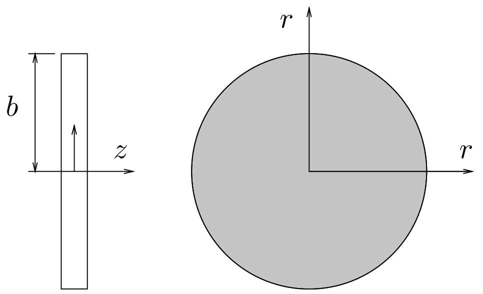
isotropic material :

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1}{8}\,\frac{\rho}{A_p}\,\omega^2 r^3 = c_1 r + \frac{c_2}{r} - \frac{1}{A_p}\,\beta r^3 \quad \text{with} \quad \beta = \frac{1}{8}\,\rho\omega^2$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \frac{(3A_p + Q_p)}{A_p}\,\beta r^2$$
$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \frac{(A_p + 3Q_p)}{A_p}\,\beta r^2$$

$$c_1 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}\left\{\frac{3A_p + Q_p}{A_p}\,b^4 + \frac{A_p - Q_p}{A_p}\,a^4\right\}$$
$$c_2 = \frac{\beta}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}\left\{\frac{A_p + Q_p}{A_p}\,a^4 b^2 - \frac{3A_p + Q_p}{A_p}\,a^2 b^4\right\}$$

B.7 Thermal load



$$f(r) = \frac{A_p + Q_p}{A_p}\,\alpha(\Delta T)_{,r}$$
$$u_r(r=0) \neq \infty$$
$$\sigma_{rr}(r=b) = 0$$

Solid disc with a radial thermal load

orthotropic material :

general solution $\quad u_r = c_1 r^{\zeta} + c_2 r^{-\zeta} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 \zeta r^{\zeta-1} - c_2 \zeta r^{-\zeta-1} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 r^{\zeta-1} + c_2 r^{-\zeta-1} + \frac{\bar{u}_r}{r}$$
$$\sigma_{rr} = (A_p\zeta + Q_p)c_1 r^{\zeta-1} - (A_p\zeta - Q_p)c_2 r^{-\zeta-1} + A_p\bar{u}_{r,r} + Q_p\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$
$$\sigma_{tt} = (Q_p\zeta + B_p)c_1 r^{\zeta-1} - (Q_p\zeta - B_p)c_2 r^{-\zeta-1} + Q_p\bar{u}_{r,r} + B_p\frac{\bar{u}_r}{r} - (B_p + Q_p)\alpha\Delta T$$

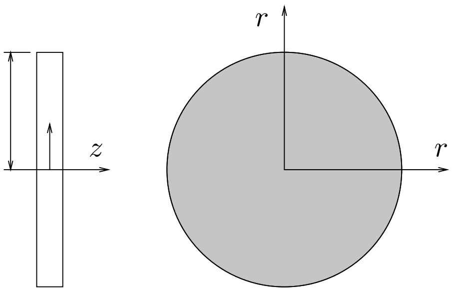
isotropic material :

general solution $\quad u_r = c_1 r + \dfrac{c_2}{r} + \bar{u}_r$

$$\varepsilon_{rr} = c_1 - c_2 r^{-2} + \bar{u}_{r,r} \quad ; \quad \varepsilon_{tt} = c_1 + c_2 r^{-2} + \frac{\bar{u}_r}{r}$$
$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} + A_p\bar{u}_{r,r} + Q_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$
$$\sigma_{tt} = (Q_p + A_p)c_1 - (Q_p - A_p)\frac{c_2}{r^2} + Q_p\bar{u}_{r,r} + A_p\,\frac{\bar{u}_r}{r} - (A_p + Q_p)\alpha\Delta T$$

B.8 Solid disc with radial temperature gradient

We only consider the isotropic case.



$u_r(r=0) \neq \infty$
$\sigma_{rr}(r=b) = 0$

Solid disc with a radial temperature gradient

$$\Delta T(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 \quad \rightarrow \quad f(r) = \frac{A_p + Q_p}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right)$$

$$u_(r) = c_1 r + \frac{c_2}{r} + \frac{A_p + Q_p}{A_p}\,\alpha\left(\tfrac{1}{3}a_1 r^2 + \tfrac{1}{4}a_2 r^3 + \tfrac{1}{5}a_3 r^4\right)$$

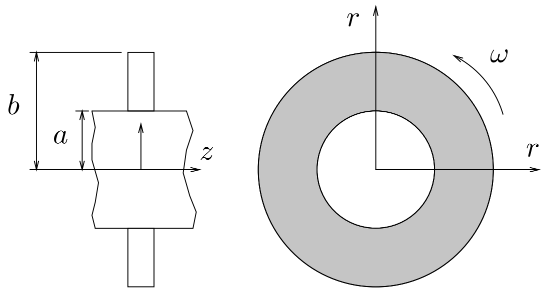
$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \alpha\left\{(A_p + Q_p)a_0 + \frac{A_p^2 - Q_p^2}{A_p}\left(\tfrac{1}{3}a_1 r + \tfrac{1}{4}a_2 r^2 + \tfrac{1}{5}a_3 r^3\right)\right\}$$

$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \alpha\left\{(A_p + Q_p)a_0 + \frac{A_p^2 - Q_p^2}{A_p}\left(\tfrac{2}{3}a_1 r + \tfrac{3}{4}a_2 r^2 + \tfrac{4}{5}a_3 r^3\right)\right\}$$

$$c_2 = 0 \qquad ; \qquad c_1 = \alpha\left\{a_0 + \frac{(A_p - Q_p)}{A_p}\left(\tfrac{1}{3}a_1 b + \tfrac{1}{4}a_2 b^2 + \tfrac{1}{5}a_3 b^3\right)\right\}$$

B.9 Disc on a rigid axis with radial temperature gradient

We only consider the isotropic case.



$u_r(r = a) = 0$
$\sigma_{rr}(r = b) = 0$

Disc on rigid axis subjected to a radial temperature gradient

$$\Delta T(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 \quad \rightarrow \quad f(r) = \frac{A_p + Q_p}{A_p}\,\alpha\left(a_1 + 2a_2 r + 3a_3 r^2\right)$$

$$u(r) = c_1 r + \frac{c_2}{r} + \frac{A_p + Q_p}{A_p}\,\alpha\left(\tfrac{1}{3}a_1 r^2 + \tfrac{1}{4}a_2 r^3 + \tfrac{1}{5}a_3 r^4\right)$$

$$\sigma_{rr} = (A_p + Q_p)c_1 - (A_p - Q_p)\frac{c_2}{r^2} - \alpha\left\{(A_p + Q_p)a_0 + \frac{A_p^2 - Q_p^2}{A_p}\left(\tfrac{1}{3}a_1 r + \tfrac{1}{4}a_2 r^2 + \tfrac{1}{5}a_3 r^3\right)\right\}$$

$$\sigma_{tt} = (A_p + Q_p)c_1 + (A_p - Q_p)\frac{c_2}{r^2} - \alpha\left\{(A_p + Q_p)a_0 + \frac{A_p^2 - Q_p^2}{A_p}\left(\tfrac{2}{3}a_1 r + \tfrac{3}{4}a_2 r^2 + \tfrac{4}{5}a_3 r^3\right)\right\}$$

$$c_1 = \frac{-\alpha(A_p + Q_p)}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}$$
$$\left\{\frac{(A_p - Q_p)}{A_p}a^2\left(\tfrac{1}{3}a_1 a + \tfrac{1}{4}a_2 a^2 + \tfrac{1}{5}a_3 a^3\right) - b^2 a_0 - \frac{(A_p - Q_p)}{A_p}b^2\left(\tfrac{1}{3}a_1 b + \tfrac{1}{4}a_2 b^2 + \tfrac{1}{5}a_3 b^3\right)\right\}$$

$$c_2 = \frac{-\alpha(A_p + Q_p)a^2 b^2}{(A_p + Q_p)b^2 + (A_p - Q_p)a^2}$$
$$\left\{\frac{(A_p + Q_p)}{A_p}\left(\tfrac{1}{3}a_1 a + \tfrac{1}{4}a_2 a^2 + \tfrac{1}{5}a_3 a^3\right) + a_0 + \frac{(A_p - Q_p)}{A_p}\left(\tfrac{1}{3}a_1 b + \tfrac{1}{4}a_2 b^2 + \tfrac{1}{5}a_3 b^3\right)\right\}$$