WR FOR AXI-SYMMETRIC DEFORMATION

1 Weighted residual formulation and FEM for axi-symmetric deformation

The equilibrium equation is transformed in a weighted residual integral according to the principle of weighted residuals. Using $dV = r d\phi dr$ and the axi-symmetry condition, the weighted residual integral becomes an integral in r only.

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r = 0 \quad \forall \quad r \quad \leftrightarrow \\
&\int_V w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\}\, dV = 0 \qquad \forall \quad w(r) \\
&2\pi t \int_{R_i}^{R_o} w\{\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + q_r\} r\, dr = 0 \qquad \forall \quad w(r)
\end{aligned}
$$

Partial integration of the first term leads to the weak form of the weighted residual integral. The right-hand side f_e represents the contribution of the external loads.

$$
\begin{aligned}
&w\sigma_{rr,r} r = w\frac{d\sigma_{rr}}{dr} r = \frac{d}{dr}(w\sigma_{rr} r) - \frac{dw}{dr}\sigma_{rr} r - w\sigma_{rr} \qquad \rightarrow \\
&\int_{R_i}^{R_o} (w_{,r}\sigma_{rr} r + w\sigma_{tt})\, dr = \int_{R_i}^{R_o} w q_r r\, dr + [w\sigma_{rr} t]_{R_i}^{R_o} = f_e
\end{aligned}
$$

1.1 Linear elastic deformation

The material behavior is described by a linear relation between the stress and strain components. The stiffness parameters A_p, B_p and Q_p can be specified for the material symmetry and for plane stress or plane strain.

$$
\left.
\begin{aligned}
\sigma_{rr} &= A_p\varepsilon_{rr} + Q_p\varepsilon_{tt} = A_p u_{r,r} + Q_p\frac{u_r}{r} \\
\sigma_{tt} &= Q_p\varepsilon_{rr} + B_p\varepsilon_{tt} = Q_p u_{r,r} + B_p\frac{u_r}{r}
\end{aligned}
\right\} \quad \rightarrow
$$

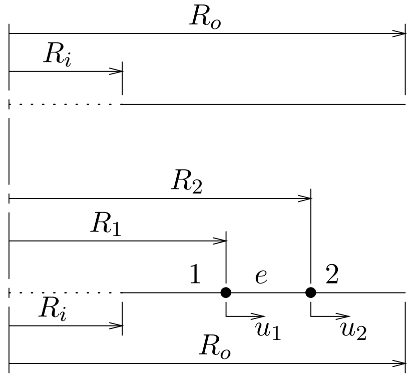
$$
\int_{R_i}^{R_o} \left\{ w_{,r}\left(A_p u_{r,r} + Q_p\frac{u_r}{r}\right) r + w\left(Q_p u_{r,r} + B_p\frac{u_r}{r}\right)\right\} dr = f_e
$$

1.2 Finite element method for an axi-symmetric ring

As always, the finite element method relies on discretisaton of the material volume, implying the weighted residual integral to be written as a sum of integrals over the indivudual elements. Unknown displacments and weighting functions are then interpolated in each element between nodal point values.

Discretisation

The disc is subdivided into ring elements, which have an inner radius R_1 and and outer radius R_2. These elements are connected in the element nodal points – in fact concentric nodal rings – and no gaps are allowed between them. The weighted residual integral can then be written as a summation of integrals over the elements.



$$\sum_{e=1}^{ne} \int_{R_1}^{R_2} \left[A_p w_{,r} u_{r,r} r + Q_p w_{,r} u_r + Q_p w u_{r,r} + B_p w \frac{1}{r} u_r \right] dr = \sum_{e=1}^{ne} f_e^e$$

Interpolation

In each element the radial displacement is written as a function of r. The coefficients are expressed in the nodal radial displacements u_1 and u_2, which leads to interpolation functions ψ_1 and ψ_2, associated with these nodes. They are a function of the radius r and are specified in section 1.3. For element type 1, the interpolation of the radial displacement is in accordance with the general solution for the homogeneous equilibrium equation. For element type 2, interpolation is done linearly between the nodal displacements.

Following the Galerkin approach, the weighting function $w(r)$ is interpolated the same way as $u_r(r)$.

$$u_r = \psi_1 u_1 + \psi_2 u_2$$

$$\text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + \psi_2 w_2$$

The interpolation for displacement and weighting function is substituted in the weighted residual integral. Derivatives of the interpolation functions w.r.t. r, are indicated as $\psi_{i,r}$. This leads to the element stiffness matrix $\underline{K}^e$ and the column with external nodal forces $\underset{\sim}{f}_e^e$.

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \int_{R_1}^{R_2} \left\{ A_p \begin{bmatrix} \psi_{1,r} \\ \psi_{2,r} \end{bmatrix} \begin{bmatrix} \psi_{1,r} & \psi_{2,r} \end{bmatrix} r + Q_p \begin{bmatrix} \psi_{1,r} \\ \psi_{2,r} \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} + \right.$$
$$\left. Q_p \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \begin{bmatrix} \psi_{1,r} & \psi_{2,r} \end{bmatrix} + B_p \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \frac{1}{r} \right\} \begin{bmatrix} u_{r1} \\ u_{r2} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \underset{\sim}{f}_e^e$$

$$
\begin{bmatrix} w_1 & w_2 \end{bmatrix} \left\{ A_p \begin{bmatrix} \psi_{1,r}\psi_{1,r} & \psi_{1,r}\psi_{2,r} \\ \psi_{2,r}\psi_{1,r} & \psi_{2,r}\psi_{2,r} \end{bmatrix} r + Q_p \begin{bmatrix} \psi_{1,r}\psi_1 & \psi_{1,r}\psi_2 \\ \psi_{2,r}\psi_1 & \psi_{2,r}\psi_2 \end{bmatrix} + \right.
$$

$$
\left. Q_p \begin{bmatrix} \psi_1\psi_{1,r} & \psi_1\psi_{2,r} \\ \psi_2\psi_{1,r} & \psi_2\psi_{2,r} \end{bmatrix} + B_p \begin{bmatrix} \psi_1\psi_1 & \psi_1\psi_2 \\ \psi_2\psi_1 & \psi_2\psi_2 \end{bmatrix} \frac{1}{r} \right\} dr \begin{bmatrix} u_{r1} \\ u_{r2} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \underset{\sim}{f}{}_e^e
$$

$$
\underset{\sim}{w}^{eT} \underline{K}^e \, \underset{\sim}{u}^e = \underset{\sim}{w}^{eT} \underset{\sim}{f}{}_e^e
$$

Integration

The element stiffness matrix has to be build by integration of functions over the element. When the interpolation function is specified, this integration can be done analytically.

$$
\begin{aligned}
K_{11}^e &= \int_{R_i}^{R_o} \left[A_p\psi_{1,r}\psi_{1,r}r + Q_p\psi_{1,r}\psi_1 + Q_p\psi_1\psi_{1,r} + B_p\psi_1\psi_1\frac{1}{r} \right] dr \\
K_{12}^e &= \int_{R_1}^{R_2} \left[A_p\psi_{1,r}\psi_{2,r}r + Q_p\psi_{1,r}\psi_2 + Q_p\psi_1\psi_{2,r} + B_p\psi_1\psi_2\frac{1}{r} \right] dr \\
K_{21}^e &= \int_{R_1}^{R_2} \left[A_p\psi_{2,r}\psi_{1,r}r + Q_p\psi_{2,r}\psi_1 + Q_p\psi_2\psi_{1,r} + B_p\psi_2\psi_1\frac{1}{r} \right] dr \\
K_{22}^e &= \int_{R_1}^{R_2} \left[A_p\psi_{2,r}\psi_{2,r}r + Q_p\psi_{2,r}\psi_2 + Q_p\psi_2\psi_{2,r} + B_p\psi_2\psi_2\frac{1}{r} \right] dr
\end{aligned}
$$

External load

The external load is the addition of a volume load and the edge loads. The latter ones can be applied directly in the edge nodes. The volume load, however, is the result of an integration procedure over the volume of the disc.

$$
f_e = \int_{R_i}^{R_o} wq_r r \, dr + [w\sigma_{rr}r]_{R_i}^{R_o} = \sum_{e=1}^{ne} \int_{R_1}^{R_2} wq_r r \, dr + [w\sigma_{rr}r]_{R_i}^{R_o} = \sum_{e=1}^{ne} q_e^e + [w\sigma_{rr}r]_{R_i}^{R_o}
$$

The contribution of the volume load can only be integrated after specification of this volume load as a function of the radius r. It is assumed here that the volume load is a centrifugal load, for which we have : $q_r = \rho\omega^2 r$.

$$
\begin{aligned}
q_e^e &= \rho\omega^2 \int_{R_1}^{R_2} wr^2 \, dr = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} r^2 \, dr = \underset{\sim}{w}^{eT} \underset{\sim}{q}^e \\
f_e^e &= \underset{\sim}{w}^{eT} \underset{\sim}{q}^e + w_o\sigma_{rr}(r = R_o)R_o - w_i\sigma_{rr}(r = R_i)R_i
\end{aligned}
$$

Assembling

Because the ring elements are placed sequentially from inside to outside in the disc, assembling the global weighted residual equation is very straightforward. The requirement that it must be satisfied for all nodal values $\underset{\sim}{w}$, results in the set of equations for the nodal displacements $\underset{\sim}{u}$.

$$\underset{\sim}{w}^T \underline{K} \, \underset{\sim}{u} = \underset{\sim}{w}^T \underset{\sim}{f}_e \quad \forall \quad \underset{\sim}{w} \qquad \Rightarrow \quad \underline{K} \, \underset{\sim}{u} = \underset{\sim}{f}_e$$

Boundary conditions

In the case of the axi-symmetric disc, we do not have to prevent rigid body motion, by suppressing nodal displacements. Nodal displacement is always associated with deformation.

1.3 Ring elements

Two elelement types can be used, which have different interpolation for the radial displacement u_r.

In element type 1 the interpolation is based on the general solution of the homogeneous differential equation for u_r in the case of isotropic material behavior. In element type 2 the interpolation is a linear function of the radial coordinate.

1.3.1 Element `elt=1`

For this element, the interpolation function is chosen in accordance with the general solution.

$$\begin{aligned}
& u_r = a_1 r + a_2 \frac{1}{r} \quad \rightarrow \quad u_{r1} = a_1 R_1 + a_2 \frac{1}{R_1} \quad ; \quad u_{r2} = a_1 R_2 + a_2 \frac{1}{R_2} \quad \rightarrow \\
& u_r = \psi_1 u_{r1} + \psi_2 u_{r2} \qquad ; \qquad u_{r,r} = \psi_{1,r} u_{r1} + \psi_{2,r} u_{r2} \\
& \psi_1 = a R_1 (-r + R_2^2 r^{-1}) \quad ; \quad \psi_2 = a R_2 (r - R_1^2 r^{-1}) \qquad \text{with} \quad a = \frac{1}{R_2^2 - R_1^2} \\
& \psi_{1,r} = a R_1 (-1 - R_2^2 r^{-2}) \quad ; \quad \psi_{2,r} = a R_2 (1 + R_1^2 r^{-2}) \\
& \\
& \text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + \psi_2 w_2
\end{aligned}$$

Element stiffness matrix

Interpolation functions and their derivatives are substituted in the integrals of the element stifness matrix and subsequently integrated. Mind that $K^e_{21} = K^e_{12}$.

$$\begin{aligned}
K^e_{11} &= \left[A_p \psi_{1,r} \psi_{1,r} r + Q_p \psi_{1,r} \psi_1 + Q_p \psi_1 \psi_{1,r} + B_p \psi_1 \psi_1 \frac{1}{r} \right] dr \\
&= a^2 R_1^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - \tfrac{1}{2}(A_p - 2Q_p + B_p) R_2^4 (R_2^{-2} - R_1^{-2}) + \right.
\end{aligned}$$

$$2(A_p - B_p)R_2^2 \ln\left(\frac{R_2}{R_1}\right)\Bigg]$$

$$\begin{aligned}
K_{12}^e &= \int_{R_1}^{R_2} \left[A_p\psi_{1,r}\psi_{2,r}r + Q_p\psi_{1,r}\psi_2 + Q_p\psi_1\psi_{2,r} + B_p\psi_1\psi_2\frac{1}{r}\right] dr \\
&= a^2R_1R_2\left[-\tfrac{1}{2}(A_p - 2Q_p + B_p)(R_2^2 - R_1^2) + \tfrac{1}{2}(A_p - 2Q_p + B_p)R_1^2R_2^2(R_2^{-2} - R_1^{-2}) - \right. \\
&\qquad \left.(A_p - B_p)(R_2^2 - R_1^2)\ln\left(\frac{R_2}{R_1}\right)\right] \\
K_{22}^e &= \int_{R_1}^{R_2} \left[A_p\psi_{2,r}\psi_{2,r}r + Q_p\psi_{2,r}\psi_2 + Q_p\psi_2\psi_{2,r} + B_p\psi_2\psi_2\frac{1}{r}\right] dr \\
&= a^2R_2^2\left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - \tfrac{1}{2}(A_p - 2Q_p + B_p)R_1^4(R_2^{-2} - R_1^{-2}) + \right. \\
&\qquad \left.2(A_p - B_p)R_1^2\ln\left(\frac{R_2}{R_1}\right)\right]
\end{aligned}$$

Centrifugal load

For a centrifugal load, the nodal forces are calculated by integration over the element.

$$\begin{aligned}
\underset{\sim}{q}^e &= \rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} r^2\,dr = \rho\omega^2 a \int_{R_1}^{R_2} \begin{bmatrix} R_1(-r^3 + R_2^2 r) \\ R_2(r^3 - R_1^2 r) \end{bmatrix} dr \\
&= \rho\omega^2 \tfrac{1}{4} \begin{bmatrix} R_1(R_1^2 + R_2^2) \\ R_2(R_1^2 + R_2^2) \end{bmatrix}
\end{aligned}$$

1.3.2 Element `elt=2`

The second element based on a linear interpolation of the radial displacement.

$$\begin{aligned}
&u_r = a_0 + a_1 r \quad \rightarrow \quad u_{r1} = a_0 + a_1R_1 \quad ; \quad u_{r2} = a_0 + a_1R_2 \quad \rightarrow \\
&u_r = \psi_1 u_{r1} + \psi_2 u_{r2} \quad ; \quad u_{r,r} = \psi_{1,r}u_1 + \psi_{2,r}u_2 \\
&\psi_1 = a(R_2 - r) \quad ; \quad \psi_2 = a(-R_1 + r) \quad ; \quad \text{with} \quad a = \frac{1}{R_2 - R_1} \\
&\psi_{1,r} = -a \quad ; \quad \psi_{2,r} = a \\
&\\
&\text{Galerkin} \quad \rightarrow \quad w = \psi_1 w_1 + w_2\psi_2
\end{aligned}$$

Element siffness matrix

Interpolation functions and their derivatives are substituted in the integrals of the element stifness matrix and subsequently integrated. Mind that $K_{21}^e = K_{12}^e$.

$$
\begin{aligned}
K_{11}^e &= \left[A_p \psi_{1,r} \psi_{1,r} r + Q_p \psi_{1,r} \psi_1 + Q_p \psi_1 \psi_{1,r} + B_p \psi_1 \psi_1 \frac{1}{r} \right] \, dr \\
&= a^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - 2(Q_p + B_p) R_2 (R_2 - R_1) + B_p R_2^2 \ln\left(\frac{R_2}{R_1} \right) \right] \\
K_{12}^e &= \int_{R_1}^{R_2} \left[A_p \psi_{1,r} \psi_{2,r} r + Q_p \psi_{1,r} \psi_2 + Q_p \psi_1 \psi_{2,r} + B_p \psi_1 \psi_2 \frac{1}{r} \right] \, dr \\
&= a^2 \left[-\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) + (Q_p + B_p)(R_2^2 - R_1^2) - B_p R_1 R_2 \ln\left(\frac{R_2}{R_1} \right) \right] \\
K_{22}^e &= \int_{R_1}^{R_2} \left[A_p \psi_{2,r} \psi_{2,r} r + Q_p \psi_{2,r} \psi_2 + Q_p \psi_2 \psi_{2,r} + B_p \psi_2 \psi_2 \frac{1}{r} \right] \, dr \\
&= a^2 \left[\tfrac{1}{2}(A_p + 2Q_p + B_p)(R_2^2 - R_1^2) - 2(Q_p + B_p) R_1 (R_2 - R_1) + B_p R_1^2 \ln\left(\frac{R_2}{R_1} \right) \right]
\end{aligned}
$$

Centrifugal load

For a centrifugal load, the nodal forces are calculated by integration over the element.

$$
\begin{aligned}
\underset{\sim}{q}^e &= \rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} r^2 \, dr = a\rho\omega^2 \int_{R_1}^{R_2} \begin{bmatrix} R_2 r^2 - r^3 \\ -R_1 r^2 + r^3 \end{bmatrix} \, dr \\
&= a\rho\omega^2 \begin{bmatrix} \frac{1}{12} R_2^4 - \frac{1}{3} R_2 R_1^3 + \frac{1}{4} R_1^4 \\ \frac{1}{12} R_1^4 - \frac{1}{3} R_1 R_2^3 + \frac{1}{4} R_2^4 \end{bmatrix}
\end{aligned}
$$

1.4 FE program `femaxi`

The Matlab program **`femaxi`** is used to analyze rings, which are subjected to various boundary conditions.

1.4.1 Thick-walled pressurized cylinder

The first example is the same as we have seen before: a cylinder subjected to an internal pressure. Parameter values are listed in the table below.

isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa	
$a = 0.25$ m	$b = 0.5$ m	$h = 0.5$ m	$E = 250$ GPa	$\nu = 0.33$

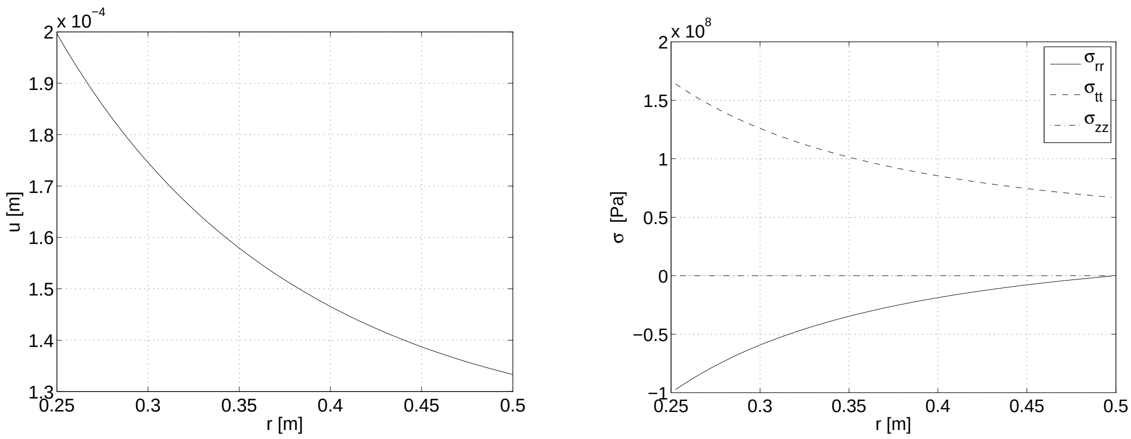


Fig. 1.1 : *Displacement and stresses in a pressurized cylinder for plane stress.*

1.4.2 Compound thick-walled pressurized cylinder

In this example, the disc is composed of two materials with different properties. The inner part is isotropic. The outer part is made of orthotropic material with an increased modulus in tangential direction, and reduced Poisson ratio's. In this case element type 2 (linear interpolation) has to be used, because type 1 interpolation field is not well suited for the orthotropic material behavior.

isotropic	plane stress	$p_i = 100$ MPa	$p_e = 0$ MPa
$a_1 = 0.25$ m	$a_2 = 0.375$ m	$E = 250$ GPa	$\nu = 0.33$
$a_2 = 0.375$ m	$b = 0.5$ m	$E1 = E$ GPa	$E2 = 10E$ GPa
$\nu_{12} = \nu/10$	$\nu_{32} = \nu/10$		

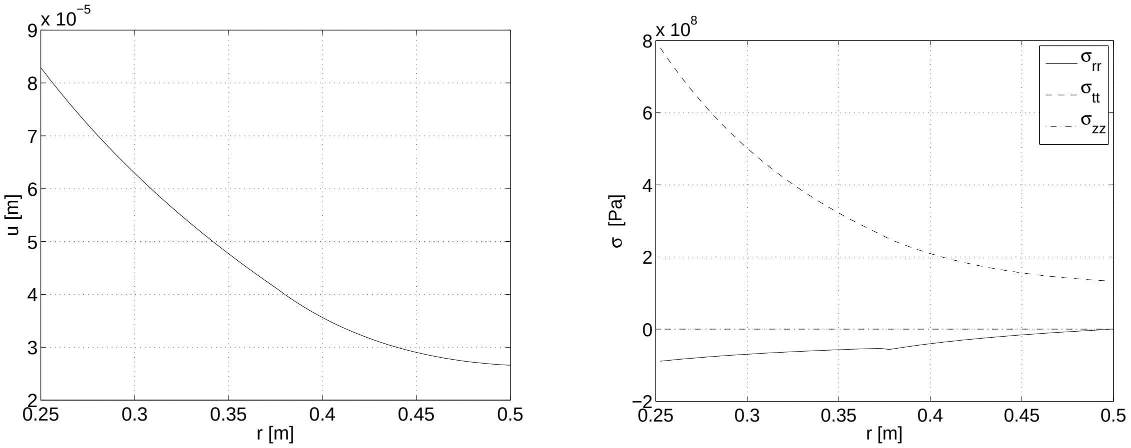


Fig. 1.2 : *Displacement and stresses in a pressurized compound cylinder for plane stress*

1.4.3 Rotating disc

The disc is rotated with an angular frequency $\omega = 6$ cycles/sec.

orthotropic	plane stress	$\omega = 6$ c/s		
$a = 0.2$ m	$b = 0.5$ m	$E = 200$ GPa	$\nu = 0.3$	$Gr = 7500$ kg/m^3

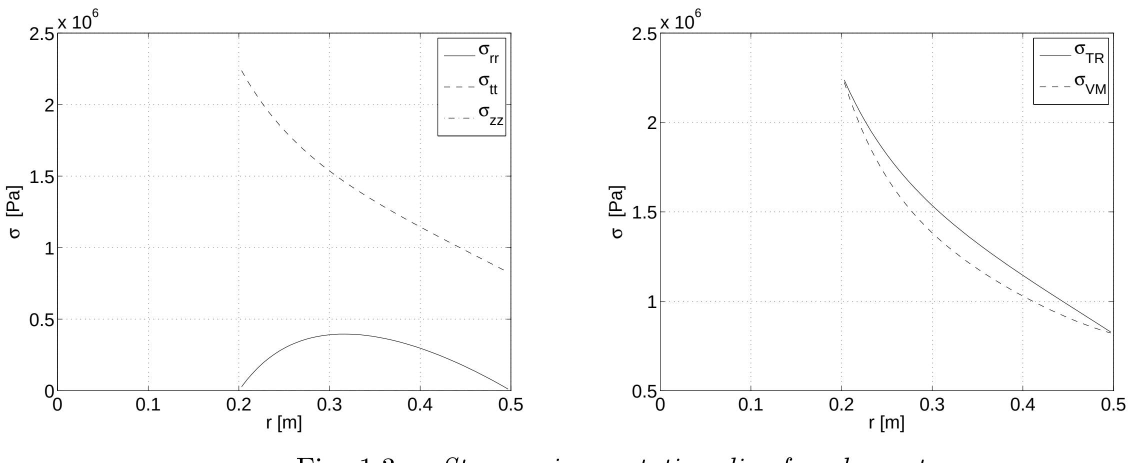


Fig. 1.3 : *Stresses in a rotating disc for plane stress.*