NUMERICAL SOLUTION PROCEDURE

1 Weighted residual formulation

Unknown variables have to be solved from the combined set of equilibrium equations and constitutive equations. Some of the equilibrium equations are partial differential equations. For the general case of large deformations and nonlinear material behavior, the equations are nonlinear. It is obvious that only for academic and very simple cases, analytic solutions exist. For more practical problems, approximate solutions must be determined with a numerical technique, of which the finite element method is widely used and will be considered here.

Application of the finite element method in continuum mechanics requires the reformulation of the equilibrium equations. They are transformed from differential equations to an integral equation, the so-called *weighted residual integral*.

First, we formulate the weighted residual integral for linear problems, so for small deformation and linear elastic material behavior. The finite element method is then explained for this case. Examples of plane stress, plane strain and axisymmetric problems will be calculated with a Matlab program.

Subsequently, we formulate the weighted residual integral for nonlinear problems, where the iterative solution procedure has to be applied. Finite element analyses can be done again with a Matlab program.

1.1 Three-dimensional deformation

For an approximation, the equilibrium equation is not satisfied exactly in each material point. The error can be "smeared out" over the material volume, using a *weighting function* $\vec{w}(\vec{x})$.

When the weighted residual integral is satisfied for each allowable weighting function $\vec{w}$, the equilibrium equation is satisfied in each point of the material.

$$\begin{array}{ll} \text{equilibrium equation} & \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0} \qquad \forall\ \vec{x} \in V \\ \text{approximation } \rightarrow \text{residual} & \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{\Delta}(\vec{x}) \neq \vec{0} \qquad \forall\ \vec{x} \in V \\ \text{weighted residual} & \int\limits_V \vec{w}(\vec{x}) \cdot \vec{\Delta}(\vec{x})\, dV = \int\limits_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} \right] dV \end{array}$$

$$\int\limits_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} \right] dV = 0 \qquad \forall \quad \vec{w}(\vec{x}) \qquad \leftrightarrow \qquad \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \vec{0} \qquad \forall\ \vec{x} \in V$$

In the weighted residual integral, one term contains the divergence of the stress tensor. This means that the integral can only be evaluated, when the derivatives of the stresses are continuous over the domain of integration. This requirement can be relaxed by applying partial integration to the term with the stress divergence. The result is the so-called weak formulation of the weighted residual integral.

Gauss theorem is used to transfer the volume integral with the term $\vec{\nabla}.(\ )$ to a surface integral. Also $\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} = \vec{n} \cdot \boldsymbol{\sigma}^c$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$ is used.

$$\left.\begin{array}{l} \displaystyle\int_V \vec{w} \cdot \left[\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} \right] dV = 0 \\ \vec{\nabla} \cdot (\boldsymbol{\sigma}^T \cdot \vec{w}) = (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}^T + \vec{w} \cdot (\vec{\nabla} \cdot \boldsymbol{\sigma}^T) \end{array} \right\} \rightarrow$$

$$\left.\begin{array}{l} \displaystyle\int_V \left[\vec{\nabla} \cdot (\boldsymbol{\sigma}^T \cdot \vec{w}) - (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}^T + \vec{w} \cdot \rho \vec{q} \right] dV = 0 \qquad \forall\, \vec{w} \\ \displaystyle\int_V \vec{\nabla} \cdot (\boldsymbol{\sigma}^T \cdot \vec{w}) = \int_V \vec{n} \cdot \boldsymbol{\sigma}^T \cdot \vec{w}\, dA = \int_A \vec{w} \cdot \vec{p}\, dA \end{array} \right\} \rightarrow$$

$$\int_V (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\, dV = \int_V \vec{w} \cdot \rho \vec{q} dV + \int_A \vec{w} \cdot \vec{p}\, dA \qquad \forall\, \vec{w}$$

$$\int_V (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall\, \vec{w}$$

1.2 Linear elastic formulation

When deformation and rotations are small, the deformation is geometrically linear. The deformed state is almost equal to the undeformed state. This implies that integration can be carried out over the undeformed volume V_0 and the undeformed area A_0.

The material behavior is described by Hooke's law, which can be substituted in the weighted residual integral, according to the displacement solution method.

The weighted residual integral is now completely expressed in the displacement $\vec{u}$. Approximate solutions can be determined with the finite element method.

$$\int_{V_0} (\vec{\nabla}\vec{w})^T : \boldsymbol{\sigma}\, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} dV_0 + \int_{A_0} \vec{w} \cdot \vec{p}\, dA_0 = f_{e0}(\vec{w}) \qquad \forall\, \vec{w}$$

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = {}^4\boldsymbol{C} : \tfrac{1}{2} \left\{ (\vec{\nabla}_0\vec{u}) + (\vec{\nabla}_0\vec{u})^T \right\} = {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^T : {}^4\boldsymbol{C} : (\vec{\nabla}_0\vec{u})\, dV_0 = \int_{V_0} \vec{w} \cdot \rho \vec{q} dV_0 + \int_{A_0} \vec{w} \cdot \vec{p}\, dA_0 = f_{e0}(\vec{w}) \qquad \forall\, \vec{w}$$

This tensor relation can be written in matrix/column notation. We use the column notation of the vector gradient as it was introduced in section **??** : $\boldsymbol{L}_{0w} = (\vec{\nabla}_0\vec{w})^c \rightarrow \underset{\approx}{L}_{0w}$ and $\boldsymbol{L}_{0u} = (\vec{\nabla}_0\vec{u})^c \rightarrow \underset{\approx}{L}_{0u}$ where $\underset{\approx}{L}$ is the column with the derivatives w.r.t. the coordinates.

$$\int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{C}} \left(\underset{\approx}{L}_{0u} \right)_t dV_0 = f_{e0}(\underset{\sim}{w}) \qquad \forall\, \underset{\sim}{w}$$