1 Total Lagrange formulation

When deformations are large – geometrically nonlinear –, the current volume of the material is unknown, which means that the weighted residual integral can not be evaluated. Transformation of this integral is always possible. Besides the integral also the gradient operator must be transformed. The configuration, which is the target of the transformation is the *reference configuration*.

The first thing we can think of is a transformation to the undeformed configuration t_0. This transformation results in the *Total Lagrange* formulation. The second Piola-Kirchhoff stress tensor is mostly used in this case to represent the stress state.

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to undeformed configuration t_0

$$\begin{array}{l} \vec{\nabla} = \boldsymbol{F}^{-c} \cdot \vec{\nabla}_0 \quad \rightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_0\vec{w})^c \cdot \boldsymbol{F}^{-1} \\ dV = \det(\boldsymbol{F}) dV_0 = J dV_0 \end{array}$$

weighted residual integral

$$\left.\begin{array}{l} \displaystyle\int_{V_0} (\vec{\nabla}_0\vec{w})^c \cdot \boldsymbol{F}^{-1} : \boldsymbol{\sigma} J\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\ \boldsymbol{P} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-c} \end{array}\right\} \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0\vec{w})^c : (\boldsymbol{P} \cdot \boldsymbol{F}^c)\, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

1.1 Iterative solution process

In the Total Lagrange formulation the weighted residual integral is transformed from the current configuration C_c to the initial undeformed configuration C_0. Unknown variables in the integral are the total deformation tensor $\boldsymbol{F}$ and the 2nd-Piola-Kirchhoff stress tensor $\boldsymbol{P}$.

To describe the essential steps of the iteration procedure, it is assumed that an approximate state C_c^* is determined with values for $\boldsymbol{F}^*$ and $\boldsymbol{P}^*$. The unknown current values are written as $\boldsymbol{F} = \boldsymbol{F}^* + \delta\boldsymbol{F}$ and $\boldsymbol{P} = \boldsymbol{P}^* + \delta\boldsymbol{P}$, where $\delta(.)$ indicates the difference between C_c^* and C_c. The iterative change of the deformation tensor $\delta\boldsymbol{F}$ can be expressed in the iterative displacement $\delta\vec{x} = \vec{u}$.

$$\left.\begin{array}{l} \displaystyle\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : (\boldsymbol{P} \cdot \boldsymbol{F}^c) \, dV_0 = f_{e0}(\vec{w}) \qquad \forall \; \vec{w}(\vec{x}) \\ \\ \boldsymbol{F} = (\vec{\nabla}_0 \vec{x})^c = \{\vec{\nabla}_0 (\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_0 \vec{x}^*)^c + (\vec{\nabla}_0 \delta\vec{x})^c = \boldsymbol{F}^* + \delta\boldsymbol{F} = \boldsymbol{F}^* + \boldsymbol{L}_{0u} \\ \boldsymbol{P} = \boldsymbol{P}^* + \delta\boldsymbol{P} \end{array} \right\} \rightarrow$$

$$\int_{V_0} (\vec{\nabla}_0 \vec{w})^c : (\boldsymbol{P}^* + \delta\boldsymbol{P}) \cdot (\boldsymbol{F}^* + \boldsymbol{L}_{0u})^c \, dV_0 = f_{e0}(\vec{w}) \qquad \forall \; \vec{w}(\vec{x})$$

It is assumed that the iterative displacement, its gradient and the stress variation are very small and then the weighted residual integral is linearized with respect to $\vec{u}$. In analogy with $\boldsymbol{L}_{0u}$, $\boldsymbol{L}_{0w} = (\vec{\nabla}_0 \vec{w})^c$ is introduced.

$$\begin{aligned}
&\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* + \delta\boldsymbol{P}) \cdot (\boldsymbol{F}^* + \boldsymbol{L}_{0u})^c \, dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\
&\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c} + \boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c + \delta\boldsymbol{P} \cdot \boldsymbol{F}^{*c}) \; dV_0 = f_{e0}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \\
&\int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c + \delta\boldsymbol{P} \cdot \boldsymbol{F}^{*c}) \; dV_0 = \\
&\qquad\qquad f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c}) \, dV_0 = r^* \qquad \forall \quad \vec{w}(\vec{x})
\end{aligned}$$

1.2 Material model

The right-hand side of the iterative equation represents the residual load. To calculate r^* and the term with $\boldsymbol{P}^*$ in the left-hand integral, the stress $\boldsymbol{P}^*(t)$ must be determined from the constitutive equation. From the material model also a relation between $\delta\boldsymbol{P}$ and $\boldsymbol{L}_{0u}$ must be derived. The iterative change of the 2nd-Piola-Kirchhoff stress $\delta\boldsymbol{P}$, must be expressed in the iterative displacement $\vec{u}$ and substituted in the iterative weighted residual integral.

$$\delta\boldsymbol{P} = {}^4\boldsymbol{M} : \boldsymbol{L}_{0u} \quad \rightarrow$$

$$\begin{aligned}
&\int_{V_0} \boldsymbol{L}_{0w} : \left(\boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c + ({}^4\boldsymbol{M} : \boldsymbol{L}_{0u}) \cdot \boldsymbol{F}^{*c}\right) \; dV_0 = \\
&\qquad\qquad f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x}) \\
&\int_{V_0} \left[\boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{L}_{0u}^c) + \boldsymbol{L}_{0w} : (\boldsymbol{F}^* \cdot {}^4\boldsymbol{M}^{lrc}) : \boldsymbol{L}_{0u}^c\right] \; dV_0 = \\
&\qquad\qquad f_{e0}(\vec{w}) - \int_{V_0} \boldsymbol{L}_{0w} : (\boldsymbol{P}^* \cdot \boldsymbol{F}^{*c}) \, dV_0 \qquad \forall \quad \vec{w}(\vec{x})
\end{aligned}$$

1.3 Matrix/column notation

We will now express the vectors and tensors in their components w.r.t. a basis of a coordinate system. A matrix-column notation is used, which is explained elsewhere. The asterisk $(\ )^*$ indicating an approximate value is omitted.

$$\int_{V_0} \left[\left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{P}} \left(\underset{\approx}{L}_{0u} \right)_t + \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \underline{\underline{M}}_{0c} \left(\underset{\approx}{L}_{0u} \right)_t \right] dV_0 =$$

$$f_{e0}(\underset{\sim}{w}) - \int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \underline{\underline{F}}_{cr} \underset{\approx}{P} \, dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

$$\int_{V_0} \left(\underset{\approx}{L}_{0w} \right)_t^T \left[\underline{\underline{P}} + \underline{\underline{F}}_{cr} \underline{\underline{M}}_{0c} \right] \left(\underset{\approx}{L}_{0u} \right)_t dV_0 = f_{e0}(\underset{\sim}{w}) - f_{i0}(\underset{\sim}{w})$$

2 Updated Lagrange formulation

In the *Updated Lagrange* formulation the reference configuration is chosen to be the start of the current increment at t_n.

$$\int_V (\vec{\nabla}\vec{w})^c : \boldsymbol{\sigma}\, dV = f_e(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

transformation to begin increment configuration t_n

$$\begin{aligned} &\vec{\nabla} = \boldsymbol{F}_n^{-c} \cdot \vec{\nabla}_n \quad \rightarrow \quad (\vec{\nabla}\vec{w})^c = (\vec{\nabla}_n\vec{w})^c \cdot \boldsymbol{F}_n^{-1} \\ &dV = \det(\boldsymbol{F}_n) dV_n \end{aligned}$$

weighted residual integral

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c \cdot \boldsymbol{F}_n^{-1} : \boldsymbol{\sigma} \det(\boldsymbol{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x}) \quad \rightarrow$$

$$\int_{V_n} (\vec{\nabla}_n\vec{w})^c : (\boldsymbol{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\boldsymbol{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

2.1 Iterative solution process

To describe the essential steps of the iteration procedure, it is assumed that an approximate state C_c^* is determined with values for $\boldsymbol{F}_n^*$, $\boldsymbol{\sigma}^*$ and the other variables. The unknown current values are written as $\boldsymbol{F}_n = (\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}$, where $\delta(.)$ indicates the difference between C_c^* and C_c, and $\boldsymbol{L}_u^* = (\vec{\nabla}^*\vec{u})^c$, with $\vec{u} = \delta\underset{\sim}{\vec{x}}$ the iterative displacement.

$$\left.\begin{aligned} &\int_{V_n} (\vec{\nabla}_n\vec{w})^c : (\boldsymbol{F}_n^{-1} \cdot \boldsymbol{\sigma}) \det(\boldsymbol{F}_n)\, dV_n = f_{en}(\vec{w}) \qquad \forall\, \vec{w}(\vec{x}) \\ &\boldsymbol{F}_n = (\vec{\nabla}_n\vec{x})^c = \{\vec{\nabla}_n(\vec{x}^* + \delta\vec{x})\}^c = (\vec{\nabla}_n\vec{x}^*)^c + (\vec{\nabla}_n\delta\vec{x})^c \\ &\quad = \boldsymbol{F}_n^* + \delta\boldsymbol{F}_n = \boldsymbol{F}_n^* + (\vec{\nabla}^*\delta\vec{x})^c \cdot (\vec{\nabla}_n\vec{x}^*)^c = \boldsymbol{F}_n^* + \boldsymbol{L}_u^* \cdot \boldsymbol{F}_n^* = (\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^* \\ &\boldsymbol{\sigma} = \boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma} \end{aligned}\right\} \rightarrow$$

$$\begin{aligned} \int_{V_n} (\vec{\nabla}_n\vec{w})^c : \left[(\boldsymbol{F}_n^*)^{-1} \cdot (\boldsymbol{I} + \boldsymbol{L}_u^*)^{-1} \cdot (\boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma}) \ \det\{(\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*\} \right] dV_n \\ = f_{en}(\vec{w}) \qquad \forall\, \vec{w}(\vec{x}) \end{aligned}$$

Assuming that the iterative displacement and its gradient are very small, the weighted residual integral can be linearized with respect to $\vec{u}$. In analogy with $\boldsymbol{L}_u^*$, $\boldsymbol{L}_w^* = (\vec{\nabla}^*\vec{w})^c$ is introduced.

$$(\boldsymbol{I} + \boldsymbol{L}_u^*)^{-1} \approx \boldsymbol{I} - \boldsymbol{L}_u^*$$

$$\det\{(\boldsymbol{I} + \boldsymbol{L}_u^*) \cdot \boldsymbol{F}_n^*\} = \det(\boldsymbol{I} + \boldsymbol{L}_u^*)\det(\boldsymbol{F}_n^*) \approx \operatorname{tr}(\boldsymbol{I} + \boldsymbol{L}_u^*)\det(\boldsymbol{F}_n^*) = (1 + \boldsymbol{I} : \boldsymbol{L}_u^*)\det(\boldsymbol{F}_n^*)$$

weighted residual integral

$$\int_{V_n} (\vec{\nabla}_n \vec{w})^c : \left[(\boldsymbol{F}_n^*)^{-1} \cdot (\boldsymbol{I} - \boldsymbol{L}_u^*) \cdot (\boldsymbol{\sigma}^* + \delta\boldsymbol{\sigma})(1 + \boldsymbol{I} : \boldsymbol{L}_u^*)\det(\boldsymbol{F}_n^*)\right] dV_n$$
$$= f_{en}(\vec{w}) \qquad \forall \quad \vec{w}(\vec{x})$$

further linearisation

$$\int_{V^*} [\boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \boldsymbol{I} : \boldsymbol{L}_u^{*c} + \boldsymbol{L}_w^* : \delta\boldsymbol{\sigma} - \boldsymbol{L}_w^* : (\boldsymbol{\sigma}^{*c} \cdot \boldsymbol{L}_u^{*c})^c] \, dV^* =$$
$$f_e^*(\vec{w}) - \int_{V^*} \boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \, dV^* = r^* \qquad \forall \quad \vec{w}(\vec{x})$$

2.2 Material model

The right-hand side of the iterative equation represents the residual load. To calculate r^* and two terms in the left-hand integral, the stress $\boldsymbol{\sigma}^*(t)$ must be determined from the constitutive equation.

The iterative change of the stress $\delta\boldsymbol{\sigma}$, must be expressed in the iterative displacement $\vec{u}$ and substituted in the iterative weighted residual integral.

$$\delta\boldsymbol{\sigma} = {}^4\boldsymbol{M} : \boldsymbol{L}_u^* \quad \rightarrow$$

$$\int_{V^*} \left[\boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \boldsymbol{I} : \boldsymbol{L}_u^{*c} + \boldsymbol{L}_w^* : {}^4\boldsymbol{M} : \boldsymbol{L}_u^* - \boldsymbol{L}_w^* : (\boldsymbol{\sigma}^{*c} \cdot \boldsymbol{L}_u^{*c})^c\right] dV^* =$$
$$f_e^*(\vec{w}) - \int_{V^*} \boldsymbol{L}_w^* : \boldsymbol{\sigma}^* \, dV^* \qquad \forall \quad \vec{w}(\vec{x})$$

2.3 Matrix/column notation

We will now express the vectors and tensors in their components w.r.t. a basis of a coordinate system. A matrix-column notation is used, which is explained elsewhere. The asterisk $(\ )^*$ indicating an approximate value is omitted.

$$\int_{V^*} \left[\left(\underset{\approx}{L}_w\right)_t^T \underset{\approx}{\sigma} \underset{\approx}{I}^T \left(\underset{\approx}{L}_u\right)_t + \left(\underset{\approx}{L}_w\right)_t^T \underline{\underline{M}} \left(\underset{\approx}{L}_u\right)_t - \left(\underset{\approx}{L}_w\right)_t^T \underline{\underline{\sigma}}_{tr} \left(\underset{\approx}{L}_u\right)_t\right] dV^* =$$
$$f_e(\underset{\sim}{w}) - \int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \underset{\approx}{\sigma} \, dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

$$\int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \left[\underset{\approx}{\sigma} \underset{\approx}{I}^T - \underline{\underline{\sigma}}_{tr} + \underline{\underline{M}}\right] \left(\underset{\approx}{L}_u\right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$

$$\int_{V^*} \left(\underset{\approx}{L}_w\right)_t^T \left[\underline{\underline{\Sigma}} + \underline{\underline{M}}\right] \left(\underset{\approx}{L}_u\right)_t dV^* = f_e(\underset{\sim}{w}) - f_i(\underset{\sim}{w})$$