SOLUTION STRATEGIES

1 Solution strategies

1.1 Governing equations

The deformation of a three-dimensional continuum in three-dimensional space is described by the displacement vector $\vec{u}$ of each material point. Due to the deformation, stresses arise and the stress state is characterized by the stress tensor $\boldsymbol{\sigma}$. For static problems, this tensor has to satisfy the equilibrium equations. Solving stresses from these equations is generally not possible and additional equations are needed, which relate stresses to deformation. These constitutive equations, which describe the material behavior, relate the stress tensor $\boldsymbol{\sigma}$ to the strain tensor $\boldsymbol{\varepsilon}$, which is a function of the displacement gradient tensor $(\vec{\nabla}\vec{u})$. Components of this strain tensor cannot be independent and are related by the compatibility equations.

$$
\begin{array}{lll}
\text{unknown variables} & & \\
\quad \text{displacements} & : & \vec{u} = \vec{u}(\vec{x}) \quad \rightarrow \quad \boldsymbol{F} = \left(\vec{\nabla}_0 \vec{x}\right)^T \quad \rightarrow \quad \boldsymbol{E}\ ,\ \boldsymbol{\varepsilon} \\
\quad \text{stresses} & : & \boldsymbol{\sigma} \qquad \rightarrow \qquad g(\boldsymbol{\sigma}) = g(\sigma_1, \sigma_2, \sigma_3) = g_t \\
\text{equations} & & \\
\quad \text{compatibility} & : & \nabla^2 \left\{ \mathrm{tr}(\boldsymbol{\varepsilon}) \right\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \boldsymbol{\varepsilon})^T = 0 \\
\quad \text{equilibrium} & : & \vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho \vec{q} = \rho \ddot{\vec{u}} \qquad ; \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \\
\quad \text{material law} & : & \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{F}) \quad \rightarrow \quad \boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}
\end{array}
$$

1.1.1 Boundary conditions

Some of the governing equations are partial differential equations, where differentiation is done w.r.t. the spatial coordinates. These differential equations can only be solved when proper boundary conditions are specified. In each boundary point of the material body, either the displacement or the load must be prescribed. It is also possible to specify a relation between displacement and load in such a point.

When the acceleration of the material points cannot be neglected, the equilibrium equation becomes the equation of motion, with $\rho\ddot{\vec{u}}$ as its right-hand term. In that case a solution can only be determined when proper initial conditions are prescribed, i.e. initial displacement, velocity or acceleration. In this section we will assume $\ddot{\vec{u}} = \vec{0}$.

$$
\begin{array}{lll}
\text{displacement} & : & \vec{u} = \vec{u}_p \qquad \forall \quad \vec{x} \in A_u \\
\text{edge load} & : & \vec{p} = \vec{n} \cdot \boldsymbol{\sigma} = \vec{p}_p \qquad \forall \quad \vec{x} \in A_p
\end{array}
$$

Saint-Venant's principle

The so-called *Saint-Venant principle* states that, if a load on a structure is replaced by a statically equivalent load, the resulting strains and stresses in the structure will only be altered near the regions where the load is applied. With this principle in mind, the real boundary conditions can often be modeled in a simplified way. Concentrated forces can for instance be replaced by distributed loads, and vice versa. Stresses and strains will only differ significantly in the neighborhood of the boundary, where the load is applied.

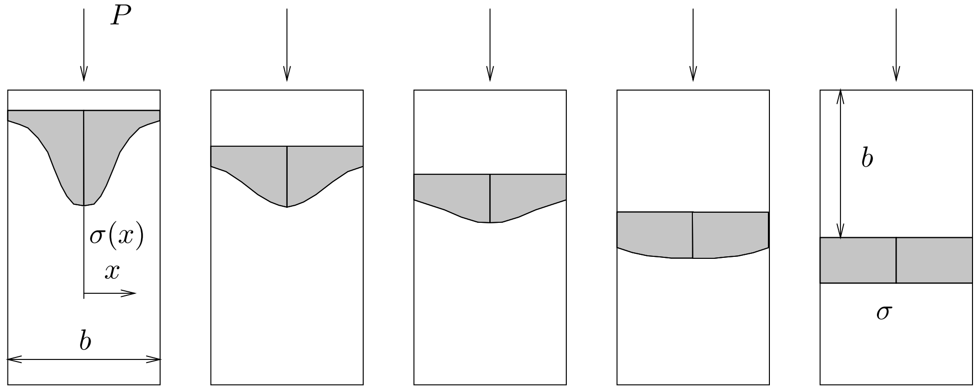

Fig. 1.1 : *Saint-Venant principle*

$$P = \int_A \sigma(x)\, dA = \sigma\, A \qquad ; \qquad A = b * t$$

1.1.2 Superposition

Under the assumption of small deformations and linear elastic material behavior, the governing equations, which must be solved to determine deformation and stresses (= solution S) are linear. When boundary conditions (fixations and loads (L)), which are needed for the solution, are also linear, the total problem is linear and the principle of superposition holds.

The principle of superposition states that the solution S for a given combined load $L = L_1 + L_2$ is the sum of the solution S_1 for load L_1 and the solution S_2 for L_2, so : $S = S_1 + S_2$.

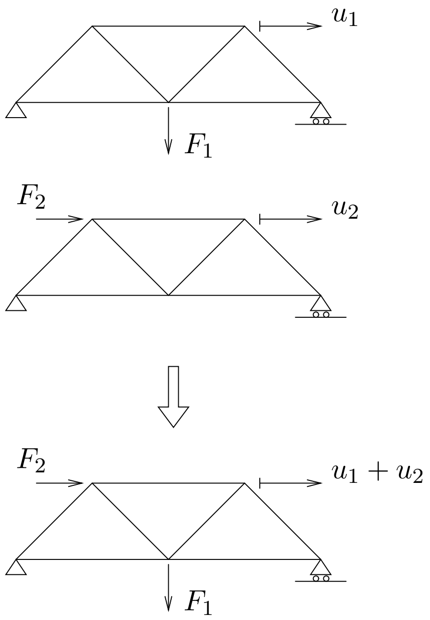

Fig. 1.2 : *Principle of superposition*

1.2 Solution : displacement method

In the *displacement method* the constitutive relation for the stress tensor is substituted in the force equilibrium equation.

Subsequently the strain tensor is replaced by its definition in terms of the displacement gradient. This results in a differential equation in the displacement $\vec{u}$, which can be solved when proper boundary conditions are specified.

In a Cartesian coordinate system the vector/tensor formulation can be replaced by index notation. It is elaborated here for the case of linear elasticity theory.

$$\left.\begin{array}{l}\vec{\nabla}\cdot\boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ \boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{c}\vec{\nabla}\cdot\left({}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\right)^T + \rho\vec{q} = \rho\ddot{\vec{u}} \\ \boldsymbol{\varepsilon} = \frac{1}{2}\left\{\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right\}\end{array}\right\} \quad \rightarrow$$

$$\vec{\nabla}\cdot\left\{{}^4\boldsymbol{C} : \left(\vec{\nabla}\vec{u}\right)\right\}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad \rightarrow \quad \vec{u} \quad \rightarrow \quad \boldsymbol{\varepsilon} \quad \rightarrow \quad \boldsymbol{\sigma}$$

Cartesian index notation

$$\left.\begin{array}{l}\sigma_{ij,j} + \rho q_i = 0_i \\ \sigma_{ij} = C_{ijlk}\varepsilon_{lk}\end{array}\right\} \quad \rightarrow \quad \left.\begin{array}{l}C_{ijkl}\varepsilon_{lk,j} + \rho q_i = 0_i \\ \varepsilon_{lk} = \frac{1}{2}\left(u_{l,k} + u_{k,l}\right)\end{array}\right\} \quad \rightarrow$$

$$C_{ijkl}u_{l,kj} + \rho q_i = 0_i \quad \rightarrow \quad u_i \quad \rightarrow \quad \varepsilon_{ij} \quad \rightarrow \quad \sigma_{ij}$$

1.2.1 Planar, Cartesian : Navier equations

The displacement method is elaborated for planar deformation in a Cartesian coordinate system. Linear deformation and linear elastic material behavior is assumed. Elimination and substitution results in two partial differential equations for the two displacement components. For the sake of simplicity, we do not consider thermal loading here.

$$\left.\begin{array}{l}
\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad ; \qquad \sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y \\[2ex]
\sigma_{xx} = A_p \varepsilon_{xx} + Q_p \varepsilon_{yy} \\
\sigma_{yy} = Q_p \varepsilon_{xx} + B_p \varepsilon_{yy} \\
\sigma_{xy} = 2K \varepsilon_{xy}
\end{array}\right\}$$

$$\left.\begin{array}{l}
A_p \varepsilon_{xx,x} + Q_p \varepsilon_{yy,x} + 2K \varepsilon_{xy,y} + \rho q_x = \rho \ddot{u}_x \\
2K \varepsilon_{xy,x} + Q_p \varepsilon_{xx,y} + B_p \varepsilon_{yy,y} + \rho q_y = \rho \ddot{u}_y
\end{array}\right\}$$

$$\left.\begin{array}{l}
A_p u_{x,xx} + K u_{x,yy} + (Q_p + K) u_{y,yx} + \rho q_x = \rho \ddot{u}_x \\
K u_{y,xx} + B_p u_{y,yy} + (Q_p + K) u_{x,xy} + \rho q_y = \rho \ddot{u}_y
\end{array}\right\}$$

1.2.2 Planar, axi-symmetric with $u_t = 0$

Many engineering problems present a rotational symmetry w.r.t. an axis. They are *axi-symmetric*. In many cases the tangential displacement is zero : $u_t = 0$. This implies that there are no shear strains and stresses.

The radial and tangential stresses are related to the radial and tangential strains by the planar material law. Material parameters are indicated as A_p, B_p and Q_p and can later be specified for a certain material and for plane strain or plane stress. With the strain-displacement relations the equation of motion can be transformed into a differential equation for the radial displacement u_r

$$\begin{array}{lll}
\text{displacements} & : & u_r = u_r(r) \quad ; \quad u_z = u_z(r,z) \\
\text{strains} & : & \varepsilon_{rr} = u_{r,r} \ ; \ \varepsilon_{tt} = \frac{1}{r} u_r \ ; \ \varepsilon_{zz} = u_{z,z} \\
\text{stresses} & : & \sigma_{tz} = 0 \quad ; \quad \sigma_{rz} \approx 0 \quad ; \quad \sigma_{tr} = 0 \\
\text{eq. of motion} & : & \sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r
\end{array}$$

$$\left.\sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} \sigma_{tt} = Q_p \varepsilon_{rr} + B_p \varepsilon_{tt} \right\} \quad \rightarrow \quad \text{eq. of motion} \quad \rightarrow$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \zeta^2 \frac{1}{r^2} u_r = f(r)$$

$$\text{with} \qquad \zeta = \sqrt{\frac{B_p}{A_p}}$$

$$\text{and} \qquad f(r) = \frac{\rho}{A_p} (\ddot{u}_r - q_r) + \frac{\Theta_{p1}}{A_p} \alpha (\Delta T)_{,r} + \frac{\Theta_{p1} - \Theta_{p2}}{A_p} \frac{1}{r} \alpha \Delta T$$

For isotropic material the coefficients A_p and B_p are the same, which implies that $\zeta = 1$ and also $\Theta_{p1} = \Theta_{p2}$.

$$\left.\begin{array}{l} \sigma_{rr,r} + \dfrac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \\ \\ \left.\begin{array}{l} \sigma_{rr} = A_p \varepsilon_{rr} + Q_p \varepsilon_{tt} \\ \sigma_{tt} = Q_p \varepsilon_{rr} + A_p \varepsilon_{tt} \end{array}\right\} \end{array}\right\}$$

$$\left.\begin{array}{l} A_p \varepsilon_{rr,r} + Q_p \varepsilon_{tt,r} + \dfrac{1}{r}\left\{(A_p - Q_p)\varepsilon_{rr} + (Q_p - A_p)\varepsilon_{tt}\right\} + \rho q_r = \rho \ddot{u}_r \\ \varepsilon_{rr} = u_{r,r} \quad ; \quad \varepsilon_{tt} = \dfrac{1}{r} u_r \end{array}\right\}$$

$$u_{r,rr} + \frac{1}{r} u_{r,r} - \frac{1}{r^2} u_r = \frac{\rho}{A_p}\, (\ddot{u}_r - q_r) = f(r)$$