GOVERNING EQUATIONS

1 Governing equations

In this chapter we will recall the equations, which have to be solved to determine the deformation of a three-dimensional linear elastic material body under the influence of an external load. The equations will be written in component notation w.r.t. a Cartesian and a cylindrical vector base and simplified for plane strain, plane stress and axi-symmetry. The material behavior is assumed to be isotropic.

1.1 Vector/tensor equations

The deformed (current) state is determined by 12 state variables : 3 displacement components and 9 stress components. These unknown quantities must be solved from 12 equations : 6 equilibrium equations and 6 constitutive equations.

With proper boundary (and initial) conditions the equations can be solved, which, for practical problems, must generally be done numerically. The compatibility equations are generally satisfied for the chosen strain-displacement relation. In some solution approaches they are used instead of the equilibrium equations.

gradient operator	:	$\vec{\nabla} = \underset{\sim}{\nabla}^T \underset{\sim}{\vec{e}}$
position	:	$\vec{x} = \underset{\sim}{x}^T \underset{\sim}{\vec{e}}$
displacement	:	$\vec{u} = \underset{\sim}{u}^T \underset{\sim}{\vec{e}}$
strain	:	$\boldsymbol{\varepsilon} = \frac{1}{2}\left\{ \left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$
compatibility	:	$\nabla^2 \{\mathrm{tr}(\boldsymbol{\varepsilon})\} - \vec{\nabla} \cdot (\vec{\nabla} \cdot \boldsymbol{\varepsilon})^T = 0$
stress	:	$\boldsymbol{\sigma} = \underset{\sim}{\vec{e}}^T \underline{\sigma}\, \underset{\sim}{\vec{e}}$
balance laws	:	$\vec{\nabla} \cdot \boldsymbol{\sigma}^T + \rho\vec{q} = \rho\ddot{\vec{u}} \quad ; \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T$
material law	:	$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \quad ; \quad \boldsymbol{\varepsilon} = {}^4\boldsymbol{C}^{-1} : \boldsymbol{\sigma} = {}^4\boldsymbol{S} : \boldsymbol{\sigma}$

1.2 Three-dimensional deformation

The vectors and tensors can be written in components with respect to a three-dimensional vector basis. For various problems in mechanics, it will be suitable to choose either a Cartesian coordinate system or a cylindrical coordinate system.

1.2.1 Cartesian components

The governing equations are written in components w.r.t. a Cartesian vector base $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$. The stresses can be represented with a Cartesian stress cube.

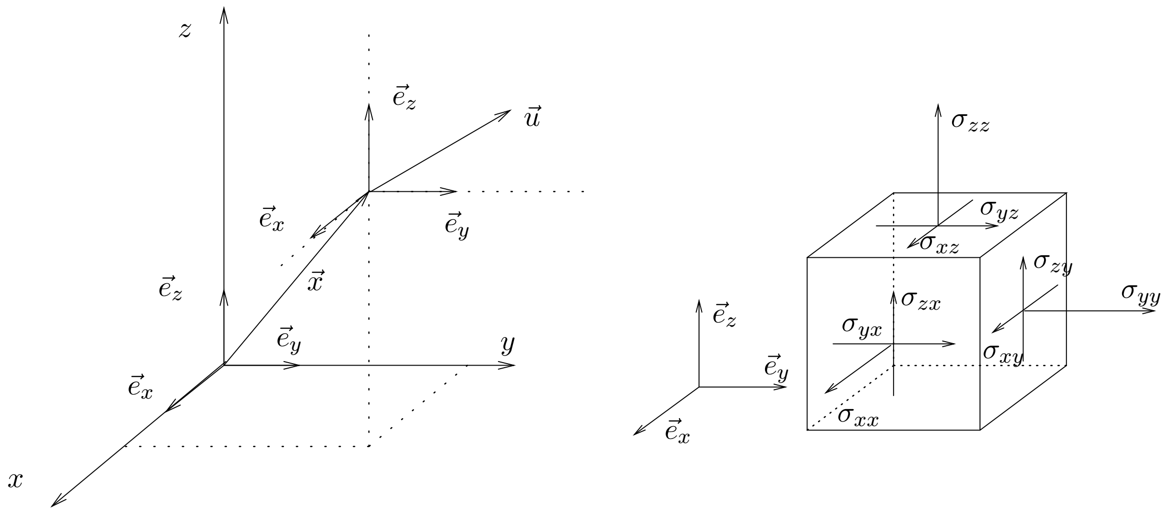

Fig. 1.1 : *Cartesian coordinate system and stress cube*

$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} x & y & z \end{array}\right] \quad ; \quad \underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array}\right] \quad ; \quad \underset{\sim}{u}^T = \left[\begin{array}{ccc} u_x & u_y & u_z \end{array}\right]$$

$$\underline{\varepsilon} = \tfrac{1}{2}\left[\begin{array}{ccc} 2u_{x,x} & u_{x,y}+u_{y,x} & u_{x,z}+u_{z,x} \\ \cdots & 2u_{y,y} & u_{y,z}+u_{z,y} \\ \cdots & \cdots & 2u_{z,z} \end{array}\right]$$

$$2\varepsilon_{xy,xy} - \varepsilon_{xx,yy} - \varepsilon_{yy,xx} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{xx,yz} + \varepsilon_{yz,xx} - \varepsilon_{zx,xy} - \varepsilon_{xy,xz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

$$\underset{\approx}{\varepsilon}^T = \left[\begin{array}{cccccc} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{xy} & \varepsilon_{yz} & \varepsilon_{zx} \end{array}\right]$$
$$\underset{\approx}{\sigma}^T = \left[\begin{array}{cccccc} \sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{array}\right]$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = \rho \ddot{u}_x \qquad (\sigma_{xy} = \sigma_{yx})$$
$$\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y = \rho \ddot{u}_y \qquad (\sigma_{yz} = \sigma_{zy})$$
$$\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zx} = \sigma_{xz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

1.2.2 Cylindrical components

The governing equations are written in components w.r.t. a cylindrical vector base $\{\vec{e}_r(\theta), \vec{e}_t(\theta), \vec{e}_z\}$, with :

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_t \qquad \text{and} \qquad \frac{d\vec{e}_t}{d\theta} = -\vec{e}_r$$

The stress components can be represented with a cylindrical stress cube.

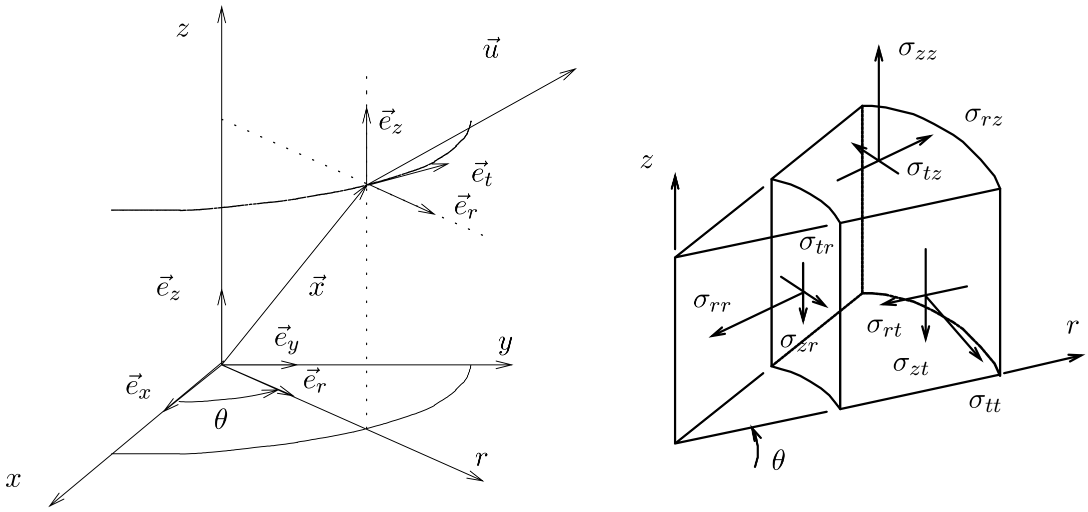

Fig. 1.2 : *Cylindrical coordinate system and stress cube*

$$\underset{\sim}{x}^T = \left[\begin{array}{ccc} r & \theta & z \end{array}\right] \quad ; \quad \underset{\sim}{\nabla}^T = \left[\begin{array}{ccc} \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \end{array}\right] \quad ; \quad \underset{\sim}{u}^T = \left[\begin{array}{ccc} u_r & u_t & u_z \end{array}\right]$$

$$\underline{\varepsilon} = \frac{1}{2}\left[\begin{array}{ccc} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \cdots & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ \cdots & \cdots & 2u_{z,z} \end{array}\right]$$

$$2\varepsilon_{rt,rt} - \varepsilon_{rr,tt} - \varepsilon_{tt,rr} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$
$$\varepsilon_{rr,tz} + \varepsilon_{tz,rr} - \varepsilon_{zr,rt} - \varepsilon_{rt,rz} = 0 \quad \rightarrow \quad \text{cyc. 2x}$$

$$\underset{\approx}{\varepsilon}^T = \left[\begin{array}{cccccc} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{zz} & \varepsilon_{rt} & \varepsilon_{tz} & \varepsilon_{zr} \end{array}\right]$$
$$\underset{\approx}{\sigma}^T = \left[\begin{array}{cccccc} \sigma_{rr} & \sigma_{tt} & \sigma_{zz} & \sigma_{rt} & \sigma_{tz} & \sigma_{zr} \end{array}\right]$$

$$\sigma_{rr,r} + \tfrac{1}{r}\,\sigma_{rt,t} + \tfrac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$
$$\sigma_{tr,r} + \tfrac{1}{r}\,\sigma_{tt,t} + \tfrac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t = \rho \ddot{u}_t \qquad (\sigma_{tz} = \sigma_{zt})$$
$$\sigma_{zr,r} + \tfrac{1}{r}\,\sigma_{zt,t} + \tfrac{1}{r}\,\sigma_{zr} + \sigma_{zz,z} + \rho q_z = \rho \ddot{u}_z \qquad (\sigma_{zr} = \sigma_{rz})$$

$$\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon} \qquad ; \qquad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma}$$

1.3 Material law

When deformations are small, every material will show linear elastic behavior. For orthotropic material there are 9 independent material constants. When there is more material symmetry, this number decreases. Finally, isotropic material can be characterized with only two material constants.

Be aware that we use now the strain components ε_{ij} and not the shear components γ_{ij}. In an earlier chapter, the parameters for orthotropic, transversally isotropic and isotropic material were rewritten in terms of engineering parameters: Young's moduli and Poisson's ratio's.

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 2K & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix} \rightarrow \underline{\underline{S}} = \underline{\underline{C}}^{-1} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}l & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}m \end{bmatrix}$$

quadratic	$B = A \; ; \; S = R \; ; \; M = L;$
transversal isotropic	$B = A \; ; \; S = R \; ; \; M = L \; ; \; K = \frac{1}{2}(A - Q)$
cubic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K \neq \frac{1}{2}(A - Q)$
isotropic	$C = B = A \; ; \; S = R = Q \; ; \; M = L = K = \frac{1}{2}(A - Q)$

1.4 Planar deformation

In many applications the loading and deformation is in one plane. The result is that the material body is in a state of plane strain or plane stress. The governing equations can than be simplified considerably.

1.4.1 Cartesian components

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left.\begin{array}{lcl} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_x = u_x(x, y) \\ u_y = u_y(x, y) \end{array}\right.$$

$$\underset{\sim}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} u_{x,x} & u_{y,y} & \frac{1}{2}(u_{x,y} + u_{y,x}) \end{bmatrix}$$

$$\underset{\sim}{\sigma}^T = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = \rho \ddot{u}_x \qquad (\sigma_{xy} = \sigma_{yx})$$

$$\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = \rho \ddot{u}_y$$

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{bmatrix}$$

1.4.2 Cylindrical components

In a plane strain situation, deformation in one direction – here the z-direction – is suppressed. In a plane stress situation, stresses on one plane – here the plane with normal in z-direction – are zero.

Eliminating σ_{zz} for plane strain and ε_{zz} for plane stress leads to a simplified Hooke's law. Also the equilibrium equation in the z-direction is automatically satisfied and has become obsolete.

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \left[\begin{array}{ccc} \varepsilon_{rr} & \varepsilon_{tt} & \varepsilon_{rt} \end{array}\right] = \left[\begin{array}{ccc} u_{r,r} & \frac{1}{r}(u_r + u_{t,t}) & \frac{1}{2}\left(\frac{1}{r}(u_{r,t} - u_t) + u_{t,r}\right) \end{array}\right]$$

$$\underset{\approx}{\sigma}^T = \left[\begin{array}{ccc} \sigma_{rr} & \sigma_{tt} & \sigma_{rt} \end{array}\right]$$

$$\sigma_{rr,r} + \tfrac{1}{r}\,\sigma_{rt,t} + \tfrac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r \qquad (\sigma_{rt} = \sigma_{tr})$$

$$\sigma_{tr,r} + \tfrac{1}{r}\,\sigma_{tt,t} + \tfrac{1}{r}\,(\sigma_{tr} + \sigma_{rt}) + \rho q_t = \rho \ddot{u}_t$$

$$\underline{\underline{C}}_p = \left[\begin{array}{ccc} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & 2K \end{array}\right] \quad ; \quad \underline{\underline{S}}_p = \left[\begin{array}{ccc} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & \frac{1}{2}k \end{array}\right]$$

Axi-symmetric + $u_t = 0$

When geometry and boundary conditions are such that we have $\dfrac{\partial(\ )}{\partial\theta} = (\ )_t = 0$ the situation is referred to as being axi-symmetric.

In many cases boundary conditions are such that there is no displacement of material points in tangential direction ($u_t = 0$). In that case we have $\varepsilon_{rt} = 0 \rightarrow \sigma_{rt} = 0$

$$\left.\begin{array}{lll} \text{plane strain} & : & \varepsilon_{zz} = \varepsilon_{rz} = \varepsilon_{tz} = 0 \\ \text{plane stress} & : & \sigma_{zz} = \sigma_{rz} = \sigma_{tz} = 0 \end{array}\right\} \qquad \left\{\begin{array}{l} u_r = u_r(r) \\ u_t = 0 \end{array}\right.$$

$$\underset{\approx}{\varepsilon}^T = \left[\begin{array}{cc} \varepsilon_{rr} & \varepsilon_{tt} \end{array}\right] = \left[\begin{array}{cc} u_{r,r} & \frac{1}{r}(u_r) \end{array}\right]$$

$$\varepsilon_{rr} = u_{r,r} = (r\varepsilon_{tt})_{,r} = \varepsilon_{tt} + r\varepsilon_{tt,r}$$

$$\underset{\approx}{\sigma}^T = \left[\begin{array}{cc} \sigma_{rr} & \sigma_{tt} \end{array}\right]$$

$$\sigma_{rr,r} + \frac{1}{r}\,(\sigma_{rr} - \sigma_{tt}) + \rho q_r = \rho \ddot{u}_r$$

$$\underline{\underline{C}}_p = \left[\begin{array}{cc} A_p & Q_p \\ Q_p & B_p \end{array}\right] \quad ; \quad \underline{\underline{S}}_p = \left[\begin{array}{cc} a_p & q_p \\ q_p & b_p \end{array}\right]$$

1.5 Inconsistency of plane stress

Although for plane stress the out-of-plane shear stresses must be zero, they are not, when calculated afterwards from the strains. This inconsistency is inherent to the plane stress assumption. Deviations must be small to render the assumption of plane stress valid.

$$\sigma_{xz} = 2K\varepsilon_{xz} = 2Ku_{z,x} \neq 0$$
$$\sigma_{yz} = 2K\varepsilon_{yz} = 2Ku_{z,y} \neq 0$$