# CONTINUUM MECHANICS

# Contents

# 1 Kinematics

The motion and deformation of a three-dimensional continuum is studied in *continuum mechanics*. A continuum is an ideal material body, where the neighborhood of a material point is assumed to be dense and fully occupied with other material points. The real microstructure of the material (molecules, crystals, particles, ...) is not considered. The deformation is also continuous, which implies that the neighborhood of a material point always consists of the same collection of material points.

Kinematics describes the transformation of a material body from its undeformed to its deformed state without paying attention to the cause of deformation. In the mathematical formulation of kinematics a Lagrangian or an Eulerian approach can be chosen. (It is also possible to follow a so-called Arbitrary-Lagrange-Euler approach.)

The undeformed state is indicated as the state at time $t_0$ and the deformed state as the state at the current time $t$. When the deformation process is time- or rate-independent, the time variable must be considered to be a fictitious time, only used to indicate subsequent moments in the deformation process.

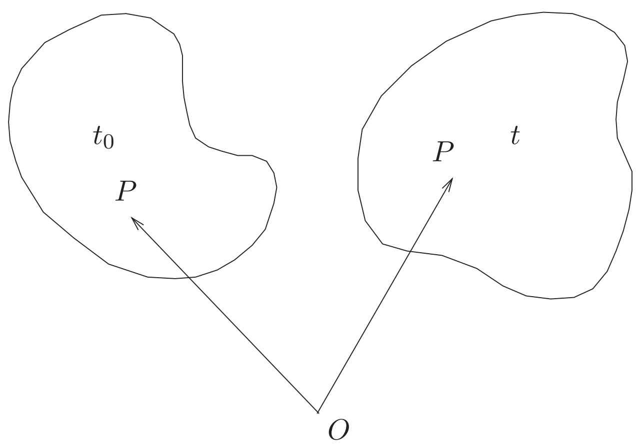



Fig. 1.1 : *Deformation of continuum*

## 1.1 Identification of points

Describing the deformation of a material body cannot be done without a proper identification of the individual material points.

### 1.1.1 Material coordinates

Each point of the material can be identified by or labeled with material coordinates. In a three-dimensional space three coordinates $\{\xi_1, \xi_2, \xi_3\}$ are needed and sufficient to identify a point uniquely. The material coordinates of a material point do never change. They can be stored in a column $\underset{\sim}{\xi} \;:\; \underset{\sim}{\xi}^T = [\; \xi_1 \quad \xi_2 \quad \xi_3 \;]$ .

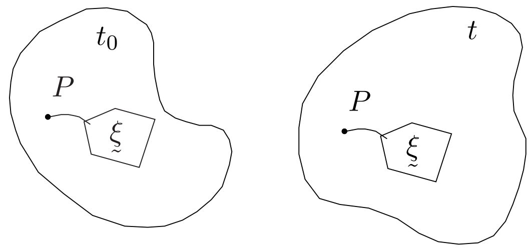



Fig. 1.2 : *Material coordinates*

### 1.1.2 Position vectors

A point of the material can also be identified with its position in space. Two position vectors can be chosen for this purpose : the position vector in the undeformed state, $\vec{x}_0$, or the position vector in the current, deformed state, $\vec{x}$. Both position vectors can be considered to be a function of the material coordinates $\underset{\sim}{\xi}$.

Each point is always identified with one position vector. One spatial position is always occupied by one material point. For a continuum the position vector is a continuous differentiable function.

Using a vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, components of the position vectors can be determined and stored in columns.

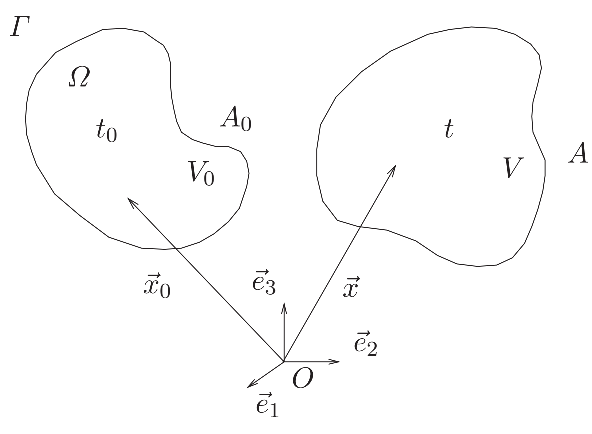



Fig. 1.3 : *Position vector*

undeformed configuration ($t_0$) $\qquad \vec{x}_0 = \vec{\chi}(\underset{\sim}{\xi}, t_0) = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3$

deformed configuration ($t$) $\qquad \vec{x} = \vec{\chi}(\underset{\sim}{\xi}, t) = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$

### 1.1.3 Euler-Lagrange

When an *Eulerian formulation* is used, all variables are determined in material points which are identified in the deformed state with their *current* position vector $\vec{x}$. When a *Lagrangian formulation* is used to describe state transformation, all variables are determined in material points which are identified in the undeformed state with their *initial* position vector $\vec{x}_0$. For a scalar quantity $a$, this can be formally written with a function $\mathcal{A}_E$ or $\mathcal{A}_L$, respectively.

The difference $da$ of a scalar quantity $a$ in two adjacent points $P$ and $Q$ can be calculated in both the Eulerian and the Lagrangian framework. This leads to the definition of two gradient operators, $\vec{\nabla}$ and $\vec{\nabla}_0$, respectively.

For a vectorial quantity $\vec{a}$, the spatial difference $d\vec{a}$ in two adjacent points, can also be calculated, using either $\vec{\nabla}_0$ or $\vec{\nabla}$. For the position vectors, the gradients result in the unity tensor $\boldsymbol{I}$.

$$
\begin{aligned}
&\text{Euler : "observer" is fixed in space} \\
&a = \mathcal{A}_E(\vec{x}, t) \\
&da = a_Q - a_P = \mathcal{A}_E(\vec{x} + d\vec{x}, t) - \mathcal{A}_E(\vec{x}, t) = d\vec{x} \cdot (\vec{\nabla} a)\Big|_t \\
&\vec{\nabla} = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \vec{e}_3 \frac{\partial}{\partial x_3}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Lagrange : "observer" follows the material} \\
&a = \mathcal{A}_L(\vec{x}_0, t) \\
&da = a_Q - a_P = \mathcal{A}_L(\vec{x}_0 + d\vec{x}_0, t) - \mathcal{A}_L(\vec{x}_0, t) = d\vec{x}_0 \cdot (\vec{\nabla}_0 a)\Big|_t \\
&\vec{\nabla}_0 = \vec{e}_1 \frac{\partial}{\partial x_{01}} + \vec{e}_2 \frac{\partial}{\partial x_{02}} + \vec{e}_3 \frac{\partial}{\partial x_{03}}
\end{aligned}
$$

$$
\begin{aligned}
&\text{position vectors} \\
&\vec{\nabla}\vec{x} = \boldsymbol{I} \qquad ; \qquad \vec{\nabla}_0 \vec{x}_0 = \boldsymbol{I}
\end{aligned}
$$

## 1.2 Time derivatives

A time derivative of a variable expresses the change of its value in time. This change can be measured in one and the same material point or in one and the same point in space. In the first case, the observer of the change follows the material, and, in the second case, he is located in a fixed spatial position.

This difference of observer position leads to two different time derivatives, the *material time derivative* and the *spatial time derivative*. Using a material time derivative is associated with the Lagrangian formulation, while in the Eulerian formulation the spatial time derivative is generally used. Below, we consider the time derivatives of a scalar vatiable $a$.

material time derivative $\quad \dfrac{Da}{Dt} = \dot{a} = \lim\limits_{\Delta t \longrightarrow 0} \dfrac{1}{\Delta t} \{A(\vec{x}_0, t + \Delta t) - A(\vec{x}_0, t)\}$

velocity of a material point $\quad \vec{v} = \vec{v}(\vec{x}_0) = \dot{\vec{x}}$

| | |
|---|---|
| spatial time derivative | $\dfrac{\delta a}{\delta t} = \lim\limits_{\Delta t \longrightarrow 0} \dfrac{1}{\Delta t} \left\{ \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\}$ |
| velocity *field* | $\vec{v} = \vec{v}(\vec{x}, t)$ |

A relation between the material and the spatial time derivative can be derived. The material velocity enters this relation and represents the velocity of the observer. The material time derivative can be written as the sum of the spatial time derivative and the convective time derivative.

$$
\begin{aligned}
\frac{Da}{Dt} &= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ A(\vec{x}_0, t + \Delta t) - A(\vec{x}_0, t) \right\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ \mathcal{A}(\vec{x} + d\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ \mathcal{A}(\vec{x} + d\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t + \Delta t) + \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\} \\
&= \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ d\vec{x} \cdot \left. (\vec{\nabla} a) \right|_{t + \Delta t} + \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\} \\
&= \lim_{\Delta t \longrightarrow 0} \left\{ \frac{d\vec{x}}{\Delta t} \cdot \left. (\vec{\nabla} a) \right|_{t + \Delta t} \right\} + \lim_{\Delta t \longrightarrow 0} \frac{1}{\Delta t} \left\{ \mathcal{A}(\vec{x}, t + \Delta t) - \mathcal{A}(\vec{x}, t) \right\} \\
&= \vec{v} \cdot (\vec{\nabla} a) + \frac{\delta a}{\delta t} \\
&= \text{(convective time derivative)} + \text{(spatial time derivative)} \\
&= \text{(material time derivative)}
\end{aligned}
$$

## 1.3 Deformation

Upon deformation, a material point changes position from $\vec{x}_0$ to $\vec{x}$. This is denoted with a displacement vector $\vec{u}$. In three-dimensional space this vector has three components : $u_1$, $u_2$ and $u_3$.

The deformation of the material can be described by the displacement vector of all the material points. This, however, is not a feasible procedure. Instead, we consider the deformation of an infinitesimal material volume in each point, which can be described with a *deformation tensor*.

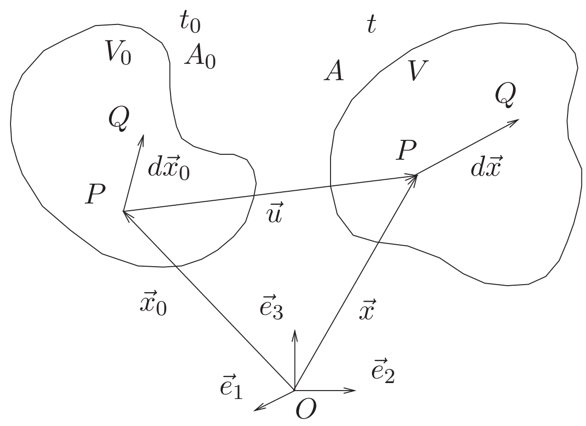



Fig. 1.4 : *Deformation of a continuum*

$$\vec{u} = \vec{x} - \vec{x}_0 = u_1\vec{e}_1 + u_2\vec{e}_2 + u_3\vec{e}_3$$

### 1.3.1 Deformation tensor

To introduce the deformation tensor, we first consider the deformation of an infinitesimal material line element, between two adjacent material points. The vector between these points in the undeformed state is $d\vec{x}_0$. Deformation results in a transformation of this vector to $d\vec{x}$, which can be denoted with a tensor, the deformation tensor $\boldsymbol{F}$. Using the gradient operator with respect to the undeformed state, the deformation tensor can be written as a gradient, which explains its much used name : *deformation gradient tensor*.

$$\begin{aligned} d\vec{x} &= \boldsymbol{F} \cdot d\vec{x}_0 \\ &= \vec{X}(\vec{x}_0 + d\vec{x}_0, \boldsymbol{t}) - \vec{X}(\vec{x}_0, \boldsymbol{t}) = d\vec{x}_0 \cdot \left(\vec{\nabla}_0 \vec{x}\right) \\ &= \left(\vec{\nabla}_0 \vec{x}\right)^c \cdot d\vec{x}_0 = \boldsymbol{F} \cdot d\vec{x}_0 \end{aligned}$$

$$\boldsymbol{F} = \left(\vec{\nabla}_0 \vec{x}\right)^c = \left[\left(\vec{\nabla}_0 \vec{x}_0\right)^c + \left(\vec{\nabla}_0 \vec{u}\right)^c\right] = \boldsymbol{I} + \left(\vec{\nabla}_0 \vec{u}\right)^c$$

In the undeformed configuration, an infinitesimal material volume is uniquely defined by three material line elements or material vectors $d\vec{x}_{01}$, $d\vec{x}_{02}$ and $d\vec{x}_{03}$. Using the deformation tensor $\boldsymbol{F}$, these vectors are transformed to the deformed state to become $d\vec{x}_1$, $d\vec{x}_2$ and $d\vec{x}_3$. These vectors span the deformed volume element, containing the same material points as in the initial volume element. It is thus obvious that $\boldsymbol{F}$ describes the transformation of the material.

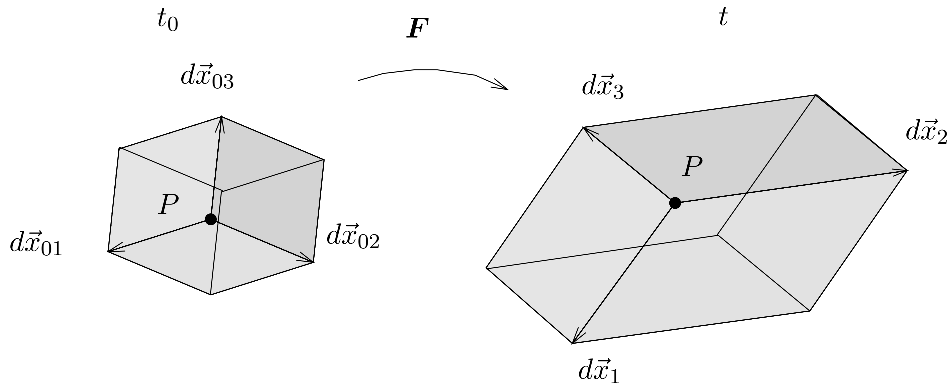



Fig. 1.5 : *Deformation tensor*

$$d\vec{x}_1 = \boldsymbol{F} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}_2 = \boldsymbol{F} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}_3 = \boldsymbol{F} \cdot d\vec{x}_{03}$$

**Volume change**

The three vectors which span the material element, can be combined in a triple product. The resulting scalar value is positive when the vectors are right-handed and represents the volume of the material element. In the undeformed state this volume is $dV_0$ and after deformation the volume is $dV$. Using the deformation tensor $\boldsymbol{F}$ and the definition of the determinant (third invariant) of a second-order tensor, the relation between $dV$ and $dV_0$ can be derived.

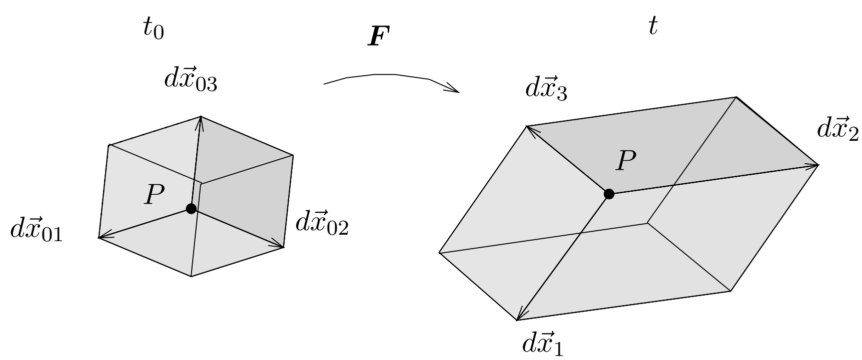



Fig. 1.6 : *Volume change*

$$\begin{aligned} dV &= d\vec{x}_1 * d\vec{x}_2 \cdot d\vec{x}_3 \\ &= (\boldsymbol{F} \cdot d\vec{x}_{01}) * (\boldsymbol{F} \cdot d\vec{x}_{02}) \cdot (\boldsymbol{F} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F})\,(d\vec{x}_{01} * d\vec{x}_{02} \cdot d\vec{x}_{03}) \\ &= \det(\boldsymbol{F})\,dV_0 \\ &= J\,dV_0 \end{aligned}$$

**Area change**

The vector product of two vectors along two material line elements represents a vector, the length of which equals the area of the parallelogram spanned by the vectors. Using the deformation tensor $\boldsymbol{F}$, the change of area during deformation can be calculated.

$$
\begin{aligned}
dA\,\vec{n} &= d\vec{x}_1 * d\vec{x}_2 = (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02}) \\
dA\,\vec{n}\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) &= (\boldsymbol{F}\cdot d\vec{x}_{01}) * (\boldsymbol{F}\cdot d\vec{x}_{02})\cdot(\boldsymbol{F}\cdot d\vec{x}_{03}) \\
&= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot d\vec{x}_{03} \qquad \forall \quad d\vec{x}_{03} \quad \rightarrow \\
dA\,\vec{n}\cdot\boldsymbol{F} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02}) \\
dA\,\vec{n} &= \det(\boldsymbol{F})(d\vec{x}_{01} * d\vec{x}_{02})\cdot\boldsymbol{F}^{-1} \\
&= \det(\boldsymbol{F})dA_0\,\vec{n}_0\cdot\boldsymbol{F}^{-1} \\
&= dA_0\,\vec{n}_0\cdot\left(\boldsymbol{F}^{-1}\det(\boldsymbol{F})\right)
\end{aligned}
$$

**Inverse deformation**

The determinant of the deformation tensor, being the quotient of two volumes, is always a positive number. This implies that the deformation tensor is regular and that the inverse $\boldsymbol{F}^{-1}$ exists. It represents the transformation of the deformed state to the undeformed state. The gradient operators $\vec{\nabla}$ and $\vec{\nabla}_0$ are related by the (inverse) deformation tensor.

$$
J = \frac{dV}{dV_0} = \det(\boldsymbol{F}) > 0 \quad \rightarrow \quad \boldsymbol{F} \text{ regular} \quad \rightarrow \quad d\vec{x}_0 = \boldsymbol{F}^{-1}\cdot d\vec{x}
$$

relation between gradient operators

$$
\boldsymbol{I} = \boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{T} \rightarrow \left(\vec{\nabla}\vec{x}\right) = \boldsymbol{F}^{-T}\cdot\left(\vec{\nabla}_0\vec{x}\right) \quad \rightarrow \quad \vec{\nabla} = \boldsymbol{F}^{-T}\cdot\vec{\nabla}_0
$$

**Homogeneous deformation**

The deformation tensor describes the deformation of an infinitesimal material volume, initially located at position $\vec{x}_0$. The deformation tensor is generally a function of the position $\vec{x}_0$. When $\boldsymbol{F}$ is not a function of position $\vec{x}_0$, the deformation is referred to as being *homogeneous*. In that case, each infinitesimal material volume shows the same deformation. The current position vector $\vec{x}$ can be related to the initial position vector $\vec{x}_0$ and an unknown rigid body translation $\vec{t}$.

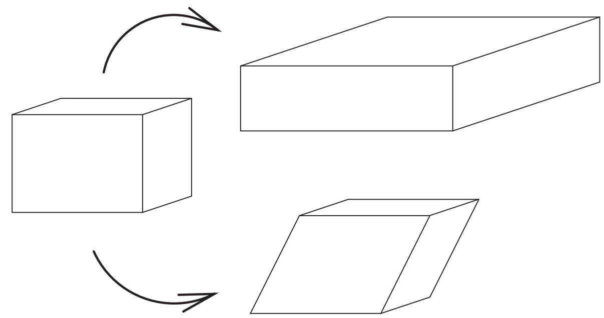


Fig. 1.7 : *Homogeneous deformation*

$$\vec{\nabla}_0 \vec{x} = \boldsymbol{F}^c = \text{uniform tensor} \quad \rightarrow \quad \vec{x} = (\vec{x}_0 \cdot \boldsymbol{F}^c) + \vec{t} = \boldsymbol{F} \cdot \vec{x}_0 + \vec{t}$$

### 1.3.2 Elongation and shear

During deformation a material line element $d\vec{x}_0$ is transformed to the line element $d\vec{x}$. The *elongation factor* or *stretch ratio* $\lambda$ of the line element, is defined as the ratio of its length after and before deformation. The elongation factor can be expressed in $\boldsymbol{F}$ and $\vec{e}_0$, the unity direction vector of $d\vec{x}_0$. It follows that the elongation is calculated from the product $\boldsymbol{F}^c \cdot \boldsymbol{F}$, which is known as the right Cauchy-Green stretch tensor $\boldsymbol{C}$.

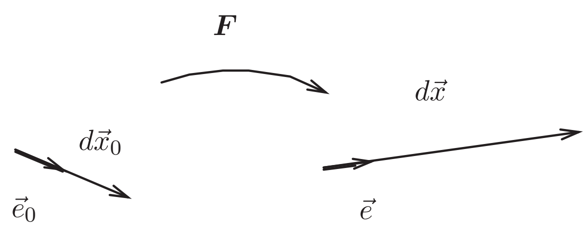



Fig. 1.8 : *Elongation of material line element*

$$\begin{aligned}
\lambda^2(\vec{e}_{01}) &= \frac{d\vec{x}_1 \cdot d\vec{x}_1}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{d\vec{x}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot d\vec{x}_{01}}{d\vec{x}_{01} \cdot d\vec{x}_{01}} = \frac{||d\vec{x}_{01}||^2}{||d\vec{x}_{01}||^2} \left(\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{01}\right) \\
&= \vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \boldsymbol{C} \cdot \vec{e}_{01}
\end{aligned}$$

We consider two material vectors in the undeformed state, $d\vec{x}_{01}$ and $d\vec{x}_{02}$, which are perpendicular. The shear deformation $\gamma$ is defined as the cosine of $\theta$, the angle between the two material vectors in the deformed state. The shear deformation can be expressed in $\boldsymbol{F}$ and $\vec{e}_{01}$ and $\vec{e}_{02}$, the unit direction vectors of $d\vec{x}_{01}$ and $d\vec{x}_{02}$. Again the shear is calculated from the right Cauchy-Green stretch tensor $\boldsymbol{C} = \boldsymbol{F}^c \cdot \boldsymbol{F}$.

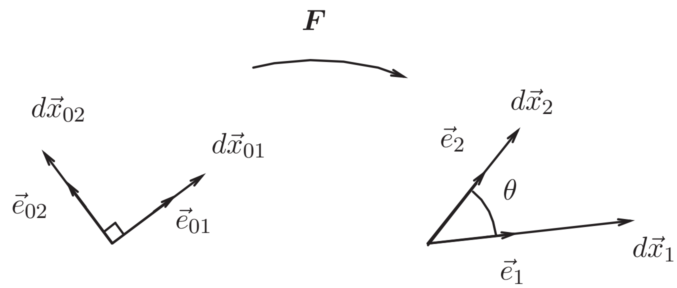



Fig. 1.9 : *Shear of two material line elements*

$$
\begin{aligned}
\gamma(\vec{e}_{01}, \vec{e}_{02}) &= \sin\left(\tfrac{\pi}{2} - \theta\right) = \cos(\theta) = \frac{d\vec{x}_1 \cdot d\vec{x}_2}{||d\vec{x}_1||\,||d\vec{x}_2||} = \frac{d\vec{x}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot d\vec{x}_{02}}{||d\vec{x}_1||\,||d\vec{x}_2||} \\
&= \frac{||d\vec{x}_{01}||\,||d\vec{x}_{02}||(\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{02})}{\lambda(\vec{e}_{01})||d\vec{x}_{01}||\lambda(\vec{e}_{02})||d\vec{x}_{02}||} = \frac{\vec{e}_{01} \cdot \boldsymbol{F}^T \cdot \boldsymbol{F} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} \\
&= \frac{\vec{e}_{01} \cdot \boldsymbol{C} \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})}
\end{aligned}
$$

### 1.3.3 Principal directions of deformation

In each point $P$ there is exactly one orthogonal material volume, which will not show any shear during deformation from $t_0$ to $t$. Rigid rotation may occur, although this is not shown in the figure.

The directions $\{1, 2, 3\}$ of the sides of the initial orthogonal volume are called *principal directions* of deformation and associated with them are the three *principal elongation factors* $\lambda_1$, $\lambda_2$ and $\lambda_3$. For this material volume the three principal elongation factors characterize the deformation uniquely. Be aware of the fact that the principal directions change when the deformation proceeds. They are a function of the time $t$.

The relative volume change $J$ is the product of the three principal elongation factors. For incompressible material there is no volume change, so the above product will have value one.

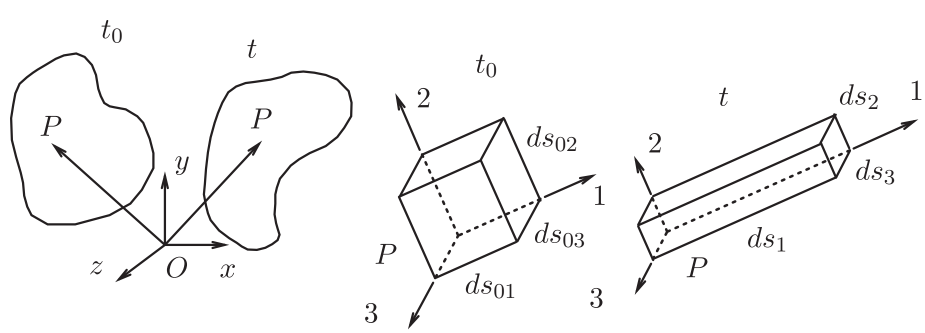



Fig. 1.10 : *Deformation of material cube with sides in principal directions*

$$\lambda_1 = \frac{ds_1}{ds_{01}} \quad ; \quad \lambda_2 = \frac{ds_2}{ds_{02}} \quad ; \quad \lambda_3 = \frac{ds_3}{ds_{03}} \quad ; \quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$$

$$J = \frac{dV}{dV_0} = \frac{ds_1 ds_2 ds_3}{ds_{01} ds_{02} ds_{03}} = \lambda_1 \lambda_2 \lambda_3$$

### 1.3.4 Strains

The elongation of a material line element is completely described by the stretch ratio $\lambda$. When there is no deformation, we have $\lambda = 1$. It is often convenient to describe the elongation with a so-called elongational strain, which is zero when there is no deformation. A strain $\varepsilon$ is defined as a function of $\lambda$, which has to satisfy certain requirements. Much used strain definitions are the linear, the logarithmic, the Green-Lagrange and the Euler-Almansi strain. One of the requirements of a strain definition is that it must linearize toward the linear strain, which is illustrated in the figure below.

| | |
|---|---|
| linear | $\varepsilon_l = \lambda - 1$ |
| logarithmic | $\varepsilon_{ln} = \ln(\lambda)$ |
| Green-Lagrange | $\varepsilon_{gl} = \frac{1}{2}(\lambda^2 - 1)$ |
| Euler-Almansi | $\varepsilon_{ea} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$ |

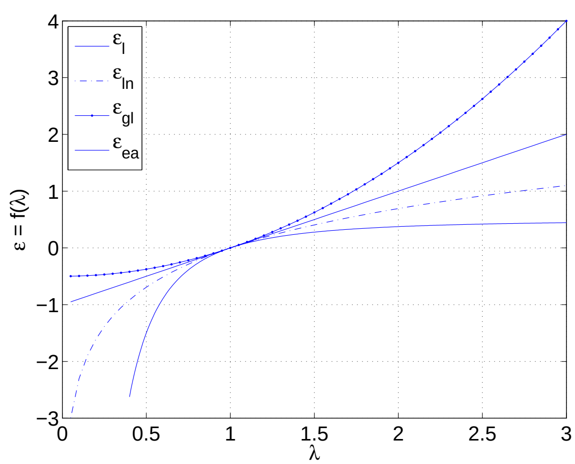



Fig. 1.11 : *Strain definitions*

### 1.3.5 Strain tensor

The Green-Lagrange strain of a line element with a known direction $\vec{e}_0$ in the undeformed state, can be calculated straightforwardly from the so-called Green-Lagrange strain tensor $\boldsymbol{E}$. Also the shear $\gamma$ can be expressed in this tensor. For other strain definitions, different strain tensors are used, which are not descussed here.

$$\tfrac{1}{2}\left\{\lambda^2(\vec{e}_{01}) - 1\right\} = \vec{e}_{01} \cdot \left\{\tfrac{1}{2}\left(\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}\right)\right\} \cdot \vec{e}_{01} = \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{01}$$

$$\gamma(\vec{e}_{01}, \vec{e}_{02}) = \frac{\vec{e}_{01} \cdot (\boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I}) \cdot \vec{e}_{02}}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} = \left[ \frac{2}{\lambda(\vec{e}_{01})\lambda(\vec{e}_{02})} \right] \vec{e}_{01} \cdot \boldsymbol{E} \cdot \vec{e}_{02}$$

$$\left. \begin{array}{l} \boldsymbol{E} = \frac{1}{2} \left( \boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I} \right) \\ \boldsymbol{F} = \left( \vec{\nabla}_0 \vec{x} \right)^T = \boldsymbol{I} + \left( \vec{\nabla}_0 \vec{u} \right)^T \end{array} \right\} \rightarrow \begin{array}{l} \boldsymbol{E} = \frac{1}{2} \left[ \left\{ \boldsymbol{I} + \left( \vec{\nabla}_0 \vec{u} \right) \right\} \cdot \left\{ \boldsymbol{I} + \left( \vec{\nabla}_0 \vec{u} \right)^T \right\} - \boldsymbol{I} \right] \\ \quad = \frac{1}{2} \left[ \left( \vec{\nabla}_0 \vec{u} \right)^T + \left( \vec{\nabla}_0 \vec{u} \right) + \left( \vec{\nabla}_0 \vec{u} \right) \cdot \left( \vec{\nabla}_0 \vec{u} \right)^T \right] \end{array}$$

### 1.3.6 Right Cauchy-Green deformation tensor

The general transformation of a material line element from the undeformed to the deformed state is uniquely described by the deformation (gradient) tensor $\boldsymbol{F}$. The true deformation consists of elongation of material line elements and mutual rotation of line elements, which is also referred to as shear.

The true deformation, represented by the expressions for $\lambda$ and $\gamma$, is described by the product $\boldsymbol{F}^c \cdot \boldsymbol{F}$, which is called the *right Cauchy-Green deformation tensor* $\boldsymbol{C}$. This important tensor has two properties, which are easily recognized : 1) it is symmetric and 2) it is positive definite.

These properties imply that $\boldsymbol{C}$ has real-valued eigenvectors and eigenvalues, of which the latter must be positive. The eigenvectors are mutually perpendicular or can be chosen to be so. Taking them as a vector basis, the tensor $\boldsymbol{C}$ can be written in spectral form.

1. symmetric $\quad \boldsymbol{C}^c = \boldsymbol{C}$

2. positive definite

$$\begin{array}{l} \vec{a} \cdot \boldsymbol{C} \cdot \vec{a} = \vec{a} \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \vec{a} = (\boldsymbol{F} \cdot \vec{a}) \cdot (\boldsymbol{F} \cdot \vec{a}) \\ \boldsymbol{F} \text{ is regular} \quad \rightarrow \quad \boldsymbol{F} \cdot \vec{a} \neq \vec{0} \quad \text{if} \quad \vec{a} \neq \vec{0} \quad \rightarrow \\ \vec{a} \cdot \boldsymbol{C} \cdot \vec{a} > 0 \quad \forall \quad \vec{a} \neq \vec{0} \end{array}$$

3. $\left. \begin{array}{l} \text{eigenvalues and eigenvectors real} \\ \text{eigenvalues positive} \\ \text{eigenvectors} \perp \text{(choice)} \end{array} \right\} \rightarrow \text{spectral representation}$

$$\boldsymbol{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$

#### Eigenvectors and eigenvalues

The physical meaning of the eigenvalues and eigenvectors of $\boldsymbol{C}$ becomes clear if we consider again the expressions for stretch and shear, but now using the spectral representation of $\boldsymbol{C}$. For these expressions to have a physical relevant meaning, the eigenvectors of $\boldsymbol{C}$ must characterize a material direction in the undeformed state. They are denoted as $\vec{n}_{0i}, i = 1, 2, 3$.

Two eigenvectors of $\boldsymbol{C}$ are mutually perpendicular and represent the direction of two material elements in the undeformed state. The shear deformation between these two material directions is zero. i.e. the material line elements remain perpendicular during deformation. They are called *principal directions of deformation* or *principal strain directions*.

The eigenvalues of $\boldsymbol{C}$ appear to be the squared stretch ratios of the material line elements oriented in the direction of the eigenvectors of $\boldsymbol{C}$. They are called the *principal elongation factors*. The right Cauchy-Green deformation tensor is fully defined in the undeformed state. It is therefore characterized as a Lagrangian tensor.

$$\boldsymbol{C} = \mu_1 \vec{m}_1 \vec{m}_1 + \mu_2 \vec{m}_2 \vec{m}_2 + \mu_3 \vec{m}_3 \vec{m}_3$$

$$\boldsymbol{C} = \mu_1 \vec{n}_{01} \vec{n}_{01} + \mu_2 \vec{n}_{02} \vec{n}_{02} + \mu_3 \vec{n}_{03} \vec{n}_{03}$$

$$\lambda(\vec{n}_{01}) = \sqrt{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{01}} = \sqrt{\mu_1} \quad ; \quad \gamma(\vec{n}_{01}, \vec{n}_{02}) = \frac{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{02}}{\sqrt{\vec{n}_{01} \cdot \boldsymbol{C} \cdot \vec{n}_{01}} \sqrt{\vec{n}_{02} \cdot \boldsymbol{C} \cdot \vec{n}_{02}}} = 0$$

$$\boldsymbol{C} = \lambda_1^2 \, \vec{n}_{01} \vec{n}_{01} + \lambda_2^2 \, \vec{n}_{02} \vec{n}_{02} + \lambda_3^2 \, \vec{n}_{03} \vec{n}_{03}$$

### 1.3.7 Right stretch tensor

Based on the right Cauchy-Green deformation tensor, a new tensor, the *right stretch tensor* $\boldsymbol{U}$, is simply defined as the square root of $\boldsymbol{C}$. It is obvious that $\boldsymbol{U}$, like $\boldsymbol{C}$, is symmetric, positive definite and regular.

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \lambda_1 \vec{n}_{01} \vec{n}_{01} + \lambda_2 \vec{n}_{02} \vec{n}_{02} + \lambda_3 \vec{n}_{03} \vec{n}_{03}$$

1. symmetric : $\boldsymbol{U}^c = \boldsymbol{U}$
2. positive definite : $\vec{a} \cdot \boldsymbol{U} \cdot \vec{a} > 0 \qquad \forall \quad \vec{a}$
3. regular : $\boldsymbol{U}^{-1} = \dfrac{1}{\lambda_1} \vec{n}_{01} \vec{n}_{01} + \dfrac{1}{\lambda_2} \vec{n}_{02} \vec{n}_{02} + \dfrac{1}{\lambda_3} \vec{n}_{03} \vec{n}_{03}$
4. $\det(\boldsymbol{C}) = \det(\boldsymbol{U} \cdot \boldsymbol{U}) = \det(\boldsymbol{F}^c \cdot \boldsymbol{F}) = \det^2(\boldsymbol{F}) \quad \rightarrow$
   $\det(\boldsymbol{U}) = \lambda_1 \lambda_2 \lambda_3 = \det(\boldsymbol{F}) = J$

The stretch tensor $\boldsymbol{U}$ can be used to transform perpendicular material line elements $d\vec{x}_{01}$, $d\vec{x}_{02}$ and $d\vec{x}_{03}$. The resulting material vectors $d\vec{x}^*_{01}$, $d\vec{x}^*_{02}$ and $d\vec{x}^*_{03}$, will have changed in length and will also be no longer perpendicular, when the original line elements do not coincide with the principal deformation directions. It can be concluded that $\boldsymbol{U}$ describes the real deformation, so elongation and shear.

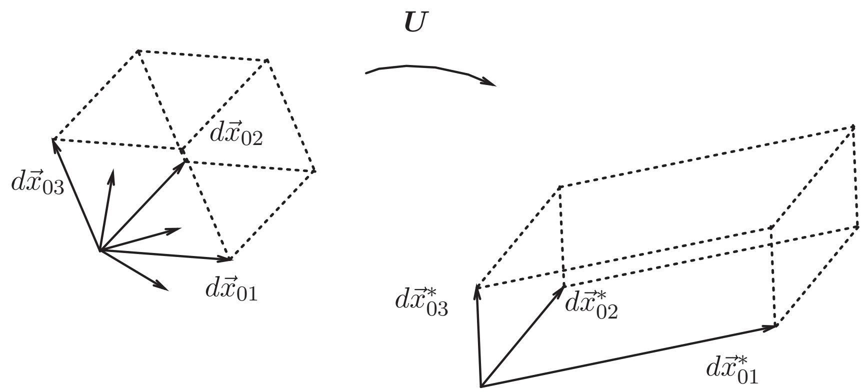



Fig. 1.12 : *Transformation by* $\boldsymbol{U}$

$$d\vec{x}^*_{01} = \boldsymbol{U} \cdot d\vec{x}_{01} \quad ; \quad d\vec{x}^*_{02} = \boldsymbol{U} \cdot d\vec{x}_{02} \quad ; \quad d\vec{x}^*_{03} = \boldsymbol{U} \cdot d\vec{x}_{03}$$

**Total transformation**

The total transformation from the undeformed to the deformed state, is not described by $\boldsymbol{U}$ but by $\boldsymbol{F}$. It seems that there must be another part of the total transformation, which is not described by $\boldsymbol{U}$. This missing link between $\boldsymbol{U}$ and $\boldsymbol{F}$ is a tensor $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$.

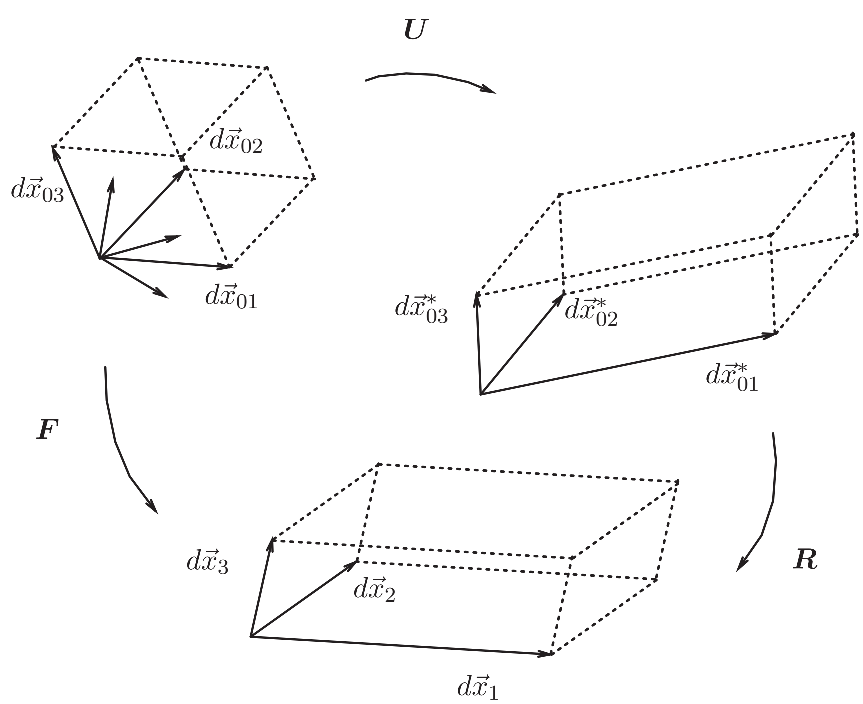



Fig. 1.13 : *Total transformation*

$$\left.\begin{array}{l} d\vec{x}^*_{01} = \boldsymbol{U} \cdot d\vec{x}_{01} \quad \rightarrow \quad d\vec{x}_{01} = \boldsymbol{U}^{-1} \cdot d\vec{x}^*_{01} \\ d\vec{x}_1 = \boldsymbol{F} \cdot d\vec{x}_{01} \end{array}\right\} \quad \rightarrow$$

$$d\vec{x}_1 = \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \cdot d\vec{x}^*_{01} = \boldsymbol{R} \cdot d\vec{x}^*_{01} \quad \rightarrow \quad \boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$$

### 1.3.8 Rotation tensor

The tensor $\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$ has some properties which renders it to have a physical meaning : it is a *rotation tensor* and describes the rigid body rotation of the material volume element during the transformation from the undeformed to the current, deformed state.

$$\boldsymbol{R} = \boldsymbol{F} \cdot \boldsymbol{U}^{-1}$$

1.

$$\begin{aligned} \boldsymbol{R}^c \cdot \boldsymbol{R} &= \boldsymbol{U}^{-c} \cdot \boldsymbol{F}^c \cdot \boldsymbol{F} \cdot \boldsymbol{U}^{-1} \\ &= \boldsymbol{U}^{-c} \cdot \boldsymbol{U} \cdot \boldsymbol{U} \cdot \boldsymbol{U}^{-1} = \boldsymbol{U}^{-c} \cdot \boldsymbol{U}^c \cdot \boldsymbol{U} \cdot \boldsymbol{U}^{-1} \\ &= \boldsymbol{I} \qquad \rightarrow \qquad \boldsymbol{R} \text{ is orthogonal} \end{aligned}$$

2.

$$\begin{aligned} \det(\boldsymbol{R}) &= \det(\boldsymbol{F} \cdot \boldsymbol{U}^{-1}) \\ &= \det(\boldsymbol{U}) \det(\boldsymbol{U}^{-1}) = \det(\boldsymbol{U} \cdot \boldsymbol{U}^{-1}) \\ &= \det(\boldsymbol{I}) = 1 \qquad \rightarrow \qquad \boldsymbol{R} \text{ is rotation tensor} \end{aligned}$$

### 1.3.9 Right polar decomposition

The total transformation described by $\boldsymbol{F}$ is decomposed into a true deformation, described by $\boldsymbol{U}$ and a rigid body rotation, described by $\boldsymbol{R}$. This decomposition is denoted as the *right polar decomposition* of the deformation tensor. This decomposition is unique and both $\boldsymbol{U}$ and $\boldsymbol{R}$ can be determined from $\boldsymbol{F}$.

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$$

### 1.3.10 Strain tensors

The stretch ratio of a material line element in the direction $\vec{e}_0$ could be determined using the right Cauchy-Green deformation tensor $\boldsymbol{C}$. For a strain definition $\varepsilon = f(\lambda)$ we would like to have a strain tensor $\boldsymbol{\varepsilon}$, such that the strain of a material line element in the direction $\vec{e}_0$ can be calculated according to : $\varepsilon(\vec{e}_0) = \vec{e}_0 \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_0$.

stretch ratio $\qquad \lambda(\vec{e}_0) = \sqrt{\vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0}$

strain tensor $\boldsymbol{\varepsilon}$

strain measure $\varepsilon(\vec{e}_0) = \vec{e}_0 \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_0 = f(\lambda(\vec{e}_0))$

shear measure $\gamma(\vec{e}_{01}, \vec{e}_{02}) = \vec{e}_{01} \cdot \boldsymbol{\varepsilon} \cdot \vec{e}_{02}$

**Linear strain tensor**

The linear strain tensor $\boldsymbol{\mathcal{E}}$ is defined as $\boldsymbol{\mathcal{E}} = \boldsymbol{U} - \boldsymbol{I}$. The linear strain of a material line element in the direction $\vec{e}_0$ cannot be calculated with this tensor. This is only possible for a line element in a principal deformation direction $\vec{n}_{0i}$.

$$\boldsymbol{\mathcal{E}} = \boldsymbol{U} - \boldsymbol{I}$$

$$\vec{e}_0 \cdot \boldsymbol{\mathcal{E}} \cdot \vec{e}_0 = \vec{e}_0 \cdot \boldsymbol{U} \cdot \vec{e}_0 - \vec{e}_0 \cdot \boldsymbol{I} \cdot \vec{e}_0 = \vec{e}_0 \cdot \boldsymbol{U} \cdot \vec{e}_0 - 1 \neq \lambda(\vec{e}_0) - 1$$

$$\vec{n}_{0i} \cdot \boldsymbol{\mathcal{E}} \cdot \vec{n}_{0i} = \vec{n}_{0i} \cdot \boldsymbol{U} \cdot \vec{n}_{0i} - 1 = \lambda(\vec{n}_{0i}) - 1 = \lambda_i - 1$$

**Logarithmic strain tensor**

The logarithmic strain tensor $\varLambda$ is defined as $\varLambda = \ln(\boldsymbol{U})$. The logarithmic strain of a material line element in the direction $\vec{e}_0$ cannot be calculated with this tensor. This is only possible for a line element in a principal deformation direction $\vec{n}_{0i}$.

$$\boldsymbol{\varLambda} = \ln(\boldsymbol{U})$$

$$\vec{e}_0 \cdot \boldsymbol{\varLambda} \cdot \vec{e}_0 = \vec{e}_0 \cdot \ln(\boldsymbol{U}) \cdot \vec{e}_0 \neq \ln(\lambda(\vec{e}_0))$$

$$\vec{n}_{0i} \cdot \boldsymbol{\varLambda} \cdot \vec{n}_{0i} = \vec{n}_{0i} \cdot \ln(\boldsymbol{U}) \cdot \vec{n}_{0i} = \ln(\lambda(\vec{n}_{0i})) = \ln(\lambda_i)$$

**Green-Lagrange strain tensor**

The Green-Lagrange strain tensor $\boldsymbol{E}$ is defined as $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$. For a material line element in the initial direction $\vec{e}_0$ the Green-Lagrange strain can be calculated using the Green-Lagrange strain tensor.

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

$$\vec{e}_0 \cdot \boldsymbol{E} \cdot \vec{e}_0 = \frac{1}{2}(\vec{e}_0 \cdot \boldsymbol{C} \cdot \vec{e}_0 - 1) = \frac{1}{2}\left(\lambda^2(\vec{e}_0) - 1\right)$$

**Infinitesimal linear strain tensor**

The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is the linearized fraction of the Green-Lagrange strain tensor $\boldsymbol{E}$. For infinitesimal displacements, the first partial derivatives of the displacement components are so small that all involved squares and products are negligible with respect to the linear terms. The non-linear terms in $\boldsymbol{E}$ can than be neglected.

For infinitesimal displacements the change in position vector of a material point is not relevant. This means that the difference between gradient operators vanishes.

$$\boldsymbol{E} = \tfrac{1}{2}\left(\boldsymbol{F}^c \cdot \boldsymbol{F} - \boldsymbol{I}\right) = \tfrac{1}{2}\left\{(\vec{\nabla}_0 \vec{u}) + (\vec{\nabla}_0 \vec{u})^c + (\vec{\nabla}_0 \vec{u}) \cdot (\vec{\nabla}_0 \vec{u})^c\right\}$$

$$\text{linearisation} \quad \rightarrow \quad \text{infinitesimal strain tensor}$$

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left\{(\vec{\nabla}_0 \vec{u}) + (\vec{\nabla}_0 \vec{u})^c\right\} = \tfrac{1}{2}\left(\boldsymbol{F} + \boldsymbol{F}^c\right) - \boldsymbol{I} = \tfrac{1}{2}\left\{(\vec{\nabla} \vec{u}) + (\vec{\nabla} \vec{u})^c\right\}$$

## 1.4 Deformation rate

The rate of deformation of a material line element is the material time derivative – we follow the same line element in time – of a material vector $d\vec{x}$ in the current state. This derivative can be related to $d\vec{x}$ with a tensor $\boldsymbol{L}$, the velocity gradient tensor. This tensor is decomposed into a symmetric and a skewsymmetric part, the *deformation rate tensor* $\boldsymbol{D}$ and the *spin tensor* $\boldsymbol{\Omega}$, respectively.

$$\begin{aligned} d\dot{\vec{x}} &= \dot{\boldsymbol{F}} \cdot d\vec{x}_0 = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot d\vec{x} = \boldsymbol{L} \cdot d\vec{x} = (\vec{\nabla}\vec{v})^c \cdot d\vec{x} \\ &= \tfrac{1}{2}\{\boldsymbol{L} + \boldsymbol{L}^c\} \cdot d\vec{x} + \tfrac{1}{2}\{\boldsymbol{L} - \boldsymbol{L}^c\} \cdot d\vec{x} \\ &= \boldsymbol{D} \cdot d\vec{x} + \boldsymbol{\Omega} \cdot d\vec{x} \end{aligned}$$

### 1.4.1 Spin tensor

The spin tensor $\boldsymbol{\Omega}$ describes only rotation rate of the material line element. This follows directly from the fact that the spin tensor is skewsymmetric and has a unique associated axial vector $\vec{\omega}$.

$$\boldsymbol{\Omega} = \tfrac{1}{2}\left\{\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} - (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c\right\} = \tfrac{1}{2}\left\{\left(\vec{\nabla}\vec{v}\right)^c - \left(\vec{\nabla}\vec{v}\right)\right\}$$

$$\boldsymbol{\Omega} = \text{skewsymmetric} \quad \rightarrow \quad \boldsymbol{\Omega} \cdot d\vec{x} = \vec{\omega} * d\vec{x} = \text{velocity} \perp d\vec{x} = \text{rotation rate}$$

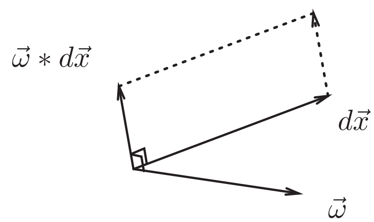



Fig. 1.14 : *Rotation rate of material line element*

The proof that a skewsymmetric tensor has an associated axial vector is repeated here.

$$\begin{aligned}
&\vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} = \vec{q} \cdot \boldsymbol{\Omega}^c \cdot \vec{q} = -\, \vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} \quad \rightarrow \\
&\vec{q} \cdot \boldsymbol{\Omega} \cdot \vec{q} = 0 \quad \rightarrow \\
&\boldsymbol{\Omega} \cdot \vec{q} = \vec{p} \quad \rightarrow \\
&\vec{q} \cdot \vec{p} = 0 \quad \rightarrow \\
&\vec{q} \perp \vec{p} \quad \rightarrow \\
&\exists \quad \vec{\omega} \quad \text{zdd} \quad \vec{p} = \vec{\omega} * \vec{q} \quad \rightarrow
\end{aligned}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q}$$

The axial vector associated with a skewsymmetric tensor is unique. Its components can be determined and expressed in the components of the skewsymmetric tensor.

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad \forall \quad \vec{q}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \tilde{\vec{e}}^T \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \tilde{\vec{e}}^T \begin{bmatrix} \Omega_{11}q_1 + \Omega_{12}q_2 + \Omega_{13}q_3 \\ \Omega_{21}q_1 + \Omega_{22}q_2 + \Omega_{23}q_3 \\ \Omega_{31}q_1 + \Omega_{32}q_2 + \Omega_{33}q_3 \end{bmatrix}$$

$$\begin{aligned}
\vec{\omega} * \vec{q} &= (\omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3) * (q_1\vec{e}_1 + q_2\vec{e}_2 + q_3\vec{e}_3) \\
&= \omega_1 q_2(\vec{e}_3) + \omega_1 q_3(-\,\vec{e}_2) + \omega_2 q_1(-\,\vec{e}_3) + \omega_2 q_3(\vec{e}_1) + \omega_3 q_1(\vec{e}_2) + \omega_3 q_2(-\,\vec{e}_1) \\
&= [\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3] \begin{bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{bmatrix}
\end{aligned}$$

$$\boldsymbol{\Omega} \cdot \vec{q} = \vec{\omega} * \vec{q} \qquad \forall \vec{q} \qquad \rightarrow \qquad \underline{\Omega} = \begin{bmatrix} 0 & -\,\omega_3 & \omega_2 \\ \omega_3 & 0 & -\,\omega_1 \\ -\,\omega_2 & \omega_1 & 0 \end{bmatrix}$$

### 1.4.2 Deformation rate tensor

The deformation rate tensor does not what its name suggests. For a random material vector $d\vec{x}$ the product $\boldsymbol{D}\cdot d\vec{x}$ is a vector which is not along $d\vec{x}$. The deformation rate tensor describes the rate of elongation but also partly the rate of rotation of $d\vec{x}$. Only for material line elements in the direction of one if its eigenvectors the tensor $\boldsymbol{D}$ describes purely elongation rate.

$$\boldsymbol{D} = \tfrac{1}{2}\left\{\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1} + (\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})^c\right\} = \left\{\left(\vec{\nabla}\vec{v}\right)^c + \left(\vec{\nabla}\vec{v}\right)\right\}$$

$$\begin{aligned}
&\boldsymbol{D} = \boldsymbol{D}^c \quad \rightarrow \quad \boldsymbol{D} = \nu_1\vec{\eta}_1\vec{\eta}_1 + \nu_2\vec{\eta}_2\vec{\eta}_2 + \nu_3\vec{\eta}_3\vec{\eta}_3 \\
&1.: \quad \text{vector } d\vec{x} \text{ along } \vec{\eta}_1 \quad : \qquad d\vec{x} = dx_1\vec{\eta}_1 \\
&\boldsymbol{D}\cdot d\vec{x} = dx_1\boldsymbol{D}\cdot\vec{\eta}_1 = dx_1\nu_1\vec{\eta}_1 = \nu_1 d\vec{x} \\
&2.: \quad \text{random vector} \quad : \qquad d\vec{x} = dx_1\vec{\eta}_1 + dx_2\vec{\eta}_2 + dx_3\vec{\eta}_3 \\
&\boldsymbol{D}\cdot d\vec{x} = dx_1\nu_1\vec{\eta}_1 + dx_2\nu_2\vec{\eta}_2 + dx_3\nu_3\vec{\eta}_3
\end{aligned}$$

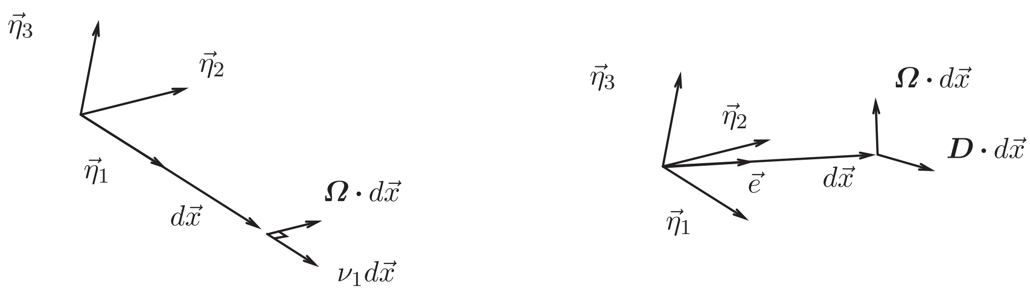



Fig. 1.15 : *Deformation rate of material line element*

### 1.4.3 Elongation rate

The elongation rate of a material line element can be expressed in the time derivative of the elongation factor $\lambda$.

$$\begin{aligned}
\lambda^2 &= \vec{e}_0\cdot\boldsymbol{C}\cdot\vec{e}_0 \quad \rightarrow \quad \frac{D}{Dt}(\lambda^2) = \frac{D}{Dt}(\vec{e}_0\cdot\boldsymbol{C}\cdot\vec{e}_0) \quad \rightarrow \\
2\lambda\dot{\lambda} &= \vec{e}_0\cdot\frac{D}{Dt}(\boldsymbol{C})\cdot\vec{e}_0 = \vec{e}_0\cdot\frac{D}{Dt}(\boldsymbol{F}^c\cdot\boldsymbol{F})\cdot\vec{e}_0 \\
&= \vec{e}_0\cdot\{\dot{\boldsymbol{F}}^c\cdot\boldsymbol{F} + \boldsymbol{F}^c\cdot\dot{\boldsymbol{F}}\}\cdot\vec{e}_0 \\
&= \vec{e}_0\cdot\boldsymbol{F}^c\cdot\{\boldsymbol{F}^{-c}\cdot\dot{\boldsymbol{F}}^c + \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\}\cdot\boldsymbol{F}\cdot\vec{e}_0 \\
&= (\boldsymbol{F}\cdot\vec{e}_0)\cdot\{(\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})^c + \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\}\cdot(\boldsymbol{F}\cdot\vec{e}_0) \\
&= (\lambda\vec{e})\cdot(2\,\boldsymbol{D})\cdot(\lambda\vec{e}) \quad \rightarrow
\end{aligned}$$

$$\frac{\dot{\lambda}}{\lambda} = \vec{e}\cdot\boldsymbol{D}\cdot\vec{e}$$

### 1.4.4 Volume change rate

The rate of change of the material volume, the material time derivative of the volume change factor $J$, is the product of $J$ itself and the trace of the deformation rate tensor $\boldsymbol{D}$. To derive this relation, we consider a material volume element in the undeformed and the deformed state. In the undeformed state the sides of the element coincide with the pricipal deformation directions $\{\vec{n}_{01}, \vec{n}_{02}, \vec{n}_{03}\}$.

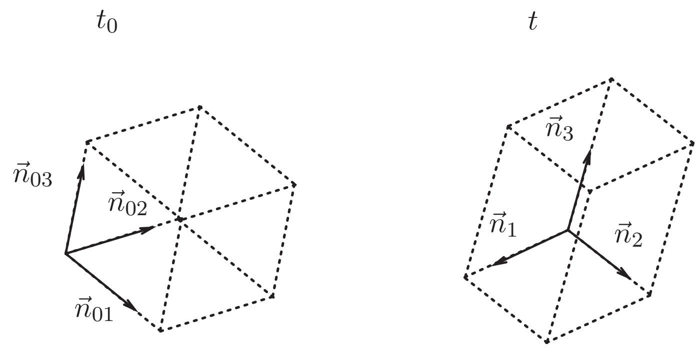



Fig. 1.16 : *Volume change rate of material cube*

$$
\begin{aligned}
\text{tr}(\boldsymbol{D}) &= \vec{n}_1 \cdot \boldsymbol{D} \cdot \vec{n}_1 + \vec{n}_2 \cdot \boldsymbol{D} \cdot \vec{n}_2 + \vec{n}_3 \cdot \boldsymbol{D} \cdot \vec{n}_3 \\
&= \frac{\dot{\lambda}_1}{\lambda_1} + \frac{\dot{\lambda}_2}{\lambda_2} + \frac{\dot{\lambda}_3}{\lambda_3} = \frac{D}{Dt}\{\ln(\lambda_1) + \ln(\lambda_2) + \ln(\lambda_3)\} = \frac{D}{Dt}\{\ln(\lambda_1\lambda_2\lambda_3)\} \\
&= \frac{D}{Dt}[\ln\{\det(\boldsymbol{U})\}] = \frac{D}{Dt}[\ln\{\det(\boldsymbol{F})\}] = \frac{D}{Dt}\{\ln(J)\} = \frac{\dot{J}}{J} \quad \rightarrow
\end{aligned}
$$

$$
\dot{J} = J\text{tr}(\boldsymbol{D}) = J\left(\vec{\nabla} \cdot \vec{v}\right)
$$

### 1.4.5 Area change rate

The rate of change of a material area $dA$ with unit normal vector $\vec{n}$ can also be expressed in the velocity gradient tensor $\boldsymbol{L}$.

$$
\begin{aligned}
\frac{D}{Dt}(dA\,\vec{n}) &= \frac{D}{Dt}\left\{\det(\boldsymbol{F})dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1}\right\} \\
&= \frac{D}{Dt}\{\det(\boldsymbol{F})\}\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} + \det(\boldsymbol{F})dA_0\vec{n}_0 \cdot \dot{\boldsymbol{F}}^{-1} \\
&= \dot{J}\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} - J\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L} \\
&= \text{tr}(\boldsymbol{L})J dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} - J\,dA_0\vec{n}_0 \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{L} \\
&= J\,\text{tr}(\boldsymbol{L})\boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 - J\,\boldsymbol{L}^c \cdot \boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 \\
&= J\,\left(\text{tr}(\boldsymbol{L})\boldsymbol{I} - \boldsymbol{L}^c\right) \cdot \boldsymbol{F}^{-c} \cdot dA_0\vec{n}_0 \\
&= \left(\text{tr}(\boldsymbol{L})\boldsymbol{I} - \boldsymbol{L}^c\right) dA\,\vec{n}
\end{aligned}
$$

## 1.5 Linear deformation

In linear elasticity theory deformations are very small. All kind of relations from general continuum mechanics theory may be linearized, resulting for instance in the linear strain tensor $\boldsymbol{\varepsilon}$, which is then fully expressed in the gradient of the displacement. The deformations are in fact so small that the geometry of the material body in the deformed state approximately equals that of the undeformed state.

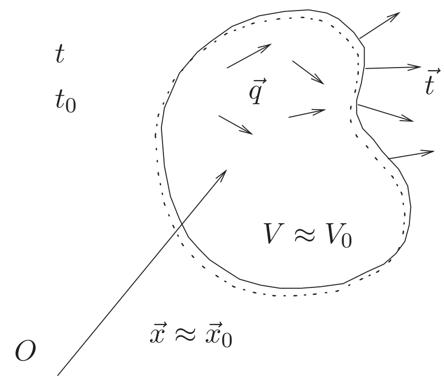



Fig. 1.17 : *Small deformation*

$$\left.\begin{array}{l} \boldsymbol{E} = \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right) + \left(\vec{\nabla}_0\vec{u}\right)\cdot\left(\vec{\nabla}_0\vec{u}\right)^T\right] \\ \text{small deformation} \quad \rightarrow \quad \left(\vec{\nabla}_0\vec{u}\right)^T = \boldsymbol{F} - \boldsymbol{I} \approx \boldsymbol{O} \end{array}\right\} \quad \rightarrow$$

$$\boldsymbol{E} \approx \frac{1}{2}\left[\left(\vec{\nabla}_0\vec{u}\right)^T + \left(\vec{\nabla}_0\vec{u}\right)\right] \approx \frac{1}{2}\left[\left(\vec{\nabla}\vec{u}\right)^T + \left(\vec{\nabla}\vec{u}\right)\right] = \boldsymbol{\varepsilon} \qquad \text{symm!}$$

Not only straining and shearing must be small to allow the use of linear strains, also the rigid body rotation must be small. This is immediately clear, when we consider the rigid rotation of a material line element $PQ$ around the fixed point $P$. The $x$- and $y$-displacement of point $Q$, $u$ and $v$ respectively, are expressed in the rotation angle $\phi$ and the length of the line element $dx_0$. The nonlinear Green-Lagrange strain is always zero. The linear strain, however, is only zero for very small rotations.

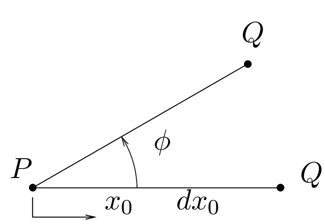



Fig. 1.18 : *Rigid rotation of a line element*

$$\left.\begin{array}{l} u = u_Q = -[dx_0 - dx_0\cos(\phi)] = [\cos(\phi) - 1]dx_0 \\ v = v_Q = [\sin(\phi)]dx_0 \end{array}\right\} \quad \rightarrow$$

$$\frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \quad ; \quad \frac{\partial v}{\partial x_0} = \sin(\phi) \quad \rightarrow$$

$$\varepsilon_{gl} = \frac{\partial u}{\partial x_0} + \tfrac{1}{2}\left(\frac{\partial u}{\partial x_0}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x_0}\right)^2 = 0$$

$$\varepsilon_l = \frac{\partial u}{\partial x_0} = \cos(\phi) - 1 \neq 0 \quad !!$$

**Elongational, shear and volume strain**

For small deformations and rotations the elongational and shear strain can be linearized and expressed in the linear strain tensor $\boldsymbol{\varepsilon}$. The volume change ratio $J$ can be expressed in linear strain components and also linearized.

$$\begin{array}{llcl}
\text{elong. strain} & \tfrac{1}{2}\left(\lambda^2(\vec{e}_{01}) - 1\right) & = & \vec{e}_{01}\cdot\boldsymbol{E}\cdot\vec{e}_{01} \\
 & & \downarrow & \\
 & \lambda(\vec{e}_{01}) - 1 & = & \vec{e}_{01}\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_{01} \\
\text{shear strain} & \gamma(\vec{e}_{01},\vec{e}_{02}) = \sin\left(\tfrac{\pi}{2} - \theta\right) & = & \left(\dfrac{2}{\lambda(\vec{e}_{01})\,\lambda(\vec{e}_{02})}\right)\vec{e}_{01}\cdot\boldsymbol{E}\cdot\vec{e}_{02} \\
 & & \downarrow & \\
 & \tfrac{\pi}{2} - \theta & = & 2\,\vec{e}_{01}\cdot\boldsymbol{\varepsilon}\cdot\vec{e}_{02} \\
\text{volume change} & J = \dfrac{dV}{dV_0} & = & \lambda_1\lambda_2\lambda_3 = (\varepsilon_1 + 1)(\varepsilon_2 + 1)(\varepsilon_2 + 1) \\
 & & \downarrow & \\
 & J & = & \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + 1 = \mathrm{tr}(\boldsymbol{\varepsilon}) + 1 \\
\text{volume strain} & J - 1 & = & \mathrm{tr}(\boldsymbol{\varepsilon})
\end{array}$$

### 1.5.1 Linear strain matrix

With respect to an orthogonal basis, the linear strain tensor can be written in components, resulting in the linear strain matrix.

Because the linear strain tensor is symmetric, it has three real-valued eigenvalues $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The strain matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the *principal strains* and the eigenvectors as the *principal strain directions*. They are equivalent to the *principal directions* of deformatieon. Line elements along these directions in the undeformed state $t_0$ do not show any shear during deformation towards the current state $t$.

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \qquad \text{with} \qquad \left\{ \begin{array}{l} \varepsilon_{21} = \varepsilon_{12} \\ \varepsilon_{32} = \varepsilon_{23} \\ \varepsilon_{31} = \varepsilon_{13} \end{array} \right.$$

principal strain matrix $\qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$

spectral form $\qquad \boldsymbol{\varepsilon} = \varepsilon_1 \vec{n}_1 \vec{n}_1 + \varepsilon_2 \vec{n}_2 \vec{n}_2 + \varepsilon_3 \vec{n}_3 \vec{n}_3$

**Cartesian components**

The linear strain components w.r.t. a Cartesian coordinate system are easily derived using the component expressions for the gradient operator and the displacement vector. For derivatives a short notation is used : $(\ )_{i,j} = \dfrac{\partial(\ )_i}{\partial x_j}$.

gradient operator $\qquad \vec{\nabla} = \vec{e}_x \dfrac{\partial}{\partial x} + \vec{e}_y \dfrac{\partial}{\partial y} + \vec{e}_z \dfrac{\partial}{\partial z}$

displacement vector $\qquad \vec{u} = u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z$

linear strain tensor $\qquad \boldsymbol{\varepsilon} = \frac{1}{2} \left\{ (\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u}) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{x,x} & u_{x,y} + u_{y,x} & u_{x,z} + u_{z,x} \\ u_{y,x} + u_{x,y} & 2u_{y,y} & u_{y,z} + u_{z,y} \\ u_{z,x} + u_{x,z} & u_{z,y} + u_{y,z} & 2u_{z,z} \end{bmatrix}$$

**Cylindrical components**

The linear strain components w.r.t. a cylindrical coordinate system are derived straightforwardly using the component expressions for the gradient operator and the displacement vector.

gradient operator $\quad \vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z}$

displacement vector $\quad \vec{u} = u_r \vec{e}_r(\theta) + u_t \vec{e}_t(\theta) + u_z \vec{e}_z$

linear strain tensor $\quad \boldsymbol{\varepsilon} = \frac{1}{2}\left\{ (\vec{\nabla}\vec{u})^c + (\vec{\nabla}\vec{u}) \right\} = \underset{\sim}{\vec{e}}^T \underline{\varepsilon}\, \underset{\sim}{\vec{e}}$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{rt} & \varepsilon_{rz} \\ \varepsilon_{tr} & \varepsilon_{tt} & \varepsilon_{tz} \\ \varepsilon_{zr} & \varepsilon_{zt} & \varepsilon_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{r,r} & \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ \frac{1}{r}(u_{r,t} - u_t) + u_{t,r} & 2\frac{1}{r}(u_r + u_{t,t}) & \frac{1}{r}u_{z,t} + u_{t,z} \\ u_{z,r} + u_{r,z} & \frac{1}{r}u_{z,t} + u_{t,z} & 2u_{z,z} \end{bmatrix}$$

**Compatibility conditions**

The six independent strain components are related to only three displacement components. Therefore the strain components cannot be independent. Six relations can be derived, which are referred to as the compatibility conditions.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial z}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\,\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yx}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\,\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \qquad \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{xy}}{\partial z^2} = \frac{\partial^2 \varepsilon_{zy}}{\partial z \partial x} + \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial y}$$

$$\frac{1}{r^2}\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} + \frac{\partial^2 \varepsilon_{tt}}{\partial r^2} - \frac{2}{r}\frac{\partial^2 \varepsilon_{rt}}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{2}{r}\frac{\partial \varepsilon_{tt}}{\partial r} - \frac{2}{r^2}\frac{\partial \varepsilon_{rt}}{\partial \theta} = 0$$

## 1.6 Special deformations

### 1.6.1 Planar deformation

It often happens that (part of) a structure is loaded in one plane. Moreover the load is often such that no bending out of that plane takes place. The resulting deformation is referred to as being planar.

Here it is assumed that the plane of deformation is the $x_1x_2$-plane. Note that in this planar deformation there still can be displacement perpendicular to the plane of deformation, which results in change of thickness.

The *in-plane* displacement components $u_1$ and $u_2$ are only a function of $x_1$ and $x_2$. The *out-of-plane* displacement $u_3$ may be a function of $x_3$ as well.

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = u_3(x_1, x_2, x_3)$$

### 1.6.2 Plane strain

When the boundary conditions and the material behavior are such that displacement of material points are only in the $x_1x_2$-plane, the deformation is referred to as *plane strain* in the $x_1x_2$-plane. Only three relevant strain components remain.

$$u_1 = u_1(x_1, x_2) \quad ; \quad u_2 = u_2(x_1, x_2) \quad ; \quad u_3 = 0$$

$$\varepsilon_{33} = 0 \quad ; \quad \gamma_{13} = \gamma_{23} = 0$$
$$\text{compatibility} \quad : \quad \varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}$$

### 1.6.3 Axi-symmetric deformation

Many man-made and natural structures have an axi-symmetric geometry, which means that their shape and volume can be constructed by virtually rotating a cross section around the axis of revolution. Points are indicated with cylindrical coordinates $\{r, \theta, z\}$. When material properties and loading are also independent of the coordinate $\theta$, the deformation and resulting stresses will be also independent of $\theta$.

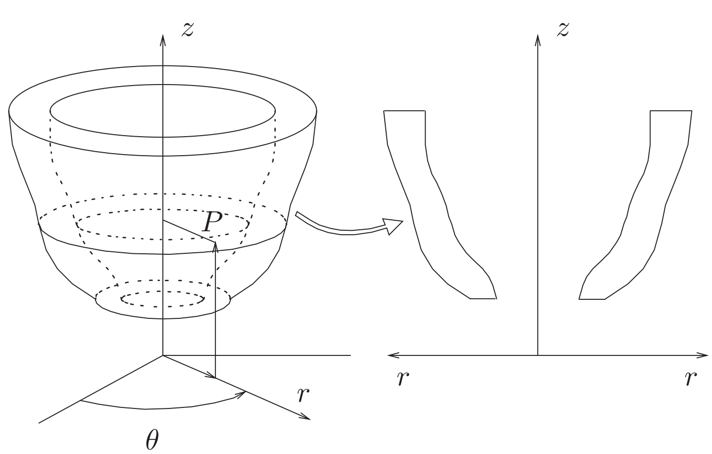



Fig. 1.19 : *Axi-symmetric deformation*

$$\frac{\partial}{\partial \theta}(\ ) = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_t(r,z)\vec{e}_t(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & -\frac{1}{r}(u_t) + u_{t,r} & u_{r,z} + u_{z,r} \\ -\frac{1}{r}(u_t) + u_{t,r} & 2\frac{1}{r}(u_r) & u_{t,z} \\ u_{z,r} + u_{r,z} & u_{t,z} & 2u_{z,z} \end{bmatrix}$$

With the *additional* assumption that no rotation around the $z$-axis takes place ($u_t = 0$), all state variables can be studied in one half of the cross section through the $z$-axis.

$$\frac{\partial}{\partial \theta}(\ ) = 0 \text{ and } u_t = 0 \qquad \rightarrow \qquad \vec{u} = u_r(r,z)\vec{e}_r(\theta) + u_z(r,z)\vec{e}_z$$

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & u_{r,z} + u_{z,r} \\ 0 & 2\frac{1}{r}(u_r) & 0 \\ u_{z,r} + u_{r,z} & 0 & 2u_{z,z} \end{bmatrix}$$

**Axi-symmetric plane strain**

When boundary conditions and material behavior are such that displacement of material points are only in the $r\theta$-plane, the deformation is referred to as *plane strain* in the $r\theta$-plane.

plane strain deformation

$$\left.\begin{array}{l} u_r = u_r(r,\theta) \\ u_t = u_t(r,\theta) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \varepsilon_{zz} = \gamma_{rz} = \gamma_{tz} = 0$$

linear strain matrix

$$\underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & u_{t,r} - \frac{1}{r}(u_t) & 0 \\ u_{t,r} - \frac{1}{r}(u_t) & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

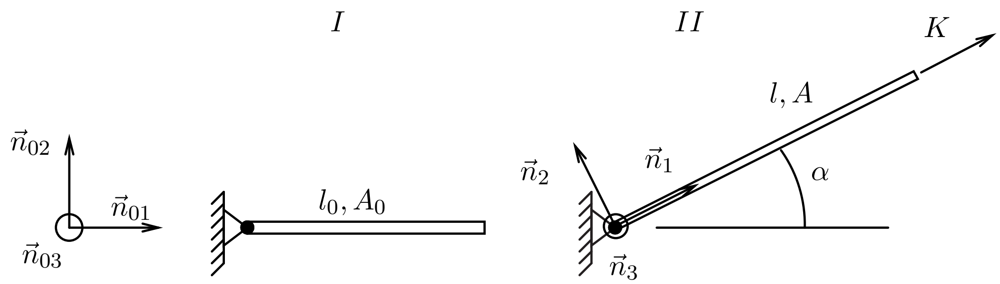
plane strain deformation with $u_t = 0$

$$\left.\begin{array}{l} u_r = u_r(r) \\ u_z = 0 \end{array}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \frac{1}{2}\begin{bmatrix} 2u_{r,r} & 0 & 0 \\ 0 & \frac{2}{r}(u_r) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

## 1.7 Examples

---

### Polar decomposition




$$
\begin{aligned}
\boldsymbol{R} &= \vec{n}_1\vec{n}_{01} + \vec{n}_2\vec{n}_{02} + \vec{n}_3\vec{n}_{03} \\
&= [\cos(\alpha)\vec{n}_{01} + \sin(\alpha)\vec{n}_{02}]\,\vec{n}_{01} + \\
&\qquad [-sin(\alpha)\vec{n}_{01} + \cos(\alpha)\vec{n}_{02}]\,\vec{n}_{02} + \vec{n}_{03}\vec{n}_{03} \\
\boldsymbol{U} &= \lambda\vec{n}_{01}\vec{n}_{01} + \mu\vec{n}_{02}\vec{n}_{02} + \mu\vec{n}_{03}\vec{n}_{03} \\
\boldsymbol{F} &= \boldsymbol{R}\cdot\boldsymbol{U} \\
&= \lambda\vec{n}_1\vec{n}_{01} + \mu\vec{n}_2\vec{n}_{02} + \mu\vec{n}_3\vec{n}_{03} \\
&= \lambda\,[\cos(\alpha)\vec{n}_{01} + \sin(\alpha)\vec{n}_{02}]\,\vec{n}_{01} + \\
&\qquad \mu\,[-sin(\alpha)\vec{n}_{01} + \cos(\alpha)\vec{n}_{02}]\,\vec{n}_{02} + \mu\vec{n}_{03}\vec{n}_{03}
\end{aligned}
$$

---

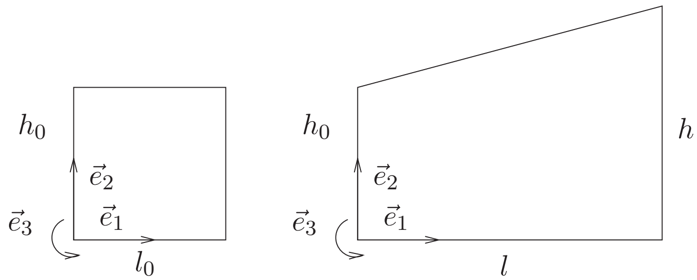
### Inhomogeneous deformation

A rectangular block of material is deformed, as shown in the figure. The basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is orthonormal. The position vector of an arbitrary material point in undeformed and deformed state, respectively is :

$$\vec{x}_0 = x_{01}\vec{e}_1 + x_{02}\vec{e}_2 + x_{03}\vec{e}_3 \quad ; \quad \vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_{03}\vec{e}_3$$

There is no deformation in $\vec{e}_3$-direction. Deformation in the 12-plane is such that straight lines remain straight during deformation.

The deformation tensor can be calculated from the relation between the coordinates of the material point in undeformed and deformed state.



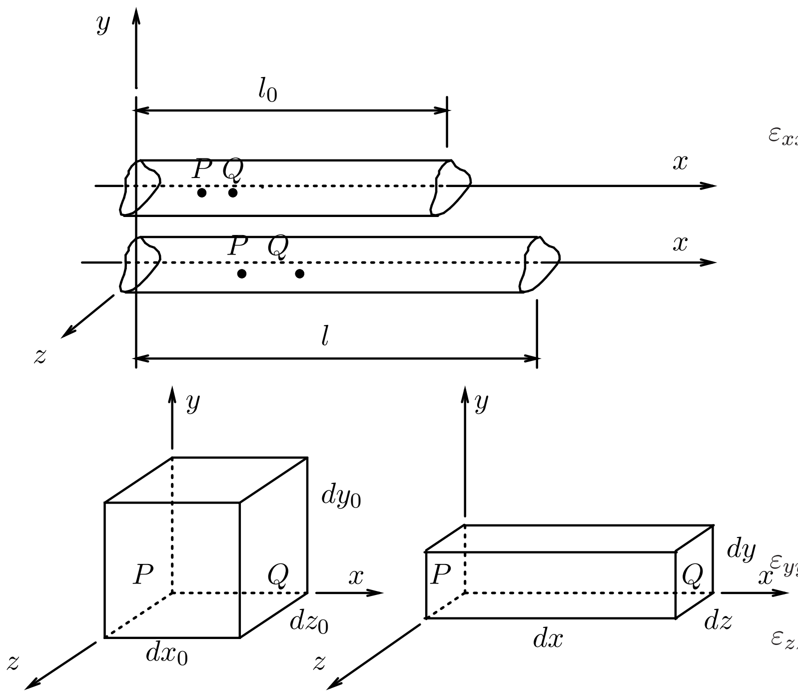

$$x_1 = \frac{l}{l_0}\,x_{01} \quad ; \quad x_2 = x_{02} + \frac{h-h_0}{h_0 l_0}\,x_{01}x_{02} \quad ; \quad x_3 = x_{03}$$

$$\begin{aligned}
\boldsymbol{F}^c &= \left(\vec{\nabla}_0 \vec{x}\right) = \left(\vec{e}_{01}\frac{\partial}{\partial x_{01}} + \vec{e}_{02}\frac{\partial}{\partial x_{02}} + \vec{e}_{03}\frac{\partial}{\partial x_{03}}\right)\left(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3\right) \\
&= \left(\vec{e}_{01}\frac{\partial}{\partial x_{01}} + \vec{e}_{02}\frac{\partial}{\partial x_{02}} + \vec{e}_{03}\frac{\partial}{\partial x_{03}}\right) \\
&\qquad \left[\left(\frac{l}{l_0}\,x_{01}\right)\vec{e}_1 + \left(x_{02} + \frac{h-h_0}{h_0 l_0}\,x_{01}x_{02}\right)\vec{e}_2 + \left(x_{03}\right)\vec{e}_3\right] \\
&= \left(\frac{l}{l_0}\right)\vec{e}_{01}\vec{e}_1 + \left(\frac{h-h_0}{h_0 l_0}x_{02}\right)\vec{e}_{01}\vec{e}_2 + \left(1 + \frac{h-h_0}{h_0 l_0}x_{01}\right)\vec{e}_{02}\vec{e}_2 + \vec{e}_{03}\vec{e}_3
\end{aligned}$$

---

**Strain $\sim$ displacement**

The strain-displacement relations for the elongation of line elements can be derived by considering the elongational deformation of an infinitesimal cube of material e.g. in a tensile test.




$$
\begin{aligned}
\varepsilon_{xx} &= \lambda_{xx} - 1 \\
&= \frac{dx}{dx_0} - 1 \\
&= \frac{dx - dx_0}{dx_0} \\
&= \frac{u_Q - u_P}{dx_0} \\
&= \frac{u(x_0 + dx_0) - u(x_0)}{dx_0} \\
&= \frac{\partial u}{\partial x_0} = \frac{\partial u}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z}
\end{aligned}
$$

Fig. 1.20 : *Homogeneous elongation of a truss*

---

**Strain $\sim$ displacement**

The strain-displacement relations for the shear of two line elements can be derived by considering the shear deformation of an infinitesimal cube of material e.g. in a torsion test.

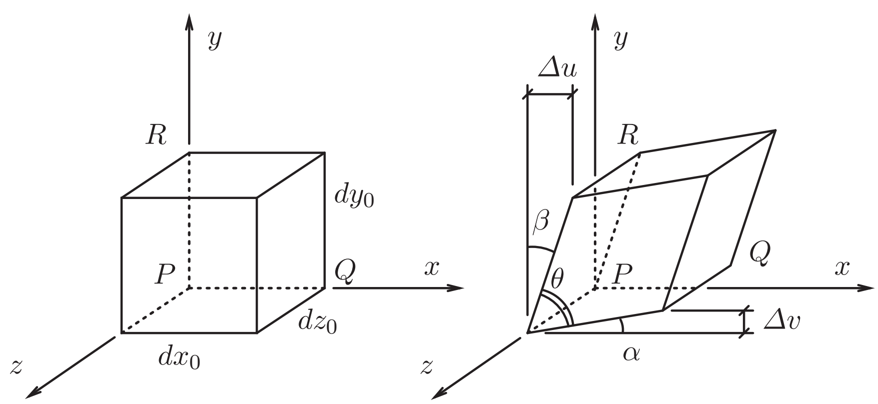



Fig. 1.21 : *Shear of a three-dimensional material cube*

$$
\begin{aligned}
\gamma_{xy} &= \tfrac{\pi}{2} - \theta_{xy} = \alpha + \beta \approx \sin(\alpha) + \sin(\beta) = \frac{\Delta v}{dx_0} + \frac{\Delta u}{dy_0} \\
&= \frac{v_Q - v_P}{dx_0} + \frac{u_R - u_P}{dy_0} = \frac{v(x_0 + dx_0) - v(x_0)}{dx_0} + \frac{u(y_0 + dy_0) - u(y_0)}{dy_0}
\end{aligned}
$$

$$= \frac{\partial v}{\partial x_0} + \frac{\partial u}{\partial y_0} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad ; \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

---

**Strain $\sim$ displacement**

Strain-displacement relations can be derived geometrically in the cylindrical coordinate system, as we did in the Cartesian coordinate system.

We consider the deformation of an infinitesimal part in the $r\theta$-plane and determine the elongational and shear strain components. The dimensions of the material volume in undeformed state are $dr \times rd\theta \times dz$.

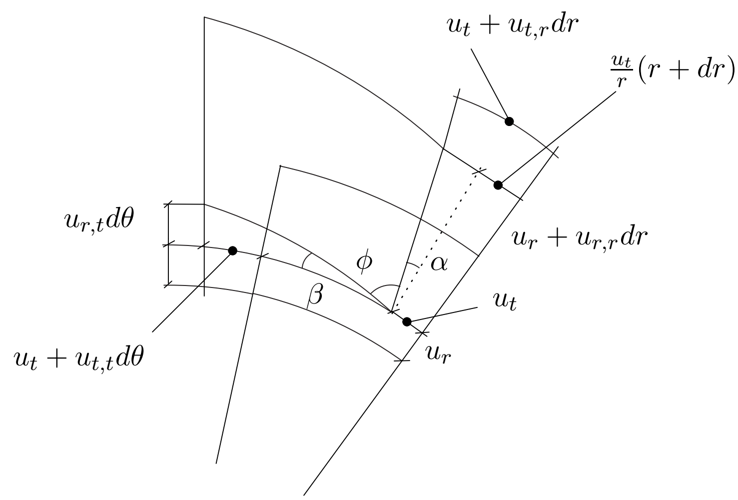



Fig. 1.22 : *Deformation of a cylindrical material volume*

$$\varepsilon_{rr} = \frac{u_{r,r}dr}{dr} = u_{r,r}$$

$$\varepsilon_{tt} = \frac{(r + u_r)d\theta - rd\theta}{rd\theta} + \frac{(u_t + u_{t,t}d\theta) - u_t}{rd\theta} = \frac{u_r}{r} + \frac{1}{r}\,u_{t,t}$$

$$\gamma_{rt} = \frac{\pi}{2} - \phi = \alpha + \beta = \left(u_{t,r} - \frac{u_t}{r}\right) + \left(\frac{1}{r}\,u_{r,t}\right)$$

---

**Strain gages**

Strain gages are used to measure strains on the surface of a thin walled pressure vessel. Three gages are glued on the surface, the second perpendicular to the first one and the third at an angle of 45$^o$ between those two. Measured strains have values $\varepsilon_{g1}, \varepsilon_{g2}$ and $\varepsilon_{g3}$.

The linear strain tensor is written in components w.r.t. the Cartesian coordinate system with its $x$-axis along the first strain gage. The components $\varepsilon_{xx}, \varepsilon_{xy}$ and $\varepsilon_{yy}$ have to be determined from the measured values.

To do this, we use the expression which gives us the strain in a specific direction, indicated by the unit vector $\vec{n}$.

$$\varepsilon_n = \vec{n} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}$$

Because we have three different directions, where the strain is known, we can write this equation three times.

$$\varepsilon_{g1} = \vec{n}_{g1} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g1} = \tilde{n}_{g1}^T \underline{\varepsilon}\, \tilde{n}_{g1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon_{xx}$$

$$\varepsilon_{g2} = \vec{n}_{g2} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g2} = \tilde{n}_{g2}^T \underline{\varepsilon}\, \tilde{n}_{g2} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon_{yy}$$

$$\varepsilon_{g3} = \vec{n}_{g3} \cdot \boldsymbol{\varepsilon} \cdot \vec{n}_{g3} = \tilde{n}_{g3}^T \underline{\varepsilon}\, \tilde{n}_{g3} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{2}(\varepsilon_{xx} + 2\varepsilon_{xy} + \varepsilon_{yy})$$

The first two equations immediately lead to values for $\varepsilon_{xx}$ and $\varepsilon_{yy}$ and the remaining unknown, $\varepsilon_{xy}$ can be solved from the last equation.

$$\left.\begin{aligned} \varepsilon_{xx} &= \varepsilon_{g1} \\ \varepsilon_{yy} &= \varepsilon_{g2} \\ \varepsilon_{xy} &= 2\varepsilon_{g3} - \varepsilon_{xx} - \varepsilon_{yy} \\ &= 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} \end{aligned}\right\} \quad \rightarrow \quad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{g1} & 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} \\ 2\varepsilon_{g3} - \varepsilon_{g1} - \varepsilon_{g2} & \varepsilon_{g2} \end{bmatrix}$$

The three gages can be oriented at various angles with respect to each other and with respect to the coordinate system. However, the three strain components can always be solved from a set of three independent equations.

---

**Shear strain**

A cube is deformed in the 12-plane by simple shear. This means that points on the upper edge move merely in 1-direction. Lines directed in 2-direction rotate over an angle $\alpha$ around the 3-axis.
The deformation matrix is :

$$\underline{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad \gamma = \tan(\alpha)$$

The right Cauchy-Green deformation matrix is

$$\underline{C} = \underline{F}^T\, \underline{F} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.
The Green-Lagrange strain matrix is

$$\underline{E} = \tfrac{1}{2}(\underline{C} - \underline{I}) = \begin{bmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & \frac{1}{2}\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal stretch ratios can be calculated from the eigenvalues of $\underline{C}$.

$$\det(\underline{C} - \underline{I}) = \det \begin{bmatrix} 1-\mu & \gamma & 0 \\ \gamma & 1+\gamma^2-\mu & 0 \\ 0 & 0 & 1-\mu \end{bmatrix} = 0 \quad \rightarrow$$

$$(1-\mu)^2(1+\gamma^2-\mu) - \gamma^2(1-\mu) = 0 \quad \rightarrow$$

$$(1-\mu)\{(1-\mu)(1+\gamma^2-\mu) - \gamma^2\} = 0 \quad \rightarrow \quad \mu_3 = 1 \quad \rightarrow \quad \lambda_3 = \sqrt{\mu_3} = 1$$

and

$$1 - \mu(2+\gamma^2) + \mu^2 = 0 \quad \rightarrow$$

$$\mu_{1,2} = \tfrac{1}{2}\left\{(2+\gamma^2) \pm \sqrt{(2+\gamma^2)^2 - 4}\right\} = \left(1+\frac{\gamma^2}{2}\right) \pm \gamma\sqrt{1+\frac{\gamma^2}{4}} \quad \rightarrow$$

$$\lambda_{1,2} = \sqrt{\left(1+\frac{\gamma^2}{2}\right) \pm \gamma\sqrt{1+\frac{\gamma^2}{4}}}$$

The linear strain matrix is

$$\underline{\varepsilon} = \begin{bmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal strains are

$$\det(\underline{\varepsilon} - \underline{I}) = \det \begin{bmatrix} -\varepsilon & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{bmatrix} = 0 \quad \rightarrow$$

$$-\varepsilon^3 + \tfrac{1}{4}\gamma^2\varepsilon = \varepsilon(-\varepsilon^2 + \tfrac{1}{4}\gamma^2) = 0 \quad \rightarrow$$

$$\varepsilon_3 = 1 \quad ; \quad \varepsilon_{1,2} = \pm\tfrac{1}{2}\gamma$$

# 2 Stresses

Kinematics describes the motion and deformation of a set of material points, considered here to be a continuous body. The cause of this deformation is not considered in kinematics. Motion and deformation may have various causes, which are collectively considered here to be external forces and moments.

Deformation of the material – not its motion alone – results in internal stresses. It is very important to calculate them accurately, because they may cause irreversible structural changes and even unallowable damage of the material.

## 2.1 Stress vector

Consider a material body in the deformed state, with edge and volume forces. The body is divided in two parts, where the cutting plane passes through the material point $P$. An edge load is introduced on both sides of the cutting plane to prevent separation of the two parts. In two associated points (= coinciding before the cut was made) in the cutting plane of both parts, these loads are of opposite sign, but have equal absolute value.

The resulting force on an area $\Delta A$ of the cutting plane in point $P$ is $\Delta\vec{k}$. The resulting force per unit of area is the ratio of $\Delta\vec{k}$ and $\Delta A$. The *stress vector* $\vec{p}$ in point $P$ is defined as the limit value of this ratio for $\Delta A \rightarrow 0$. So, obviously, the stress vector is associated to both poin $P$ and the cutting plane through this point.

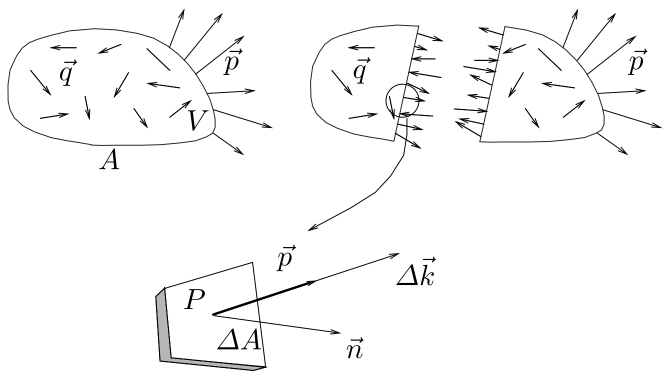



Fig. 2.23 : *Cross-sectional stresses and stress vector on a plane*

$$\vec{p} = \lim_{\Delta A \longrightarrow 0} \frac{\Delta\vec{k}}{\Delta A}$$

### 2.1.1 Normal stress and shear stress

The stress vector $\vec{p}$ can be written as the sum of two other vectors. The first is the *normal stress vector* $\vec{p}_n$ in the direction of the unity normal vector $\vec{n}$ on $\Delta A$. The second vector is in the plane and is called the *shear stress vector* $\vec{p}_s$.

The length of the normal stress vector is the *normal stress* $p_n$ and the length of the shear stress vector is the *shear stress* $p_s$.

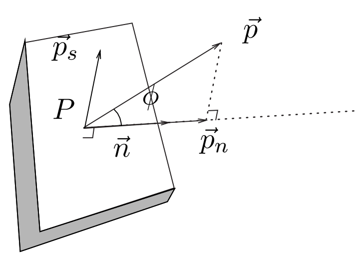



Fig. 2.24 : *Stress vector, normal strss and shear stress*

| | | |
|---|---|---|
| normal stress | : | $p_n = \vec{p} \cdot \vec{n}$ |
| tensile stress | : | positive ($\phi < \frac{\pi}{2}$) |
| compression stress | : | negative ($\phi > \frac{\pi}{2}$) |
| normal stress vector | : | $\vec{p}_n = p_n \vec{n}$ |
| shear stress vector | : | $\vec{p}_s = \vec{p} - \vec{p}_n$ |
| shear stress | : | $p_s = \lVert\vec{p}_s\rVert = \sqrt{\lVert\vec{p}\rVert^2 - p_n^2}$ |

## 2.2 Cauchy stress tensor

The stress vector can be calculated, using the *stress tensor* $\boldsymbol{\sigma}$, which represents the stress state in point $P$. The plane is identified by its unity normal vector $\vec{n}$. The stress vector is calculated according to Cauchy's theorem, which states that in each material point such a stress tensor must uniquely exist. ($\exists!$ : there exists only one.)

$$\text{Theorem of Cauchy :} \quad \exists! \quad \text{tensor} \quad \boldsymbol{\sigma} \quad \text{such that} \quad : \quad \vec{p} = \boldsymbol{\sigma} \cdot \vec{n}$$

### 2.2.1 Cauchy stress matrix

With respect to an orthogonal basis, the Cauchy stress tensor $\boldsymbol{\sigma}$ can be written in components, resulting in the Cauchy stress matrix $\underline{\sigma}$, which stores the components of the Cauchy stress tensor w.r.t. an orthonormal vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. The components of the Cauchy stress matrix are components of stress vectors on the planes with unit normal vectors in the coordinate directions.

With our definition, the first index of a stress component indicates the direction of the stress vector and the second index indicates the normal of the plane where it is loaded. As an example, the stress vector on the plane with $\vec{n} = \vec{e}_1$ is considered.

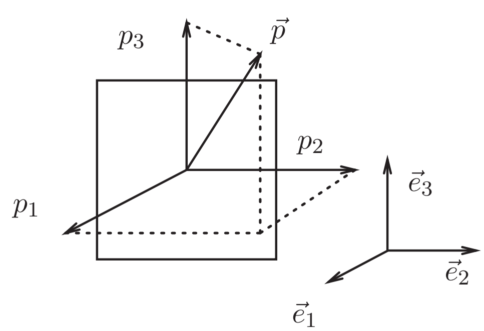


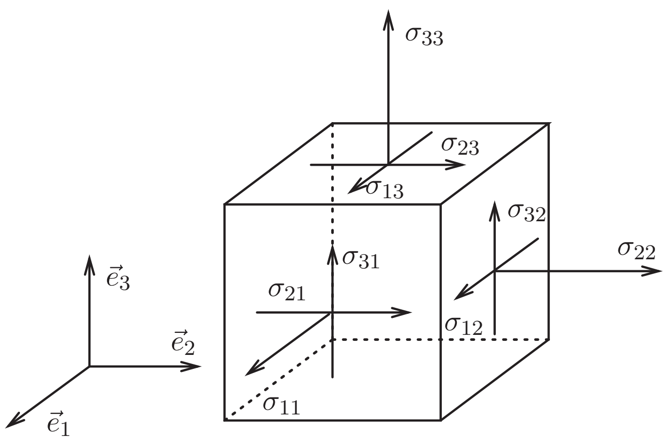

Fig. 2.25 : *Components of stress vector on a plane*

$$\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} \quad \rightarrow \quad \tilde{\vec{e}}^T \tilde{p} = \tilde{\vec{e}}^T \underline{\sigma}\ \tilde{\vec{e}} \cdot \tilde{\vec{e}}^T \tilde{n} = \tilde{\vec{e}}^T \underline{\sigma}\ \tilde{n}$$

$$\vec{n} = \vec{e}_1 \quad \rightarrow$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

The components of the Cauchy stress matrix can be represented as normal and shear stresses on the side planes of a stress cube.

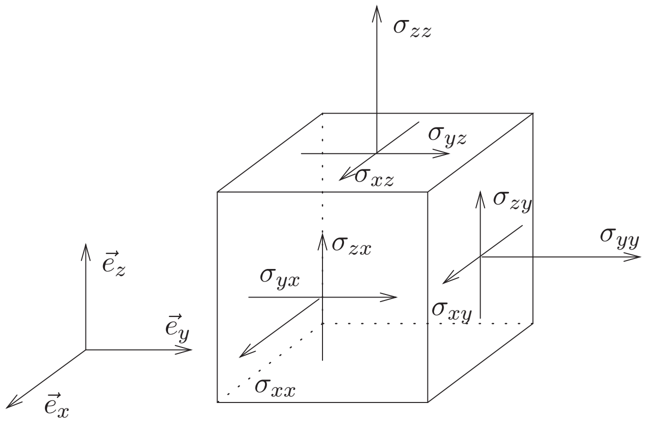



$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Fig. 2.26 : *Stress cube*

**Cartesian components**

In the Cartesian coordinate system the stress cube sides are parallel to the Cartesian coordinate axes. Stress components are indicated with the indices $x$, $y$ and $z$.

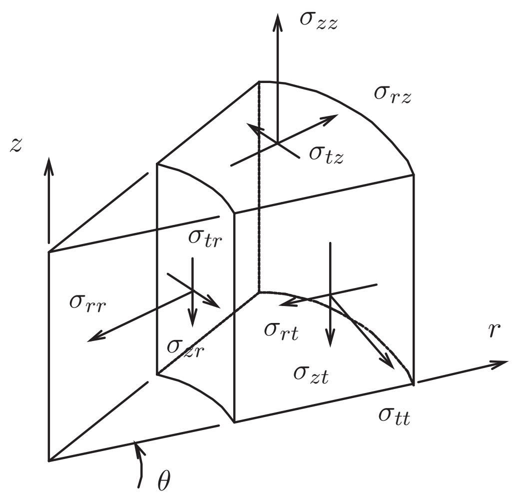



$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Fig. 2.27 : *Cartesian stress cube*

**Cylindrical components**

In the cylindrical coordinate system the stress 'cube' sides are parallel to the cylindrical coordinate axes. Stress components are indicated with the indices $r$, $t$ and $z$.




$$\underline{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} & \sigma_{rz} \\ \sigma_{tr} & \sigma_{tt} & \sigma_{tz} \\ \sigma_{zr} & \sigma_{zt} & \sigma_{zz} \end{bmatrix}$$

Fig. 2.28 : *Cylindrical stress "cube"*

### 2.2.2 Principal stresses and directions

It will be shown later that the stress tensor is symmetric. This means that it has three real-valued eigenvalues $\{\sigma_1, \sigma_2, \sigma_3\}$ and associated eigenvectors $\{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. The eigenvectors are normalized to have unit length and they are mutually perpendicular, so they constitute an orthonormal vector base. The stress matrix w.r.t. this vector base is diagonal.

The eigenvalues are referred to as the principal stresses and the eigenvectors as the principal stress directions. The stress cube with the normal principal stresses is referred to as the principal stress cube.

Using the spectral representation of $\boldsymbol{\sigma}$, it is easily shown that the stress tensor changes as a result of a rigid body rotation $\boldsymbol{Q}$.

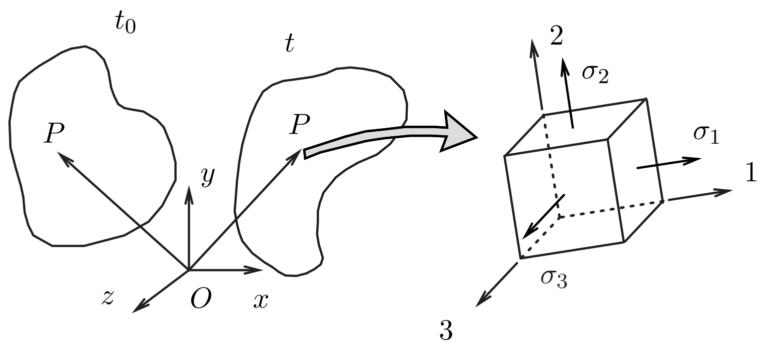



Fig. 2.29 : *Principal stress cube with principal stresses*

spectral form

$$\left.\begin{array}{l} \boldsymbol{\sigma} \cdot \vec{n}_1 = \sigma_1 \vec{n}_1 \\ \boldsymbol{\sigma} \cdot \vec{n}_2 = \sigma_2 \vec{n}_2 \\ \boldsymbol{\sigma} \cdot \vec{n}_3 = \sigma_3 \vec{n}_3 \end{array}\right\} \rightarrow \boldsymbol{\sigma} = \sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3$$

principal stress matrix

$$\underline{\sigma}_P = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

**Stress transformation**

We consider the two-dimensional plane with principal stress directions coinciding with the unity vectors $\vec{e}_1$ and $\vec{e}_2$. The principal stresses are $\sigma_1$ and $\sigma_2$. On a plane which is rotated anti-clockwise from $\vec{e}_1$ over an angle $\alpha < \frac{\pi}{2}$ the stress vector $\vec{p}$ and its normal and shear components can be calculated. They are indicated as $\sigma_\alpha$ and $\tau_\alpha$ respectively.

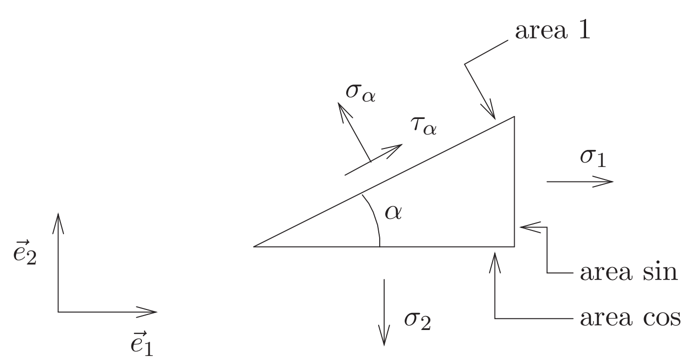


Fig. 2.30 : *Normal and shear stress on a plane*

$$\begin{aligned}
&\boldsymbol{\sigma} = \sigma_1 \vec{e}_1 + \sigma_2 \vec{e}_2 \\
&\vec{n} = -\sin(\alpha)\vec{e}_1 + \cos(\alpha)\vec{e}_2 \\
&\vec{p} = \boldsymbol{\sigma} \cdot \vec{n} = -\sigma_1 \sin(\alpha)\vec{e}_1 + \sigma_2 \cos(\alpha)\vec{e}_2 \\
&\sigma_\alpha = \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) \\
&\tau_\alpha = (\sigma_2 - \sigma_1)\sin(\alpha)\cos(\alpha)
\end{aligned}$$

**Mohr's circles of stress**

From the relations for the normal and shear stress on a plane in between two principal stress planes, a relation between these two stresses and the principal stresses can be derived. The resulting relation is the equation of a circle in the $\sigma_\alpha\tau_\alpha$-plane, referred to as Mohr's circle for stress. The radius of the circle is $\frac{1}{2}(\sigma_1 - \sigma_2)$. The coordinates of its center are $\{\frac{1}{2}(\sigma_1 + \sigma_2), 0\}$.

Stresses on a plane, which is rotated over $\alpha$ w.r.t. a principal stress plane, can be found in the circle by rotation over $2\alpha$.

Because there are three principal stresses and principal stress planes, there are also three stress circles. It can be proven that each stress state is located on one of the circles or in the shaded area.

$$\begin{aligned}
\sigma_\alpha &= \sigma_1 \sin^2(\alpha) + \sigma_2 \cos^2(\alpha) \\
&= \sigma_1(\tfrac{1}{2} - \tfrac{1}{2}\cos(2\alpha)) + \sigma_2(\tfrac{1}{2} + \tfrac{1}{2}\cos(2\alpha)) \\
&= \tfrac{1}{2}(\sigma_1 + \sigma_2) - \tfrac{1}{2}(\sigma_1 - \sigma_2)\cos(2\alpha) \quad \rightarrow
\end{aligned}$$

$$(1) \qquad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2 \cos^2(2\alpha)$$

$$\tau_\alpha = -\cos(\alpha)\sin(\alpha)\sigma_1 + \cos(\alpha)\sin(\alpha)\sigma_2 = \tfrac{1}{2}(\sigma_2 - \sigma_1)\sin(2\alpha) \quad \rightarrow$$

$$(2) \qquad \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_2 - \sigma_1)\right\}^2 \sin^2(2\alpha)$$

$$(1) + (2) \quad \rightarrow \quad \left\{\sigma_\alpha - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right\}^2 + \tau_\alpha^2 = \left\{\tfrac{1}{2}(\sigma_1 - \sigma_2)\right\}^2$$

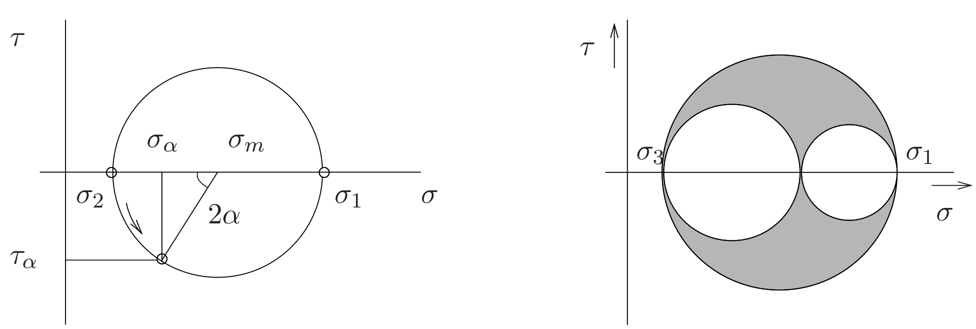



Fig. 2.31 : *Mohr's circles*

That there are three circles can be demonstrated by considering a random stress state $\{\sigma, \tau\}$ in the $\sigma\tau$-plane. The stress circles are subsequentially translated by superposition of a hydrostatic stress $-\frac{1}{2}(\sigma_1 + \sigma_3)$, $-\frac{1}{2}(\sigma_2 + \sigma_1)$ and $-\frac{1}{2}(\sigma_3 + \sigma_2)$. With the use of the stress vector $\vec{p}$ and the stress matrix $\underline{\sigma}^*$, resulting after superposition, it can be proven that the stress state is inside the largest stress circle and outside the other two.

inside $\sigma_1, \sigma_3$-circle

$$\begin{aligned} \{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_3)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} &= \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\ &= n_1^2\alpha^2 + n_2^2\beta^2 + n_3^2\alpha^2 \end{aligned}$$
$$\text{with} \quad \beta^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \leq \alpha^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_1 + \sigma_3)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \leq \alpha^2$$

outside $\sigma_2, \sigma_3$-circle

$$\begin{aligned} \{\sigma - \tfrac{1}{2}(\sigma_3 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} &= \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\ &= n_1^2\beta^2 + n_2^2\alpha^2 + n_3^2\alpha^2 \end{aligned}$$
$$\text{with} \quad \beta^2 = \left(\sigma_1 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_3 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$$

outside $\sigma_1, \sigma_2$-circle

$$\begin{aligned} \{\sigma - \tfrac{1}{2}(\sigma_1 + \sigma_2)\}^2 + \tau^2 = \sigma^2 + \tau^2 = ||\vec{p}||^2 = \vec{p} \cdot \vec{p} &= \underset{\sim}{n}^T \underline{\sigma}^T \underline{\sigma}\, \underset{\sim}{n} \\ &= n_1^2\alpha^2 + n_2^2\alpha^2 + n_3^2\beta^2 \end{aligned}$$
$$\text{with} \quad \beta^2 = \left(\sigma_3 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \geq \alpha^2 = \left(\sigma_2 - \tfrac{1}{2}(\sigma_1 + \sigma_2)\right)^2 \quad \rightarrow \quad \sigma^2 + \tau^2 \geq \alpha^2$$

## 2.3 Special stress states

Some special stress states are illustrated here. Stress components are considered in the Cartesian coordinate system.

### 2.3.1 Uni-axial stress

An unidirectional stress state is what we have in a tensile bar or truss. The axial load $N$ in a cross-section (area $A$ in the deformed state) is the integral of the axial stress $\sigma$ over $A$. For homogeneous material the stress is uniform in the cross-section and is called the *true* or *Cauchy stress*. When it is assumed to be uniform in the cross-section, it is the ratio of $N$ and $A$. The *engineering stress* is the ratio of $N$ and the initial cross-sectional area $A_0$, which makes calculation easy, because $A$ does not have to be known. For small deformations it is obvious that $A \approx A_0$ and thus that $\sigma \approx \sigma_n$.

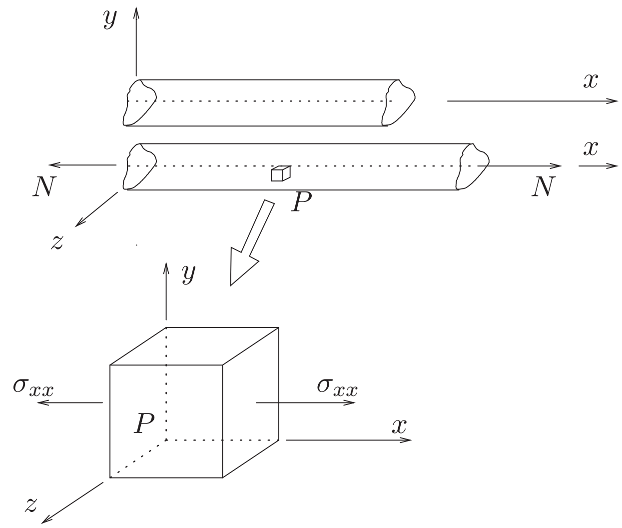
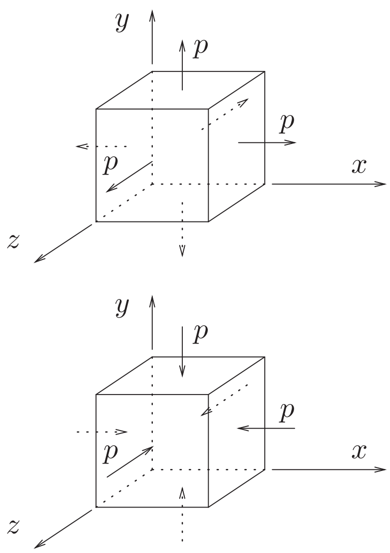



Fig. 2.32 : *Stresses on a small material volume in a tensile bar*

| | |
|---|---|
| true or Cauchy stress | $\sigma = \dfrac{N}{A} = \sigma_{xx} \quad \rightarrow \quad \boldsymbol{\sigma} = \sigma_{xx}\vec{e}_x\vec{e}_x$ |
| engineering stress | $\sigma_n = \dfrac{N}{A_0}$ |

### 2.3.2 Hydrostatic stress

A hydrostatic loading of the material body results in a hydrostatic stress state in each material point $P$. This can again be indicated by stresses (either tensile or compressive) on a stress cube. The three stress variables, with the same value, are normal to the faces of the stress cube.




$$\sigma_{xx} = p \qquad \sigma_{yy} = p \qquad \sigma_{zz} = p$$

$$\boldsymbol{\sigma} = p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)$$

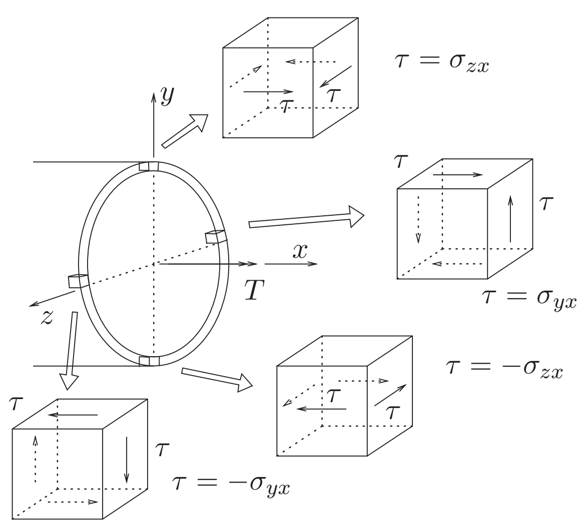
$$\sigma_{xx} = -p \qquad \sigma_{yy} = -p \qquad \sigma_{zz} = -p$$

$$\boldsymbol{\sigma} = -p(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z)$$

Fig. 2.33 : *Stresses on a material volume under hydrostatic loading*

### 2.3.3 Shear stress

The axial torsion of a thin-walled tube (radius $R$, wall thickness $t$) is the result of an axial torsional moment (torque) $T$. This load causes a shear stress $\tau$ in the cross-sectional wall. Although this shear stress has the same value in each point of the cross-section, the stress cube looks differently in each point because of the circumferential direction of $\tau$.




$$\tau = \frac{T}{2\pi R^2 t}$$

Fig. 2.34 : *Stresses on a small material volume in the wall of a tube under shear loading*

$$\boldsymbol{\sigma} = \tau(\vec{e}_i\vec{e}_j + \vec{e}_j\vec{e}_i) \text{ with } i \neq j$$

### 2.3.4 Plane stress

When stresses on a plane perpendicular to the 3-direction are zero, the stress state is referred to as *plane stress* w.r.t. the 12-plane. Only three stress components are relevant in this case.

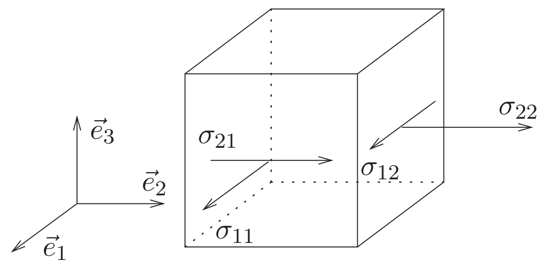



Fig. 2.35 : *Stress cube for plane stress in 12-plane*

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \quad \rightarrow \quad \boldsymbol{\sigma} \cdot \vec{e}_3 = \vec{0} \quad \rightarrow$$
$$\text{relevant stresses :} \quad \sigma_{11}, \sigma_{22}, \sigma_{12}$$

## 2.4 Resulting force on arbitrary material volume

A material body with volume $V$ and surface area $A$ is loaded with a volume load $\vec{q}$ per unit of mass and by a surface load $\vec{p}$ per unit of area. Taking a random part of the continuum with volume $\bar{V}$ and edge $\bar{A}$, the resulting force can be written as an integral over the volume, using Gauss' theorem. The load $\rho\vec{q}$ is a volume load per unit of volume, where $\rho$ is the density of the material.

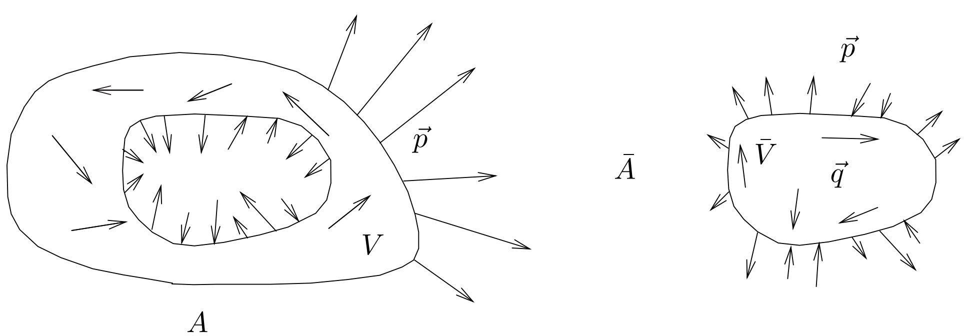



Fig. 2.36 : *Forces on a random section of a material body*

$$\vec{K} = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{p}\, dA = \int_{\bar{V}} \rho\vec{q}\, dV + \int_{\bar{A}} \vec{n} \cdot \boldsymbol{\sigma}^T\, dA$$

$$\text{Gauss theorem} \quad : \quad \int_{\bar{A}} \vec{n} \cdot (\,)\, dA = \int_{\bar{V}} \vec{\nabla} \cdot (\,)\, dV \quad \rightarrow$$

$$\vec{K} = \int_{\bar{V}} [\rho \vec{q} + \vec{\nabla} \cdot \boldsymbol{\sigma}^T]\, dV$$

## 2.5 Resulting moment on arbitrary material volume

The resulting moment about a fixed point of the forces working in volume and edge points of a random part of the continuum body can be calculated by integration.

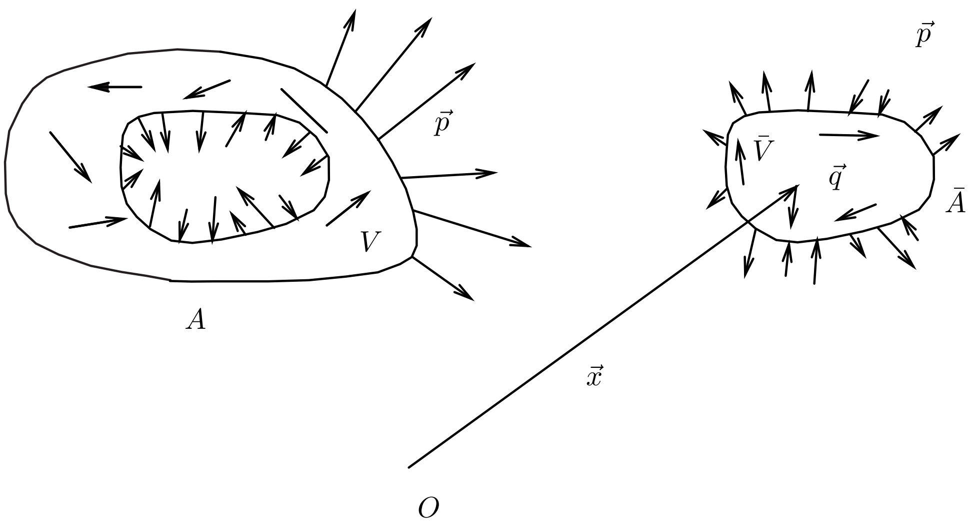



Fig. 2.37 : *Moments of forces on a random section of a material body*

$$\vec{M}_O = \int_{\bar{V}} \vec{x} * \rho \vec{q}\, dV + \int_{\bar{A}} \vec{x} * \vec{p}\, dA$$

### Resulting moment on total body

Obviously we can also calculate the resulting moment for the whole material volume. By introducing a special point $R$ other than the origin, the resulting moment can be expressed in the resulting moment about this point anf the moment of the resulting forces about this point. Often the resulting moment is considered with respect to the center of mass $M$ with position $\vec{x}_M$.

$$\begin{aligned} \vec{M}_O &= \int_V \vec{x} * \rho \vec{q}\, dV + \int_A \vec{x} * \vec{p}\, dA \\ &= \int_V (\vec{x}_R + \vec{r}) * \rho \vec{q}\, dV + \int_A (\vec{x}_R + \vec{r}) * \vec{p}\, dA \end{aligned}$$

$$
\begin{aligned}
&= \vec{x}_R * \int\limits_V \rho \vec{q}\, dV + \vec{x}_R * \int\limits_A \vec{p}\, dA + \int\limits_V \vec{r} * \rho \vec{q}\, dV + \int\limits_A \vec{r} * \vec{p}\, dA \\
&= \vec{x}_R * \vec{K} + \vec{M}_R \\
&= \vec{x}_M * \vec{K} + \vec{M}_M \quad \rightarrow
\end{aligned}
$$

$$
\vec{M}_R = (\vec{x}_M - \vec{x}_R) * \vec{K} + \vec{M}_M = \vec{r}_M * \vec{K} + \vec{M}_M
$$

## 2.6 Example

### Principal stresses and stress directions

The stress state in a material point $P$ is characterized by the stress tensor $\boldsymbol{\sigma}$, which is given in comonents with respect to an orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ :

$$\boldsymbol{\sigma} = 10\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 10\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3$$

The principal stresses are the eigenvalues of the tensor, which can be calculated as follows :

$$\det \begin{bmatrix} 10-\sigma & 6 & 0 \\ 6 & 10-\sigma & 0 \\ 0 & 0 & 1-\sigma \end{bmatrix} = 0 \quad \rightarrow$$

$$(10-\sigma)^2(1-\sigma) - 36(1-\sigma) = 0$$

$$(1-\sigma)\{(10-\sigma)^2 - 36\} = 0$$

$$(1-\sigma)(16-\sigma)(4-\sigma) = 0 \quad \rightarrow \quad \sigma_1 = 16 \quad ; \quad \sigma_2 = 4 \quad ; \quad \sigma_3 = 1$$

The eigenvectors are the principal stress directions.

$$\sigma_1 = 16 \quad \rightarrow \quad \begin{bmatrix} -6 & 6 & 0 \\ 6 & -6 & 0 \\ 0 & 0 & -15 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow$$

$$\left. \begin{array}{l} \alpha_1 = \alpha_2 \quad ; \quad \alpha_3 = 0 \\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \end{array} \right\} \quad \rightarrow \quad \vec{n}_1 = \tfrac{1}{2}\sqrt{2}\vec{e}_1 + \tfrac{1}{2}\sqrt{2}\vec{e}_2$$

$$\text{idem} \;:\quad \vec{n}_2 = -\tfrac{1}{2}\sqrt{2}\vec{e}_1 + \tfrac{1}{2}\sqrt{2}\vec{e}_2 \quad ; \quad \vec{n}_3 = \vec{e}_3$$

The average or hydrostatic stress can be calculated, leading to the hydrostatic stress tensor. The deviatoric stress tensor is the difference of the total stress tensor and the hydrostatic stress tensor.

$$\sigma_m = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) = 7$$

$$\boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}$$

$$\begin{aligned} \boldsymbol{\sigma}^d &= \boldsymbol{\sigma} - \boldsymbol{\sigma}^h = \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} \\ &= \{10\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 10\vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3\} - 7\boldsymbol{I} \\ &= 3\vec{e}_1\vec{e}_1 + 6(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1) + 3\vec{e}_2\vec{e}_2 - 6\vec{e}_3\vec{e}_3 \end{aligned}$$

# 3 Balance or conservation laws

In every physical process, so also during deformation of continuum bodies, some general accepted physical laws have to be obeyed : the conservation laws. During deformation the total *mass* has to be preserved and also the total *momentum* and *moment of momentum*. Because we do not consider dissipation and thermal effects, we will not discuss the conservation law for total energy.

## 3.1 Balance of mass

The mass of each finite, randomly chosen volume of material points in the continuum body must remain the same during the deformation process. Because we consider here a finite volume, this is the so-called *global* version of the mass conservation law.

From the requirement that this global law must hold for every randomly chosen volume, the *local version* of the conservation law can be derived. This derivation uses an integral transformation, where the integral over the volume $\bar{V}$ in the deformed state is transformed into an integral over the volume $\bar{V}_0$ in the undeformed state. From the requirement that the resulting integral equation has to be satisfied for each volume $\bar{V}_0$, the local version of the mass balance results.

The local version, which is also referred to as the *continuity equation*, can also be derived directly by considering the mass $dM$ of the infinitesimal volume $dV$ of material points.

The time derivative of the mass conservation law is also used frequently. Because we focus attention on the same material particles, a so-called *material time derivative* is used, which is indicated as $(\dot{\ })$.

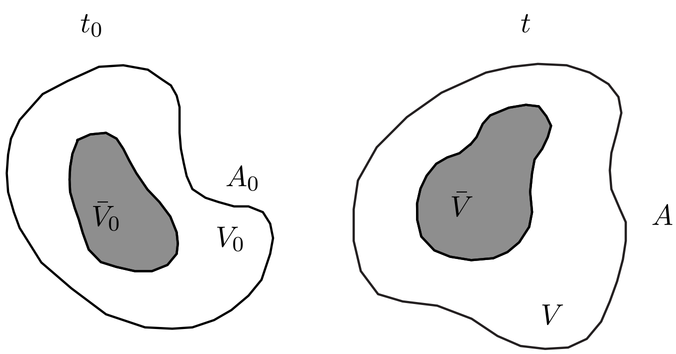



Fig. 3.38 : *Random volume in undeformed and deformed state*

$$\int_{\bar{V}} \rho \, dV = \int_{\bar{V}_0} \rho_0 \, dV_0 \quad \forall \, \bar{V} \quad \rightarrow \quad \int_{\bar{V}_0} (\rho J - \rho_0) \, dV_0 = 0 \quad \forall \, \bar{V}_0 \quad \rightarrow$$

$$\rho J = \rho_0 \qquad \forall \quad \vec{x} \in V(t)$$

$$dM = dM_0 \quad \rightarrow \quad \rho dV = \rho_0 dV_0 \quad \rightarrow \quad \rho J = \rho_0 \quad \forall \, \vec{x} \in V(t) \quad \rightarrow \quad \dot{\rho} J + \rho \dot{J} = 0$$

## 3.2 Balance of momentum

According to the balance of momentum law, a point mass $m$ which has a velocity $\vec{v}$, will change its momentum $\vec{i} = m\vec{v}$ under the action of a force $\vec{K}$. Analogously, the total force working on a randomly chosen volume of material points equals the change of the total momentum of the material points inside the volume. In the balance law, again a material time derivative is used, because we consider the same material points. The total force can be written as a volume integral of volume forces and the divergence of the stress tensor.

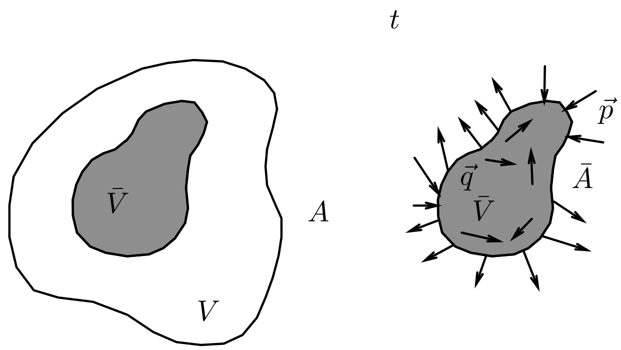



Fig. 3.39 : *Forces on random section of a material body*

$$\begin{aligned}
\vec{K} &= \frac{D\vec{i}}{Dt} = \frac{D}{Dt}\int_{\bar{V}} \rho\vec{v}\,dV \qquad \forall \quad \bar{V} \quad \rightarrow \\
&= \frac{D}{Dt}\int_{\bar{V}_0} \rho\vec{v}J\,dV_0 = \int_{\bar{V}_0} \frac{D}{Dt}(\rho\vec{v}J)\;dV_0 \qquad \forall \quad \bar{V}_0 \\
&= \int_{\bar{V}_0} \left(\dot{\rho}\vec{v}J + \rho\dot{\vec{v}}J + \rho\vec{v}\dot{J}\right)\,dV_0 \qquad \forall \quad \bar{V}_0 \\
&\qquad\qquad \text{mass balance} \quad : \quad \dot{\rho}J + \rho\dot{J} = 0 \quad \rightarrow \\
&= \int_{\bar{V}_0} \rho\dot{\vec{v}}J\,dV_0 = \int_{\bar{V}} \rho\dot{\vec{v}}\,dV \qquad \forall \quad \bar{V}
\end{aligned}$$

$$\int_{\bar{V}} \left(\rho\vec{q} + \vec{\nabla}\cdot\boldsymbol{\sigma}^c\right)\,dV = \int_{\bar{V}} \rho\dot{\vec{v}}\,dV \qquad \forall \quad \bar{V}$$

From the requirement that the global balance law must hold for every randomly chosen volume of material points, the local version of the balance of momentum can be derived, which must hold in every material point. In the derivation an integral transformation is used.

The local balance of momentum law is also called the *equation of motion*. For a stationary process, where the material velocity $\vec{v}$ in a fixed spatial point does not change, the equation is simplified. For a static process, where there is no acceleration of masses, the *equilibrium equation* results.

| | | |
|---|---|---|
| local version : equation of motion | $\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\dot{\vec{v}} = \rho\dfrac{\delta\vec{v}}{\delta t} + \rho\vec{v}\cdot\left(\vec{\nabla}\vec{v}\right)$ | $\forall\ \vec{x}\in V(t)$ |
| stationary $\left(\dfrac{\delta\vec{v}}{\delta t} = 0\right)$ | $\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \rho\vec{v}\cdot\left(\vec{\nabla}\vec{v}\right)$ | |
| static : equilibrium equation | $\vec{\nabla}\cdot\boldsymbol{\sigma}^c + \rho\vec{q} = \vec{0}$ | |

### 3.2.1 Cartesian and cylindrical components

The equilibrium equation can be written in components w.r.t. a Cartesian vector basis. This results in three partial differential equations, one for each coordinate direction.

$$
\begin{aligned}
\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x &= 0 \\
\sigma_{yx,x} + \sigma_{yy,y} + \sigma_{yz,z} + \rho q_y &= 0 \\
\sigma_{zx,x} + \sigma_{zy,y} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}
$$

Writing tensor and vectors in components w.r.t. a cylindrical vector basis is more elaborative because the cylindrical base vectors $\vec{e}_r$ and $\vec{e}_t$ are a function of the coordinate $\theta$, so they have to be differentiated, when expanding the divergence term.

$$
\begin{aligned}
\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r &= 0 \\
\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \sigma_{tz,z} + \rho q_t &= 0 \\
\sigma_{zr,r} + \frac{1}{r}\sigma_{zt,t} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z &= 0
\end{aligned}
$$

## 3.3 Balance of moment of momentum

The balance of moment of momentum states that the total moment about a fixed point of all forces working on a randomly chosen volume of material points ($\vec{M}_O$), equals the change of the total moment of momentum of the material points inside the volume, taken w.r.t. the same fixed point ($\vec{L}_O$).

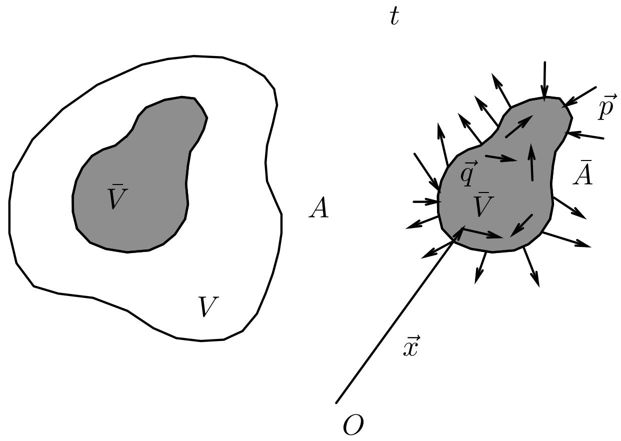



Fig. 3.40 : *Moment of forces on a random section of a material body*

$$\begin{aligned}
\vec{M}_O &= \frac{D\vec{L}_O}{Dt} = \frac{D}{Dt}\int\limits_{\bar{V}} \vec{x} * \rho\vec{v}\, dV \qquad \forall \quad \bar{V} \\
&= \frac{D}{Dt}\int\limits_{\bar{V}_0} \vec{x} * \rho\vec{v} J\, dV_0 = \int\limits_{\bar{V}_0} \frac{D}{Dt}\left(\vec{x} * \rho\vec{v} J\right)\, dV_0 \qquad \forall \quad \bar{V}_0 \\
&= \int\limits_{\bar{V}_0} \left(\dot{\vec{x}} * \rho\vec{v} J + \vec{x} * \dot{\rho}\vec{v} J + \vec{x} * \rho\dot{\vec{v}} J + \vec{x} * \rho\vec{v}\dot{J}\right)\, dV_0 \qquad \forall \quad \bar{V}_0
\end{aligned}$$

$$\left.\begin{array}{ll} \text{mass balance} \quad : & \dot{\rho}J + \rho\dot{J} = 0 \\ \dot{\vec{x}} * \vec{v} = \vec{v} * \vec{v} = \vec{0} & \end{array}\right\} \rightarrow$$

$$= \int\limits_{\bar{V}_0} \vec{x} * \rho\dot{\vec{v}} J\, dV_0 = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

$$\int\limits_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int\limits_{\bar{A}} \vec{x} * \vec{p}\, dA = \int\limits_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \qquad \forall \quad \bar{V}$$

For the analysis of the dynamics of deformable and rigid bodies the balance law is often reformulated, such that its motion is the superposition a rotation about the center of rotation $R$ and its translation. In appendix **??** this is elaborated.

To derive a local version, the integral over the area $\bar{A}$ has to be transformed to an integral over the enclosed volume $\bar{V}$. In this derivation, the operator ${}^3\boldsymbol{\epsilon}$ is used, which is defined such that

$$\vec{a} * \vec{b} = {}^3\boldsymbol{\epsilon} : \vec{a}\,\vec{b}$$

holds for all vectors $\vec{a}$ and $\vec{b}$.

$$\int_{\bar{A}} \vec{x} * \vec{p}\, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\vec{p})\, dA = \int_{\bar{A}} {}^3\boldsymbol{\epsilon} : (\vec{x}\,\boldsymbol{\sigma} \cdot \vec{n})\, dA = \int_{\bar{A}} \vec{n} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dA$$

$$= \int_{\bar{V}} \vec{\nabla} \cdot \{{}^3\boldsymbol{\epsilon} : (\vec{x}\boldsymbol{\sigma})\}^c\, dV$$

$$= \int_{\bar{V}} \vec{\nabla} \cdot \{(\vec{x}\boldsymbol{\sigma})^c : {}^3\boldsymbol{\epsilon}^c\}\, dV = \int_{\bar{V}} \vec{\nabla} \cdot \{(\boldsymbol{\sigma}^c\vec{x}) : {}^3\boldsymbol{\epsilon}^c\}\, dV$$

$$= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} \cdot (\vec{\nabla} \cdot \vec{x}) : {}^3\boldsymbol{\epsilon}^c\right] dV$$

$$= \int_{\bar{V}} \left[(\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\vec{x} : {}^3\boldsymbol{\epsilon}^c + \boldsymbol{\sigma} : {}^3\boldsymbol{\epsilon}^c\right] dV = \int_{\bar{V}} \left[{}^3\boldsymbol{\epsilon} : \vec{x}(\vec{\nabla} \cdot \boldsymbol{\sigma}^c) + {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\right] dV$$

$$= \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c + \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\, dV$$

Substitution in the global version and using the local balance of momentum, leads to the local version of the balance of moment of momentum.

$$\int_{\bar{V}} \vec{x} * \rho\vec{q}\, dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV + \int_{\bar{V}} \vec{x} * (\vec{\nabla} \cdot \boldsymbol{\sigma}^c)\, dV = \int_{\bar{V}} \vec{x} * \rho\dot{\vec{v}}\, dV \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} \vec{x} * \left[\rho\vec{q} + (\vec{\nabla} \cdot \boldsymbol{\sigma}^c) - \rho\dot{\vec{v}}\right] dV + \int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow$$

$$\int_{\bar{V}} {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c\, dV = \vec{0} \quad \forall \quad \bar{V} \quad \rightarrow \quad {}^3\boldsymbol{\epsilon} : \boldsymbol{\sigma}^c = \vec{0} \quad \forall \quad \vec{x} \in \bar{V}$$

$$\begin{bmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\sigma}^c = \boldsymbol{\sigma} \qquad \forall \quad \vec{x} \in V(t)$$

### 3.3.1 Cartesian and cylindrical components

With respect to a Cartesian or cylindrical basis the symmetry of the stress tensor results in three equations.

$$\underline{\sigma} = \underline{\sigma}^T \qquad \rightarrow$$

| | | | | | | |
|---|---|---|---|---|---|---|
| Cartesian | : | $\sigma_{xy} = \sigma_{yx}$ | ; | $\sigma_{yz} = \sigma_{zy}$ | ; | $\sigma_{zx} = \sigma_{xz}$ |
| cylindrical | : | $\sigma_{rt} = \sigma_{tr}$ | ; | $\sigma_{tz} = \sigma_{zt}$ | ; | $\sigma_{zr} = \sigma_{rz}$ |

## 3.4 Balance of energy

The first law of thermodynamics states that the total amount of energy supplied to a material body is converted to kinetic energy ($U_k$) and internal energy ($U_i$). The supplied energy is considered to be 1) work done by external mechanical loads ($U_e$), and 2) thermal energy supplied by internal sources or external fluxes ($U_t$). The internal energy can be of very different character, such as elastically stored energy and dissipated energy due to plastic deformation, viscous effects, crack growth, etcetera.

$$\frac{D}{Dt}(U_e + U_t) = \frac{D}{Dt}(U_k + U_i)$$

### 3.4.1 Mechanical energy

When a point load $\vec{k}$ is applied in a material point and the point moves with a velocity $\vec{v}$, the work of the load per unit of time is $\dot{U}_e = \vec{k} \cdot \vec{v}$. For a random volume $\bar{V}$ with edge $\bar{A}$ inside a material body the mechanical work of all loads per unit of time can be calculated. Using Gauss' theorem, this work can be written as an integral over the volume $\bar{V}$. Also the equation of motion is used to arrive at the final result.

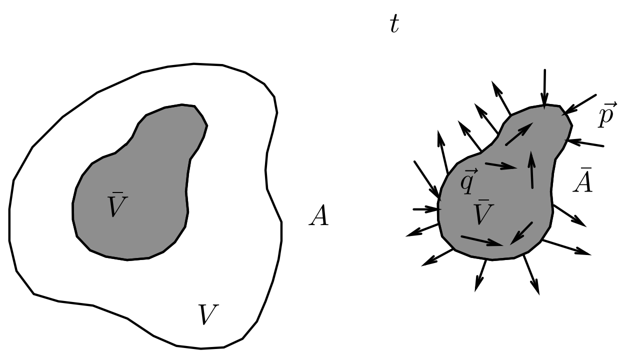



Fig. 3.41 : *Mechanical load on a material volume*

$$\dot{U}_e = \int_{\bar{V}} \rho\vec{q}\cdot\vec{v}\,dV + \int_{\bar{A}} \vec{p}\cdot\vec{v}\,dA = \int_{\bar{V}} \{\rho\vec{q}\cdot\vec{v} + \vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{v})\}\,dV$$

$$\begin{aligned}\vec{\nabla}\cdot(\boldsymbol{\sigma}^c\cdot\vec{v}) &= (\vec{\nabla}\cdot\boldsymbol{\sigma}^c)\cdot\vec{v} + \boldsymbol{\sigma} : (\vec{\nabla}\vec{v}) \\ &= \rho\dot{\vec{v}}\cdot\vec{v} - \rho\vec{q}\cdot\vec{v} + \boldsymbol{\sigma} : \boldsymbol{D} + \boldsymbol{\sigma} : \boldsymbol{\Omega}\end{aligned}$$

$$= \int_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \boldsymbol{\sigma} : \boldsymbol{D})\,dV$$

### 3.4.2 Thermal energy

Thermal energy can be produced by internal sources. The heat production per unit of mass is $r$ [J kg$^{-1}$].

Heat can flow in or out of a material body or in the body from one part to another. In a material point $P$ the heat flux vector is $\vec{H}$ [J]. The heat flux density vector *in* $P$ *through* a plane with area $\Delta A$ is

$$\vec{h} = \lim_{\Delta A \longrightarrow 0} \frac{\vec{H}}{\Delta A} \qquad\qquad [\text{J m}^{-2}]$$

The resulting heat flux *in* $P$ *through* the plane is $\vec{n} \cdot \vec{h}$ [J m$^{-2}$] , where $\vec{n}$ is the unit normal vector on the plane.

For a random volume $\bar{V}$ having edge $\bar{A}$ with unit normal outward vector $\vec{n}$, the increase in thermal energy at time $t$ is $\dot{U}_t$.

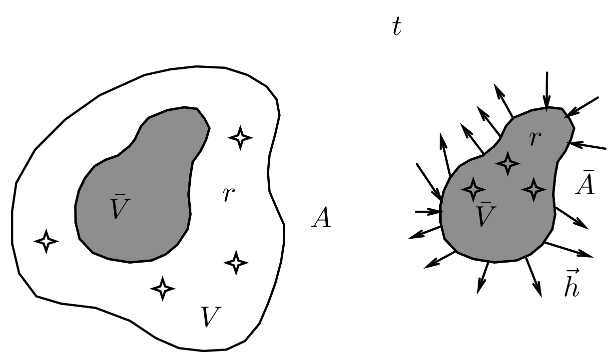



Fig. 3.42 : *Heat sources in and heat flux into a material volume*

$$\dot{U}_t = \int_{\bar{V}} \rho r \, dV - \int_{\bar{A}} \vec{n} \cdot \vec{h} \, dA = \int_{\bar{V}} (\rho r - \vec{\nabla} \cdot \vec{h}) \, dV$$

### 3.4.3 Kinetic energy

The kinetic energy of a point mass $m$ with velocity $\vec{v}$ is

$$U_k = \tfrac{1}{2} \, m \, ||\vec{v}||^2 = \tfrac{1}{2} \, m \, \vec{v} \cdot \vec{v}$$

For a random volume $\bar{V}$ of material points, having density $\rho$ and velocity $\vec{v}$, the total kinetic energy $U_k$ can be calculated by intergration.

$$U_k(t) = \int_{\bar{V}} \tfrac{1}{2} \, \rho \, \vec{v} \cdot \vec{v} \, dV \quad \rightarrow \quad \dot{U}_k = \frac{D}{Dt} \int_{\bar{V}} \tfrac{1}{2} \, \rho \, \vec{v} \cdot \vec{v} \, dV = \frac{D}{Dt} \int_{\bar{V}_0} \tfrac{1}{2} \, \rho \, \vec{v} \cdot \vec{v} J \, dV_0$$

$$= \tfrac{1}{2} \int_{\bar{V}_0} \left\{ \dot{\rho} \, \vec{v} \cdot \vec{v} J + 2\rho \, \dot{\vec{v}} \cdot \vec{v} J + \rho \, \vec{v} \cdot \vec{v} \dot{J} \right\} dV_0$$

$$= \int_{\bar{V}_0} \rho \, \dot{\vec{v}} \cdot \vec{v} J \, dV_0 = \int_{\bar{V}} \rho \, \dot{\vec{v}} \cdot \vec{v} \, dV$$

### 3.4.4 Internal energy

The internal energy per unit of mass is $\phi$. The total internal energy of all material points in a random volume $\bar{V}$ of a material body, $U_i$, can be calculated by integration.

$$
\begin{aligned}
U_i(t) = \int\limits_{\bar{V}} \rho\phi \, dV \quad \rightarrow \quad \dot{U}_i &= \frac{D}{Dt}\int\limits_{\bar{V}} \rho\phi \, dV = \frac{D}{Dt}\int\limits_{\bar{V}_0} \rho\phi J \, dV_0 \\
&= \int\limits_{\bar{V}_0} \left\{ \dot{\rho}\phi J + \rho\dot{\phi}J + \rho\phi\dot{J} \right\} dV_0 \\
&= \int\limits_{\bar{V}} \rho\dot{\phi} \, dV
\end{aligned}
$$

### 3.4.5 Energy balance

The energy balance or first law of thermodynamics for a random volume of material points in a material body, can be written as an integral equation. It is the global form of the balance law, because a finite volume is considered.

$$
\dot{U}_e + \dot{U}_t = \dot{U}_k + \dot{U}_i
$$

$$
\int\limits_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla}\cdot\vec{h}) \, dV = \int\limits_{\bar{V}} (\rho\dot{\vec{v}}\cdot\vec{v} + \rho\dot{\phi}) \, dV \qquad \forall \quad \bar{V}
$$

$$
\int\limits_{\bar{V}} \rho\dot{\phi} \, dV = \int\limits_{\bar{V}} (\boldsymbol{\sigma} : \boldsymbol{D} + \rho r - \vec{\nabla}\cdot\vec{h}) \, dV \qquad \forall \quad \bar{V}
$$

### 3.4.6 Energy equation

The local version of the energy balance, also called the *energy equation*, is easily derived by taking into account the fact that the global version must be valid for each volume $\bar{V}$.

The specific internal energy $\phi$ can be written as the product of the specific heat $C_p$ (assumed to be constant here) and the absolute temperature $T$.

The heat flux density $\vec{h}$ is often related to the temperature gradient $\vec{\nabla}T$ accoding to Fourier's law, with the thermal conductivity tensor $\boldsymbol{K}$.

$$
\left.
\begin{array}{ll}
\rho\dot{\phi} + \vec{\nabla}\cdot\vec{h} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r & \forall \quad \vec{x} \in V(t) \\
\dot{\phi} = C_p\dot{T} \quad (C_p : \text{specific heat}) &
\end{array}
\right\} \rightarrow
$$

$$\left.\begin{array}{ll}\rho C_p\dot{T} + \vec{\nabla}\cdot\vec{h} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r & \forall \quad \vec{x} \in V(t) \\ \text{Fourier's law} \quad \vec{h} = -\boldsymbol{K}\cdot(\vec{\nabla}T) & \end{array}\right\} \rightarrow$$

$$\rho C_p\dot{T} - \vec{\nabla}\cdot\left\{\boldsymbol{K}\cdot\left(\vec{\nabla}T\right)\right\} = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r$$
$$\rho C_p\dot{T} - (\vec{\nabla}\cdot\boldsymbol{K})\cdot(\vec{\nabla}T) - \boldsymbol{K} : (\vec{\nabla}\vec{\nabla}T)^c = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r$$

homogeneous conductivity : $\vec{\nabla}\cdot\boldsymbol{K} = \vec{0} \quad \rightarrow$

$$\rho C_p\dot{T} - \boldsymbol{K} : (\vec{\nabla}\vec{\nabla}T)^c = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r$$

$$\rho C_p\frac{\delta T}{\delta t} + \rho C_p\vec{v}\cdot(\vec{\nabla}T) - \boldsymbol{K} : (\vec{\nabla}\vec{\nabla}T)^c = \boldsymbol{\sigma} : \boldsymbol{D} + \rho r$$

**Energy equation : Cartesian components**

orthotropic conductivity $\boldsymbol{K} = k_x\vec{e}_x\vec{e}_x + k_y\vec{e}_y\vec{e}_y + k_z\vec{e}_z\vec{e}_z$

$$\begin{aligned}\rho C_p\frac{\delta T}{\delta t} &+ \rho C_p\left(v_xT_{,x} + v_yT_{,y} + v_zT_{,z}\right) - k_xT_{,xx} - k_yT_{,yy} - k_zT_{,zz} \\ &= \sigma_{xx}v_{x,x} + \sigma_{yy}v_{y,y} + \sigma_{zz}v_{z,z} + \rho r\end{aligned}$$

two-dimensional (in $xy$-plane only)

$$\rho C_p\frac{\delta T}{\delta t} + \rho C_p\left(v_xT_{,x} + v_yT_{,y}\right) - k_xT_{,xx} - k_yT_{,yy} = \sigma_{xx}v_{x,x} + \sigma_{yy}v_{y,y} + \rho r$$

one-dimensional ( in $x$-direction only; $k_x = k$ )

$$\rho C_p\frac{\delta T}{\delta t} + \rho C_pv_xT_{,x} - kT_{,xx} = \sigma_{xx}v_{x,x} + \rho r$$

no convection ($v_x = v_{x,x} = 0$)

$$\rho C_p\frac{\delta T}{\delta t} - kT_{,xx} = \rho r$$

no heat source/sinc

$$\rho C_p\frac{\delta T}{\delta t} - kT_{,xx} = 0$$

stationary

$$T_{,xx} = 0$$

solution

$$T(x) = c_0 + c_1x$$

**Energy equation : Cylindrical components**

orthotropic conductivity $\boldsymbol{K} = k_r\vec{e}_r\vec{e}_r + k_t\vec{e}_t\vec{e}_t + k_z\vec{e}_z\vec{e}_z$

$$
\begin{aligned}
&\rho C_p\frac{\delta T}{\delta t} + \rho C_p(v_r\vec{e}_r + v_t\vec{e}_t + v_z\vec{e}_z)\cdot(T_{,r}\vec{e}_r + \frac{1}{r}T_{,t}\vec{e}_t + T_{,z}\vec{e}_z) \\
&\qquad - \left(\vec{e}_r\frac{\partial}{\partial r} + \vec{e}_t\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z}\right)\left(\vec{e}_r\frac{\partial T}{\partial r} + \vec{e}_t\frac{1}{r}\frac{\partial T}{\partial \theta} + \vec{e}_z\frac{\partial T}{\partial z}\right) : (k_r\vec{e}_r\vec{e}_r + k_t\vec{e}_t\vec{e}_t + k_z\vec{e}_z\vec{e}_z) \\
&\qquad = \sigma_{rr}v_{r,r} + \sigma_{tt}v_{t,t} + \sigma_{zz}v_{z,z} + \rho r \\
&\rho C_p\frac{\delta T}{\delta t} + \rho C_p(v_rT_{,r} + \frac{1}{r}v_tT_{,t} + v_zT_{,z}) \\
&\quad - \Big(\vec{e}_r\vec{e}_rT_{,rr} - \vec{e}_r\vec{e}_t\frac{1}{r^2}\,T_{,t} + \vec{e}_r\vec{e}_t\frac{1}{r}\,T_{,rt} + \vec{e}_r\vec{e}_zT_{,rz} + \vec{e}_t\vec{e}_t\frac{1}{r}T_{,r} + \vec{e}_t\vec{e}_r\frac{1}{r}T_{,tr} - \vec{e}_t\vec{e}_r\frac{1}{r^2}T_{,t} + \vec{e}_t\vec{e}_t\frac{1}{r^2}T_{,tt} + \\
&\quad \vec{e}_t\vec{e}_z\frac{1}{r}T_{,tz} + \vec{e}_z\vec{e}_rT_{,zr} + \vec{e}_z\vec{e}_t\frac{1}{r}T_{,zt} + \vec{e}_z\vec{e}_zT_{,zz}\Big) : (k_r\vec{e}_r\vec{e}_r + k_t\vec{e}_t\vec{e}_t + k_z\vec{e}_z\vec{e}_z) = \sigma_{rr}v_{r,r} + \sigma_{tt}v_{t,t} + \sigma_{zz}v_{z,z} + \rho r \\
&\rho C_p\frac{\delta T}{\delta t} + \rho C_p(v_rT_{,r} + \frac{1}{r}v_tT_{,t} + v_zT_{,z}) \\
&\quad - \left(T_{,rr}k_r + \frac{1}{r}T_{,r}k_t + \frac{1}{r^2}T_{,tt}k_t + T_{,zz}k_z\right) = \sigma_{rr}v_{r,r} + \sigma_{tt}v_{t,t} + \sigma_{zz}v_{z,z} + \rho r
\end{aligned}
$$

**Axi-symmetry**

$$\rho C_p\frac{\delta T}{\delta t} + \rho C_p\left(v_rT_{,r} + v_zT_{,z}\right) - \left(T_{,rr}k_r + \frac{1}{r}T_{,r}k_t + T_{,zz}k_z\right) = \sigma_{rr}v_{r,r} + \sigma_{zz}v_{z,z} + \rho r$$

two-dimensional ( polar; $rt$-plane)

$$\rho C_p\frac{\delta T}{\delta t} + \rho C_pv_rT_{,r} - k_rT_{,rr} - k_t\frac{1}{r}T_{,r} = \sigma_{rr}v_{r,r} + \rho r$$

stationary ( $\frac{\delta T}{\delta t} = 0$ )

$$\rho C_pv_rT_{,r} - k_rT_{,rr} - k_t\frac{1}{r}T_{,r} = \sigma_{rr}v_{r,r} + \rho r$$

isotropic thermal conductivity ( $k_r = k_t = k$ )

$$\rho C_pv_rT_{,r} - kT_{,rr} - k\frac{1}{r}T_{,r} = \sigma_{rr}v_{r,r} + \rho r$$

no convection ( $v_r = v_{r,r} = 0$ )

$$-kT_{,rr} - k\frac{1}{r}T_{,r} = \rho r$$

no heat source/sinc

$$T_{,rr} + \frac{1}{r}T_{,r} = 0$$

solution

$$T = r^a \quad \rightarrow \quad a(a-1)r^{a-2} + r^{-1}ar^{a-1} = 0 \quad \rightarrow \quad a = 0 \quad \rightarrow \quad T = c_0 + c_1r$$

### 3.4.7 Mechanical power for three-dimensional deformation

Elastic deformation of a three-dimensional continuum leads to storage of elastic energy, which can be calculated per unit of undeformed ($W_0$) or deformed ($W$) volume. Different expressions for the strain rate can than be combined with different stress tensors, which are all a function of the Cauchy stress tensor $\boldsymbol{\sigma}$. The starting point is the change of stored energy per unit of deformed volume.

$$\dot{W} = \boldsymbol{\sigma} : \boldsymbol{D} \qquad \boldsymbol{\sigma} = \text{Cauchy stress tensor}$$

$$\begin{aligned} \dot{W}_0 &= [J\boldsymbol{\sigma}] : \boldsymbol{D} \\ &= \boldsymbol{\kappa} : \boldsymbol{D} \qquad \boldsymbol{\kappa} = \text{Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \boldsymbol{D} = J\boldsymbol{\sigma} : \tfrac{1}{2}\left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} + (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c\right) = \\ &= J\boldsymbol{\sigma} : \left(\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}\right) : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{F}} = \boldsymbol{S} : \dot{\boldsymbol{U}} \\ &= \boldsymbol{S} : \dot{\boldsymbol{\mathcal{E}}} \qquad \boldsymbol{S} = \text{1st-Piola-Kirchhoff stress tensor} \end{aligned}$$

$$\begin{aligned} \dot{W}_0 &= J\boldsymbol{\sigma} : \boldsymbol{D} = J\boldsymbol{\sigma} : \left(\boldsymbol{F}^{-c} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1}\right) = J\left(\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-c}\right) : \dot{\boldsymbol{E}} \\ &= \boldsymbol{P} : \dot{\boldsymbol{E}} \qquad \boldsymbol{P} = \text{2nd-Piola-Kirchhoff stress tensor} \end{aligned}$$

## 3.5 Special equilibrium states

The three-dimensional equilibrium equations can be simplified for special deformation or stress states, such as plane strain, plane stress and axisymmetric cases.

### Planar deformation

It is assumed here that the $z$-direction is the direction where either the strain or the stress is zero. Only stresses and strains in the plane perpendicular to the $z$-direction remain to be determined from equilibrium. The strain or stress in the $z$-direction can be calculated afterwards, either directly from the material law or iteratively during the solution procedure.

Cartesian components

$$\begin{aligned} &\sigma_{xx,x} + \sigma_{xy,y} + \rho q_x = 0 \\ &\sigma_{yx,x} + \sigma_{yy,y} + \rho q_y = 0 \\ &\sigma_{xy} = \sigma_{yx} \end{aligned}$$

cylindrical components

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}\sigma_{rt,t} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{1}{r}\sigma_{tt,t} + \frac{1}{r}(\sigma_{tr} + \sigma_{rt}) + \rho q_t = 0
\end{aligned}
$$

$$
\sigma_{rt} = \sigma_{tr}
$$

**Axisymmetric deformation**

In many cases the geometry, boundary conditions and material behavior is such that no state variable depends on the circumferential coordinate $\theta$ : $\frac{\partial}{\partial\theta} = 0$. For such axisymmetric deformations, the equilibrium equations can be simplified considerably.

In many axisymmetric deformations the boundary conditions are such that there is no displacement in the circumferential direction : $u_t = 0$. In these cases there are only four relevant strain and stress components and only three equilibrium equations.

When boundary conditions and material behavior are such that displacement of material points are only in the $r\theta$-plane, the deformation is referred to as *plane strain* in the $r\theta$-plane.

When stresses on a plane perpendicular to the $z$-direction are zero, the stress state is referred to as *plane stress* w.r.t. the $r\theta$-plane.

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \sigma_{rz,z} + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \sigma_{tz,z} + \rho q_t = 0 \qquad \text{(if } u_t \neq 0) \\
&\sigma_{zr,r} + \frac{1}{r}\sigma_{zr} + \sigma_{zz,z} + \rho q_z = 0 \\
&\sigma_{rt} = \sigma_{tr} \quad ; \quad \sigma_{tz} = \sigma_{zt} \qquad \text{(if } u_t \neq 0) \\
&\sigma_{zr} = \sigma_{rz}
\end{aligned}
$$

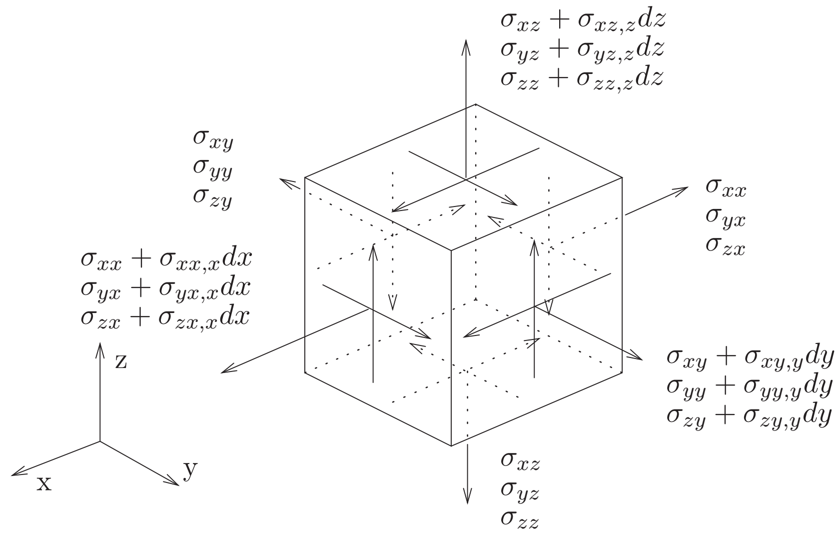
planar

$$
\begin{aligned}
&\sigma_{rr,r} + \frac{1}{r}(\sigma_{rr} - \sigma_{tt}) + \rho q_r = 0 \\
&\sigma_{tr,r} + \frac{2}{r}(\sigma_{tr}) + \rho q_t = 0 \qquad \text{(if } u_t \neq 0) \\
&\sigma_{rt} = \sigma_{tr} \qquad \text{(if } u_t \neq 0)
\end{aligned}
$$

## 3.6 Examples

**Equilibrium of forces : Cartesian**

The equilibrium equations in the three coordinate directions can be derived by considering the force equilibrium of the Cartesian stress cube.



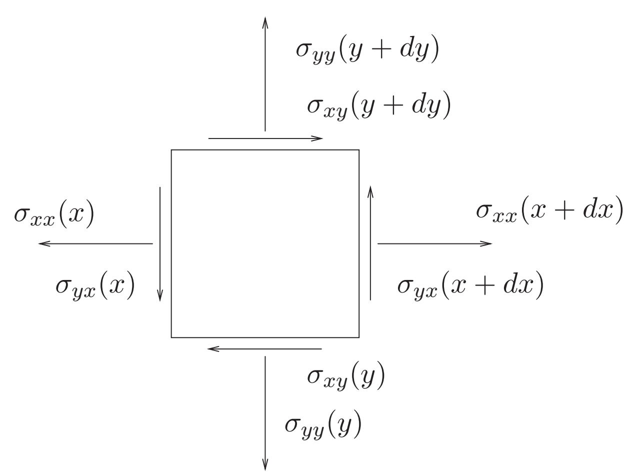

$$(\sigma_{xx} + \sigma_{xx,x}dx)dydz + (\sigma_{xy} + \sigma_{xy,y}dy)dxdz + (\sigma_{xz} + \sigma_{xz,z}dz)dxdy -$$
$$(\sigma_{xx})dydz - (\sigma_{xy})dxdz - (\sigma_{xz})dxdy + \rho q_x dxdydz = 0$$

$$\sigma_{xx,x} + \sigma_{xy,y} + \sigma_{xz,z} + \rho q_x = 0$$

---

**Equilibrium of moments : Cartesian**

The forces, working on the Cartesian stress cube, have a moment w.r.t. a certain point in space. The sum of all the moments must be zero. We consider the moments of forces in the $xy$-plane w.r.t. the $z$-axis through the center of the cube. Anti-clockwise moments are positive.



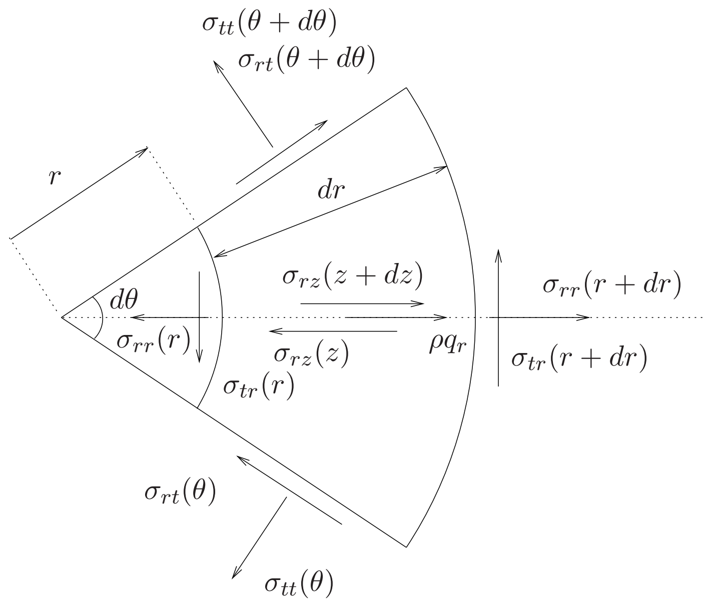
$$\sigma_{yx} dydz\, \tfrac{1}{2}dx + \sigma_{yx} dydz\, \tfrac{1}{2}dx + \sigma_{yx,x} dxdydz \tfrac{1}{2}dx$$
$$- \sigma_{xy} dxdz\, \tfrac{1}{2}dy - \sigma_{xy} dxdz\, \tfrac{1}{2}dy - \sigma_{xy,x} dxdydz\, \tfrac{1}{2}dy = 0$$

$$\sigma_{yx} - \sigma_{xy} = 0 \qquad \rightarrow \quad \sigma_{yx} = \sigma_{xy}$$

---

**Equilibrium of forces : cylindrical**

The equilibrium equations in the three coordinate directions can be derived by considering the force equilibrium of the cylindrical stress 'cube'. Here only the equilibrium in $r$-direction is considered. The stress components are a function of the three cylindrical coordinates $r$, $\theta$ and $z$, but only the relevant (changing) ones are indicated.




$$-\sigma_{rr}(r) r d\theta dz - \sigma_{rz}(z) r dr d\theta - \sigma_{rt}(\theta) dr dz - \sigma_{tt}(\theta) dr \tfrac{1}{2} d\theta dz$$
$$+ \sigma_{rr}(r + dr)(r + dr) d\theta dz + \sigma_{rz}(z + dz) r dr d\theta$$
$$+ \sigma_{rt}(\theta + d\theta) dr dz - \sigma_{tt}(\theta + d\theta) dr \tfrac{1}{2} d\theta dz + \rho q_r r dr d\theta dz = 0$$

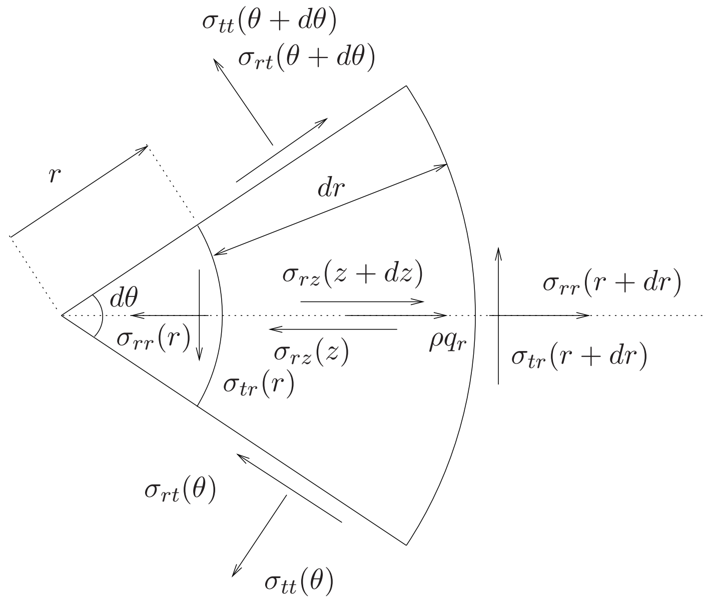
$$\sigma_{rr,r} r dr d\theta dz + \sigma_{rr} dr d\theta dz + \sigma_{rz,z} r dr d\theta dz + \sigma_{rt,t} dr d\theta dz$$
$$- \sigma_{tt}(\theta) dr d\theta dz + \rho q_r dr d\theta dz = 0$$

$$\sigma_{rr,r} + \frac{1}{r}\,\sigma_{rr} + \sigma_{rz,z} + \frac{1}{r}\,\sigma_{rt,t} - \frac{1}{r}\,\sigma_{tt} + \rho q_r = 0$$

---

**Equilibrium of moments : cylindrical**

The forces, working on the cylindrical stress cube, have a moment w.r.t. a certain point in space. The sum of all the moments mus be zero. We consider the moments of forces in the $r\theta$-plane w.r.t. the $z$-axis through the center of the cube.



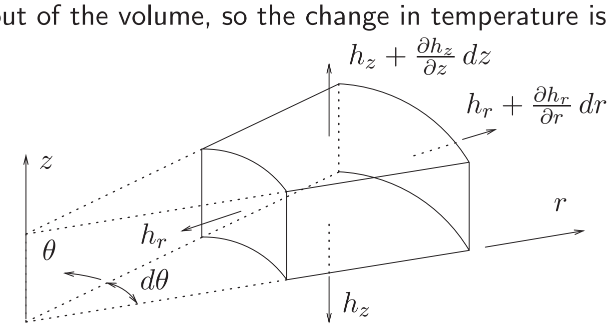

$$\sigma_{tr}(r) r d\theta dz \, \tfrac{1}{2} dr + \sigma_{tr}(r + dr)(r + dr) d\theta dz \, \tfrac{1}{2} dr$$
$$- \sigma_{rt}(\theta) dr dz \, \tfrac{1}{2} r d\theta - \sigma_{rt}(\theta + d\theta) dr dz \, \tfrac{1}{2} r d\theta = 0$$

$$\sigma_{tr} r dr d\theta dz - \sigma_{rt} r dr d\theta dz = 0 \qquad \rightarrow \qquad \sigma_{tr} = \sigma_{rt}$$

---

**Equilibrium of heat flow : cylindrical**

Heat is flowing out of the volume, so the change in temperature is negative.



$$-\rho C_p r d\theta dr dz \frac{\partial T}{\partial t} = \left( h_r + \frac{\partial h_r}{\partial r} dr \right) (r + dr) d\theta dz - h_r r d\theta dz + \left( h_z + \frac{\partial h_z}{\partial z} \right) r d\theta dr - h_z r d\theta dz$$

linearization

$$-\rho C_p r d\theta dr dz \frac{\partial T}{\partial t} = h_r dr d\theta dz + \frac{\partial h_r}{\partial r} r d\theta dr dz + \frac{\partial h_z}{\partial z} r d\theta dr dz \quad \rightarrow$$

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial h_r}{\partial r} - \frac{1}{r} h_r - \frac{\partial h_z}{\partial z}$$

$$h_r = -k \frac{\partial T}{\partial r} \quad ; \quad h_z = -k \frac{\partial T}{\partial z} \qquad \rightarrow \quad \rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial r^2} + k \frac{1}{r} \frac{\partial T}{\partial r} + k \frac{\partial^2 T}{\partial z^2}$$

# 4 Constitutive equations

Stresses must always satisfy the balance laws, which are considered to be laws of physics in the non-quantum world, where we live our lives together with our materials and structures. Balance laws must apply to each material, of which the deformation is studied. It is obvious, however, that various materials will behave very differently, when subjected to the same external loads. This behavior must be incorporated in the continuum mechanics theory and is therefore modelled mathematically. The resulting equations are referred to as *constitutive equations*. They can not fully be derived from physical principles, although the theory of *thermodynamics* tells us a lot of how they must look like. The real mathematical formulation of the material laws is however based on experimental observations of the deformation of the material.

In later sections, the behavior of a wide range of materials is modelled and used in a three-dimensional context. In this chapter, the more general aspects of constitutive equations are discussed.

## 4.1 Equations and unknowns

Although it is obvious that material laws must be incorporated to describe the behavior of different materials, they are also needed from a purely mathematical point of view. This has to do with the number of unknown variables and the number of equations, from which they must be solved. Obviously, the number of equations has to be the same as the number of unknowns.

The local balance laws for mass, momentum and moment of momentum have to be satisfied in every material point of the continuum body at every time during the deformation process.

The mass balance law is a scalar equation. The balance of momentum or equation of motion is a partial differential equation. It is a vector equation. The balance of moment of momentum is a tensor equation.

The unknown variables, which appear in the balance laws, are the density $\rho$ of the material, the position vector $\vec{x}$ of the material point and the stress tensor $\boldsymbol{\sigma}$. The continuity equation can be used to express the density in the deformation tensor $\boldsymbol{F}$, which is known, when the position $\vec{x}$ of the material point is known. So we can skip the mass balance from our equation set and the density from the set of unknowns.

The moment of momentum equation can be used directly to state that there are only 6 unknown stress components instead of 9. So we loose three equations and three unknowns. The number of unknowns is now 9 and the number of equations is 3, so 6 constitutive equations are required. These equations are relations between the stress components and the components of the position vector.

| | |
|---|---|
| mass | $\rho J = \rho_0$ |
| momentum | $\vec{\nabla} \cdot \boldsymbol{\sigma}^c + \rho \vec{q} = \rho \dot{\vec{v}}$ |
| moment of momentum | $\boldsymbol{\sigma}^c = \boldsymbol{\sigma}$ |

| | |
|---|---|
| density | $\rho$ |
| position vector | $\vec{x}$ |
| Cauchy stress tensor | $\boldsymbol{\sigma}$ |

The number of unknowns is now 9 and the number of equations is 3, so 6 constitutive equations are required. These equations are relations between the stress components and the components of the position vector.

$$\boldsymbol{\sigma} = \boldsymbol{N}(\vec{x})$$

## 4.2 General constitutive equation

The most general constitutive equation states that the stress tensor in point $\vec{x}$ at (the current) time $t$, is a function of the position of all material points at every previous time in the deformation process. This implies that the complete deformation history of all points is needed to calculate the current stress in each material point.

This constitutive equation is far to general to be useful. In the following it will be specified by incorporating assumptions about the material behavior. In practice these assumptions must of course be based on experimental observations.

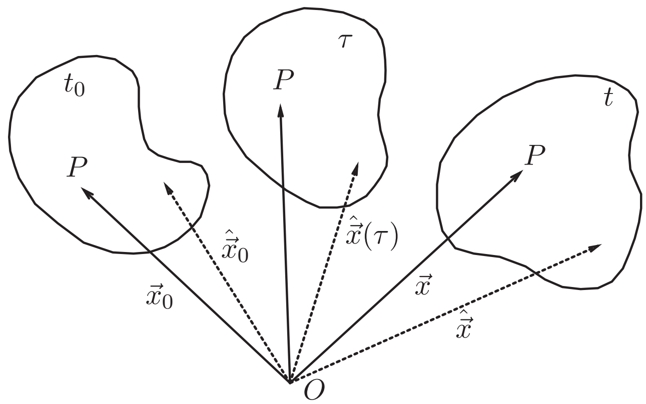



Fig. 4.43 : *Deformation history of a continuum*

$$\boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}\{\hat{\vec{x}}, \tau \mid \forall\ \hat{\vec{x}} \in V;\ \forall\ \tau \leq t\}$$

### 4.2.1 Locality

A wide range of materials and deformation processes allow the assumption of locality. In that case the stress in a point $\vec{x}$ is determined by the position of points in its direct neighborhood, so points with position vector $\vec{x} + d\vec{x}$. This can be written in terms of the deformation tensor $\boldsymbol{F}$.

$$\left.\begin{array}{l} \boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}\{\hat{\vec{x}}, \tau \mid \forall\ \hat{\vec{x}} \in V\ ;\ \forall\ \tau \leq t\} \\ \hat{\vec{x}} = \vec{x} + d\vec{x} = \vec{x} + \boldsymbol{F}(\vec{x}) \cdot d\vec{x}_0 \end{array}\right\} \quad \rightarrow \quad \boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}(\vec{x}, \boldsymbol{F}(\vec{x},\tau), \tau \mid \forall\ \tau \leq t)$$

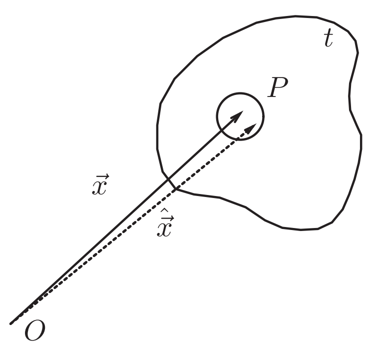



Fig. 4.44 : *Deformation with local influence*

### 4.2.2 Frame indifference

The stress state in a material point will not change when the material body is translated and/or rotated without (extra) deformation, i.e. when it moves as a rigid body. The variables in the constitutive equation may, however, change. The constitutive equation must be formulated such that these changes do not affect the stress state in a material point.

A rigid body translation is described with a displacement vector, which is equal for all material points. The stress is not allowed to change, so it is easily seen that the constitutive function $\boldsymbol{N}$ cannot depend on the position $\vec{x}$.

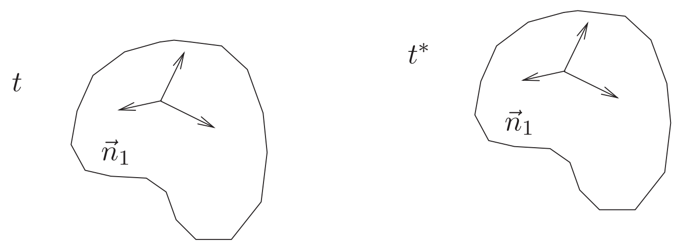


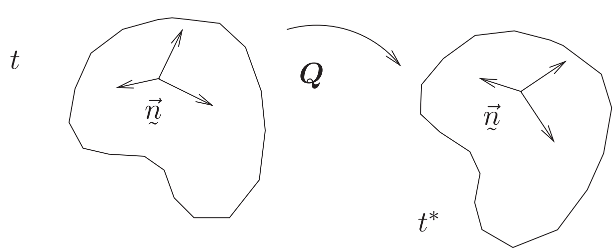

Fig. 4.45 : *Rigid body translation of a continuum*

$$\boldsymbol{\sigma}(\vec{x},t) = \boldsymbol{N}(\boldsymbol{F}(\vec{x},\tau),\tau \mid \forall\ \tau \leq t)$$

The symmetric Cauchy stress tensor can be written in spectral form. When the deformed body is subjected to a rigid rotation, described by the rotation tensor $\boldsymbol{Q}$, the principal stresses do not change, but the principal directions do. This means that the Cauchy stress tensor changes due to rigid rotation of the material.

The deformation tensor $\boldsymbol{F}$ will also change as a consequence of rigid rotation, which can be easily seen from the polar decomposition.

The relation between $\boldsymbol{\sigma}^*$ and $\boldsymbol{F}^*$ must be the same as that between $\boldsymbol{\sigma}$ and $\boldsymbol{F}$, which results in a requirement for the constitutive equation. (We skip the $\vec{x}$-dependency of $\boldsymbol{F}$.)




$$\vec{n}_1^* = \boldsymbol{Q}\cdot\vec{n}_1$$
$$\vec{n}_2^* = \boldsymbol{Q}\cdot\vec{n}_2$$
$$\vec{n}_3^* = \boldsymbol{Q}\cdot\vec{n}_3$$

Fig. 4.46 : *Rigid body rotation of a continuum*

$$\begin{aligned}
\boldsymbol{\sigma} &= \sigma_1\vec{n}_1\vec{n}_1 + \sigma_2\vec{n}_2\vec{n}_2 + \sigma_3\vec{n}_3\vec{n}_3 \\
\boldsymbol{\sigma}^* &= \sigma_1\vec{n}_1^*\vec{n}_1^* + \sigma_2\vec{n}_2^*\vec{n}_2^* + \sigma_3\vec{n}_3^*\vec{n}_3^* \\
&= \sigma_1\boldsymbol{Q}\cdot\vec{n}_1\vec{n}_1\cdot\boldsymbol{Q}^c + \sigma_2\boldsymbol{Q}\cdot\vec{n}_2\vec{n}_2\cdot\boldsymbol{Q}^c + \sigma_3\boldsymbol{Q}\cdot\vec{n}_3\vec{n}_3\cdot\boldsymbol{Q}^c
\end{aligned}$$

$$= \boldsymbol{Q} \cdot [\sigma_1 \vec{n}_1 \vec{n}_1 + \sigma_2 \vec{n}_2 \vec{n}_2 + \sigma_3 \vec{n}_3 \vec{n}_3] \cdot \boldsymbol{Q}^c = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c$$

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} \quad \rightarrow \quad \boldsymbol{F}^* = \boldsymbol{R}^* \cdot \boldsymbol{U} = \boldsymbol{Q} \cdot \boldsymbol{R} \cdot \boldsymbol{U} \quad \rightarrow \quad \boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F}$$

objectivity requirement

$$\boldsymbol{Q}(t) \cdot \boldsymbol{N} \left( \boldsymbol{F}(\tau) \mid \forall \ \tau \leq t \right) \cdot \boldsymbol{Q}^c(t) = \boldsymbol{N} \left( \boldsymbol{Q} \cdot \boldsymbol{F}(\tau) \mid \forall \ \tau \leq t \right) \qquad \forall \quad \boldsymbol{Q}$$

$$\sigma = C\boldsymbol{E} = C\tfrac{1}{2} \left( \boldsymbol{C} - \boldsymbol{I} \right) = C\tfrac{1}{2} \left( \boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I} \right)$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F}$$
$$\boldsymbol{E}^* = \tfrac{1}{2} \left( \boldsymbol{F}^T \cdot \boldsymbol{Q}^T \cdot \boldsymbol{Q} \cdot \boldsymbol{F} - \boldsymbol{I} \right) = \tfrac{1}{2} \left( \boldsymbol{F}^T \cdot \boldsymbol{F} - \boldsymbol{I} \right) = \boldsymbol{E}$$

$$\boldsymbol{\sigma}^* = C\boldsymbol{E}$$

NOT OBJECTIVE

$$\sigma = C\boldsymbol{A} = C\tfrac{1}{2} \left( \boldsymbol{B} - \boldsymbol{I} \right) = C\tfrac{1}{2} \left( \boldsymbol{F} \cdot \boldsymbol{F}^T - \boldsymbol{I} \right)$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F}$$
$$\boldsymbol{A}^* = \tfrac{1}{2} \left( \boldsymbol{Q} \cdot \boldsymbol{F} \cdot \boldsymbol{F}^T \cdot \boldsymbol{Q}^T - \boldsymbol{I} \right) = \tfrac{1}{2} \boldsymbol{Q} \cdot \left( \boldsymbol{F} \cdot \boldsymbol{F}^T - \boldsymbol{I} \right) \cdot \boldsymbol{Q}^T = \boldsymbol{Q} \cdot \boldsymbol{A} \cdot \boldsymbol{Q}^c$$

$$\boldsymbol{\sigma}^* = C\boldsymbol{A}^*$$

OBJECTIVE

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta \boldsymbol{D}$$
$$\boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^c) \qquad \text{with} \qquad \boldsymbol{L} = \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1}$$

$$\boldsymbol{\sigma}^* = \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^T$$

$$\boldsymbol{F}^* = \boldsymbol{Q} \cdot \boldsymbol{F} \quad ; \quad \boldsymbol{F}^{*^{-1}} = \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c \quad ; \quad \dot{\boldsymbol{F}}^* = \dot{\boldsymbol{Q}} \cdot \boldsymbol{F} + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}}$$
$$\boldsymbol{L}^* = (\dot{\boldsymbol{Q}} \cdot \boldsymbol{F} + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}}) \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c = \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c$$
$$\boldsymbol{D}^* = \tfrac{1}{2} \left[ \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^c + \boldsymbol{Q} \cdot (\dot{\boldsymbol{F}} \cdot \boldsymbol{F}^{-1})^c \cdot \boldsymbol{Q}^c \right]$$
$$\boldsymbol{Q} \cdot \boldsymbol{Q}^c = \boldsymbol{I} \quad \rightarrow \quad \dot{\boldsymbol{Q}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{Q}}^c$$

$$= \boldsymbol{Q} \cdot \boldsymbol{D} \cdot \boldsymbol{Q}^c$$

$$\boldsymbol{\sigma}^* = -\,p\boldsymbol{I} + 2\eta \boldsymbol{D}^*$$

OBJECTIVE

## 4.3 Invariant stress tensor

For convenient constitutive modeling where stress (rate) is related to deformation (rate), we need stress tensors which are invariant with rigid rotation. Also their time derivative must answer this requirement.

A stress tensor $\boldsymbol{S} = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c$ can be defined, where $\boldsymbol{A}$ is to be specified later, but always has to obey $\boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c$. It follows that the stress tensor $\boldsymbol{S}$ is invariant for rigid rotations.

$$\boldsymbol{S} = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c$$

$$\left.\begin{array}{l} \boldsymbol{S}^* = \boldsymbol{A}^* \cdot \boldsymbol{\sigma}^* \cdot \boldsymbol{A}^{*c} = \boldsymbol{A}^* \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{A}^{*c} \\ \text{define} \qquad \boldsymbol{A}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c \end{array}\right\} \quad \rightarrow$$

$$\boldsymbol{S}^* = \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c = \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c = \boldsymbol{S}$$

$$\boldsymbol{S} = \text{invariant for rigid rotation}$$

Also its time derivative $\dot{\boldsymbol{S}}$ is inveriant.

$$\begin{aligned}
\dot{\boldsymbol{S}} &= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{A}}^c \\
\dot{\boldsymbol{S}}^* &= \dot{\boldsymbol{A}}^* \cdot \boldsymbol{\sigma}^* \cdot \boldsymbol{A}^{*c} + \boldsymbol{A}^* \cdot \dot{\boldsymbol{\sigma}}^* \cdot \boldsymbol{A}^{*c} + \boldsymbol{A}^* \cdot \boldsymbol{\sigma}^* \cdot \dot{\boldsymbol{A}}^{*c} \\
&= (\dot{\boldsymbol{A}} \cdot \boldsymbol{Q}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c) \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot (\dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{Q}^c + \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c) \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot (\boldsymbol{Q} \cdot \dot{\boldsymbol{A}}^c + \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c) \\
&= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{A}}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c \\
&= \dot{\boldsymbol{A}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{A}^c + \\
&\qquad \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \dot{\boldsymbol{Q}}^c \cdot \boldsymbol{Q} \cdot \boldsymbol{A}^c + \boldsymbol{A} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^c \cdot \dot{\boldsymbol{Q}} \cdot \boldsymbol{A}^c \\
&= \dot{\boldsymbol{S}} \qquad \rightarrow \quad \dot{\boldsymbol{S}} = \text{invariant for rigid rotation}
\end{aligned}$$

The time derivative of $\boldsymbol{S}$ can also be expressed in the Cauchy stress tensor and its rate. As a short notation the Cauchy stress rate $\overset{\odot}{\boldsymbol{\sigma}}$ is introduced, which is a function of $\dot{\boldsymbol{\sigma}}$, $\boldsymbol{A}$ and $\dot{\boldsymbol{A}}$. This tensor has the same transformation upon rigid body rotation than the Cauchy stress tensor $\boldsymbol{\sigma}$.

$$\begin{aligned}
\boldsymbol{S} &= \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c \\
\dot{\boldsymbol{S}} &= \dot{\boldsymbol{A}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{A}}^c \\
&= \boldsymbol{A}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c + \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{A}^c \\
&= \boldsymbol{A}\cdot\left\{(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c + \dot{\boldsymbol{\sigma}}\right\}\cdot\boldsymbol{A}^c = \boldsymbol{A}\cdot\overset{\odot}{\boldsymbol{\sigma}}\cdot\boldsymbol{A}^c
\end{aligned}$$

$$\overset{\odot}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c$$

$$\begin{aligned}
\overset{\odot}{\boldsymbol{\sigma}}{}^* &= \dot{\boldsymbol{\sigma}}^* + (\boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^*)\cdot\boldsymbol{\sigma}^* + \boldsymbol{\sigma}^*\cdot(\boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^*)^c \\
&\qquad \boldsymbol{A}^* = \boldsymbol{A}\cdot\boldsymbol{Q}^c \quad\rightarrow\quad \boldsymbol{A}^{*-1} = \boldsymbol{A}^{-1*} = \boldsymbol{Q}\cdot\boldsymbol{A}^{-1} \\
&\qquad \dot{\boldsymbol{A}}^* = \dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c + \boldsymbol{A}\cdot\dot{\boldsymbol{Q}}^c \\
&\qquad \boldsymbol{A}^{-1*}\cdot\dot{\boldsymbol{A}}^* = \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c \\
&= \dot{\boldsymbol{\sigma}}^* + \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{Q}^c\cdot\boldsymbol{\sigma}^* + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c\cdot\boldsymbol{\sigma}^* + \\
&\qquad \boldsymbol{\sigma}^*\cdot\boldsymbol{Q}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{Q}^c + \boldsymbol{\sigma}^*\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^c \\
&= \dot{\boldsymbol{Q}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\dot{\boldsymbol{Q}}^c + \\
&\qquad \boldsymbol{Q}\cdot\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\dot{\boldsymbol{Q}}^c\cdot\boldsymbol{Q}^c\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c + \\
&\qquad \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c\cdot\boldsymbol{Q}^c + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^c\cdot\dot{\boldsymbol{Q}}\cdot\boldsymbol{Q}^c \\
&= \boldsymbol{Q}\cdot[\dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c]\cdot\boldsymbol{Q}^c \\
&= \boldsymbol{Q}\cdot\overset{\odot}{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^c \qquad\rightarrow\quad \overset{\odot}{\boldsymbol{\sigma}} = \text{objective}
\end{aligned}$$

## 4.4 Objective rates and associated tensors

The tensor $\boldsymbol{A}$ is now specified, which results in some alternative invariant stress tensors. With each tensor a so-called objective rate of the Cauchy stress tensor is associated. choosing $\boldsymbol{A} \in \{\boldsymbol{F}^{-1}, \boldsymbol{Q}^{-1}, \boldsymbol{F}^c, \boldsymbol{R}^c\}$ results in the Truesdell, Jaumann, Cotter-Rivlin and Dienes tensor and rate.

| | |
|---|---|
| general tensor | $\boldsymbol{S} = \boldsymbol{\sigma}_O = \boldsymbol{A}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{A}^c$ |
| | $\dot{\boldsymbol{S}} = \dot{\boldsymbol{\sigma}}_O = \boldsymbol{A}\cdot\overset{\odot}{\boldsymbol{\sigma}}_O\cdot\boldsymbol{A}^c$ |
| general rate | $\overset{\odot}{\boldsymbol{\sigma}}_O = \dot{\boldsymbol{\sigma}} + (\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot(\boldsymbol{A}^{-1}\cdot\dot{\boldsymbol{A}})^c$ |
| Truesdell tensor | $\boldsymbol{\sigma}_T = \boldsymbol{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{-c}$ |
| | $\dot{\boldsymbol{\sigma}}_T = \boldsymbol{F}^{-1}\cdot\overset{\odot}{\boldsymbol{\sigma}}_T\cdot\boldsymbol{F}^{-c}$ |
| Truesdell rate | $\overset{\odot}{\boldsymbol{\sigma}}_T = \overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{L}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot\boldsymbol{L}^c$ |

| | |
|---|---|
| Jaumann tensor | $\boldsymbol{\sigma}_J = \boldsymbol{Q}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{Q}^{-c} \quad \text{with} \quad \dot{\boldsymbol{Q}} = \boldsymbol{\Omega} \cdot \boldsymbol{Q}$ |
| | $\dot{\boldsymbol{\sigma}}_J = \boldsymbol{Q}^{-1} \cdot \overset{\odot}{\boldsymbol{\sigma}}_J \cdot \boldsymbol{Q}^{-c}$ |
| Jaumann rate | $\overset{\odot}{\boldsymbol{\sigma}}_J = \overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}^c$ |
| | |
| Cotter-Rivlin tensor | $\boldsymbol{\sigma}_C = \boldsymbol{F}^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}$ |
| | $\dot{\boldsymbol{\sigma}}_C = \boldsymbol{F}^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_C \cdot \boldsymbol{F}$ |
| Cotter-Rivlin rate | $\overset{\odot}{\boldsymbol{\sigma}}_C = \overset{\triangle}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{L}^c \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{L}$ |
| | |
| Dienes tensor | $\boldsymbol{\sigma}_D = \boldsymbol{R}^c \cdot \boldsymbol{\sigma} \cdot \boldsymbol{R} \quad \text{with} \quad \boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U}$ |
| | $\dot{\boldsymbol{\sigma}}_D = \boldsymbol{R}^c \cdot \overset{\odot}{\boldsymbol{\sigma}}_D \cdot \boldsymbol{R}$ |
| Dienes rate | $\overset{\odot}{\boldsymbol{\sigma}}_D = \overset{\diamond}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^c) \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot (\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^c)^c$ |

# 5 Linear elastic material

For linear elastic material behavior the stress tensor $\boldsymbol{\sigma}$ is related to the linear strain tensor $\boldsymbol{\varepsilon}$ by the constant fourth-order stiffness tensor ${}^4\boldsymbol{C}$ :

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}$$

The relevant components of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ w.r.t. an orthonormal vector basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are stored in columns $\underset{\approx}{\sigma}$ and $\underset{\approx}{\varepsilon}$. Note that we use double "waves" to indicate that the columns contain components of a second-order tensor.

$$\underset{\approx}{\sigma}^T = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{21}\ \sigma_{23}\ \sigma_{32}\ \sigma_{31}\ \sigma_{13}]$$
$$\underset{\approx}{\varepsilon}^T = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{21}\ \varepsilon_{23}\ \varepsilon_{32}\ \varepsilon_{31}\ \varepsilon_{13}]$$

The relation between these columns is given by the $9 \times 9$ matrix $\underline{\underline{C}}$, which stores the components of ${}^4\boldsymbol{C}$ and is referred to as the material stiffness matrix. Note again the use of double underscore to indicate that the matrix contains components of a fourth-order tensor.

| | | | |
|---|---|---|---|
| tensor notation | $\boldsymbol{\sigma} = {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}$ | | |
| index notation | $\sigma_{ij} = C_{ijkl}\varepsilon_{lk}$ | ; | $i, j, k, l \in \{1, 2, 3\}$ |
| matrix notation | $\underset{\approx}{\sigma} = \underline{\underline{C}}\,\underset{\approx}{\varepsilon}$ | | |

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

The stored energy per unit of volume is :

$$W = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} = \left[\tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}\right]^c = \tfrac{1}{2}\,\boldsymbol{\varepsilon} : {}^4\boldsymbol{C}^c : \boldsymbol{\varepsilon}$$

which implies that ${}^4\boldsymbol{C}$ is total-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^c$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^T$.

As the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^c$, the tensor ${}^4\boldsymbol{C}$ must be left-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{lc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{LT}$. As also the strain tensor is symmetric, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^c$, the constitutive relation can be written with a $6 \times 6$ stiffness matrix.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

specific energy $\quad W = \frac{1}{2}\underset{\approx}{\varepsilon}^T \underline{\underline{C}}\, \underset{\approx}{\varepsilon} \quad \rightarrow \quad$ symmetry $\quad \boxed{\underline{\underline{C}} = \underline{\underline{C}}^T}$

**Symmetric stresses**

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{21} \\ \sigma_{23} \\ \sigma_{32} \\ \sigma_{31} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2111} & C_{2122} & C_{2133} & C_{2121} & C_{2112} & C_{2132} & C_{2123} & C_{2113} & C_{2131} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3211} & C_{3222} & C_{3233} & C_{3221} & C_{3212} & C_{3232} & C_{3223} & C_{3213} & C_{3231} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \\ C_{1311} & C_{1322} & C_{1333} & C_{1321} & C_{1312} & C_{1332} & C_{1323} & C_{1313} & C_{1331} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\sigma_{ij} = \sigma_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

**Symmetric strains**

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1112} & C_{1132} & C_{1123} & C_{1113} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2212} & C_{2232} & C_{2223} & C_{2213} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3312} & C_{3332} & C_{3323} & C_{3313} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1212} & C_{1232} & C_{1223} & C_{1213} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2312} & C_{2332} & C_{2323} & C_{2313} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3112} & C_{3132} & C_{3123} & C_{3113} & C_{3131} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{23} \\ \varepsilon_{32} \\ \varepsilon_{31} \\ \varepsilon_{13} \end{bmatrix}$$

$$\boxed{\varepsilon_{ij} = \varepsilon_{ji}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

**Symmetric material parameters**

The components of $\underline{\underline{C}}$ must be determined experimentally, by prescribing strains and measuring stresses and vice versa. It is clear that only the summation of the components in the 4th, 5th and 6th columns can be determined and for that reason, it is assumed that the stiffness tensor is right-symmetric : ${}^4\boldsymbol{C} = {}^4\boldsymbol{C}^{rc}$ or equivalently $\underline{\underline{C}} = \underline{\underline{C}}^{RT}$.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & [C_{1121} + C_{1112}] & [C_{1132} + C_{1123}] & [C_{1113} + C_{1131}] \\ C_{2211} & C_{2222} & C_{2233} & [C_{2221} + C_{2212}] & [C_{2232} + C_{2223}] & [C_{2213} + C_{2231}] \\ C_{3311} & C_{3322} & C_{3333} & [C_{3321} + C_{3312}] & [C_{3332} + C_{3323}] & [C_{3313} + C_{3331}] \\ C_{1211} & C_{1222} & C_{1233} & [C_{1221} + C_{1212}] & [C_{1232} + C_{1223}] & [C_{1213} + C_{1231}] \\ C_{2311} & C_{2322} & C_{2333} & [C_{2321} + C_{2312}] & [C_{2332} + C_{2323}] & [C_{2313} + C_{2331}] \\ C_{3111} & C_{3122} & C_{3133} & [C_{3121} + C_{3112}] & [C_{3132} + C_{3123}] & [C_{3113} + C_{3131}] \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{C_{ijkl} = C_{ijlk}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

**Shear strain**

To restore the symmetry of the stiffness matrix, the factor 2 in the last three columns is swapped to the column with the strain components. The shear components are replaced by the shear strains : $2\varepsilon_{ij} = \gamma_{ij}$. This leads to a symmetric stiffness matrix $\underline{\underline{C}}$ with 21 independent components.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 2C_{1121} & 2C_{1132} & 2C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & 2C_{2221} & 2C_{2232} & 2C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & 2C_{3321} & 2C_{3332} & 2C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & 2C_{1221} & 2C_{1232} & 2C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & 2C_{2321} & 2C_{2332} & 2C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & 2C_{3121} & 2C_{3132} & 2C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\boxed{2\varepsilon_{ij} = \gamma_{ij}}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

## 5.1 Material symmetry

Almost all materials have some material symmetry, originating from the micro structure, which implies that the number of independent material parameters is reduced. The following names refer to increasing material symmetry and thus to decreasing number of material parameters :

To reduce the number of elasticity parameters, we assume a coordinate system attached to the symmetry axes or planes in the material.

monoclinic → orthotropic → quadratic → transversal isotropic → cubic → isotropic

### 5.1.1 Triclinic

In a triclinic material there is no symmetry. Therefore there are 21 material parameters to be determined from independent experimental test setups. This is practically not feasible.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & C_{1132} & C_{1113} \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & C_{2232} & C_{2213} \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & C_{3332} & C_{3313} \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & C_{1232} & C_{1213} \\ C_{2311} & C_{2322} & C_{2333} & C_{2321} & C_{2332} & C_{2313} \\ C_{3111} & C_{3122} & C_{3133} & C_{3121} & C_{3132} & C_{3113} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

21 material parameters

### 5.1.2 Monoclinic

In each material point of a monoclinic material there is one symmetry plane, which we take here to be the 12-plane. Strain components w.r.t. two vector bases $\underset{\sim}{\vec{e}} = [\vec{e}_1 \ \vec{e}_2 \ \vec{e}_3]^T$ and $\underset{\sim}{\vec{e}}^* = [\vec{e}_1 \ \vec{e}_2 \ -\vec{e}_3]^T$ must result in the same stresses. It can be proved that all components of the stiffness matrix, with an odd total of the index 3, must be zero. This implies :

$$C_{2311} = C_{2322} = C_{2333} = C_{2321} = C_{3111} = C_{3122} = C_{3133} = C_{3121} = 0$$

A monoclinic material is characterized by 13 material parameters. In the figure the directions with equal properties are indicated with an equal number of lines.

Monoclinic symmetry is found in e.g. gypsum ($CaSO_4 2H_2O$).

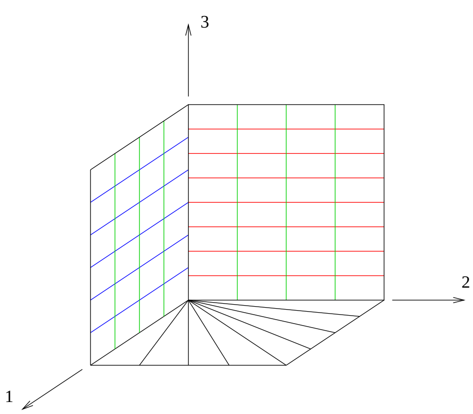



Fig. 5.47 : *One symmetry plane for monoclinic material symmetry*

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1121} & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & C_{2221} & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & C_{3321} & 0 & 0 \\ C_{1211} & C_{1222} & C_{1233} & C_{1221} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{2332} & C_{2313} \\ 0 & 0 & 0 & 0 & C_{3132} & C_{3113} \end{bmatrix}$$

13 material parameters

### 5.1.3 Orthotropic

In a point of an orthotropic material there are three symmetry planes which are perpendicular. We choose them here to coincide with the Cartesian coordinate planes. In addition to the implications for monoclinic symmetry, we can add the requirements

$$C_{1112} = C_{2212} = C_{3312} = C_{3123} = 0$$

An orthotropic material is characterized by 9 material parameters. In the stiffness matrix, they are now indicated as $A, B, C, Q, R, S, K, L$ and $M$.

Orthotropic symmetry is found in orthorhombic crystals (e.g. cementite, $Fe_3C$) and in composites with fibers in three perpendicular directions.

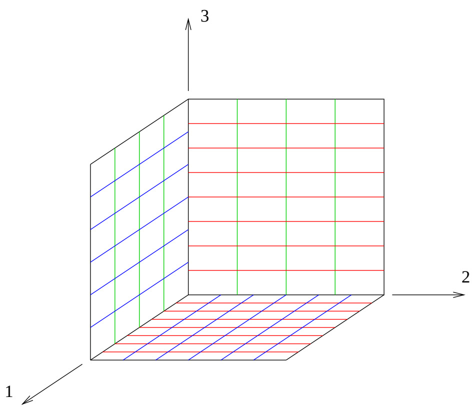



Fig. 5.48 : *Three symmetry planes for orthotropic material symmetry*

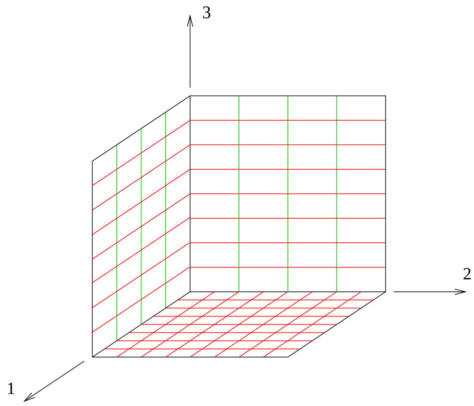
$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix}$$

| 9 | material parameters |
|---|---|

### 5.1.4 Quadratic

If in an orthotropic material the properties in two of the three symmetry planes are the same, the material is referred to as quadratic. Here we assume the behavior to be identical in the $\vec{e}_1$- and the $\vec{e}_2$-directions, however there is no isotropy in the 12-plane. This implies : $A = B$, $S = R$ and $M = L$. Only 6 material parameters are needed to describe the mechanical material behavior.

Quadratic symmetry is found in tetragonal crystals e.g. $TiO_2$ and white tin $Sn\beta$.




$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

| 6 | material parameters |
|---|---|

Fig. 5.49 : *Quadratic material*

### 5.1.5 Transversal isotropic

When the material behavior in the 12-plane is isotropic, an additional relation between parameters can be deduced. To do this, we consider a pure shear deformation in the 12-plane, where a shear stress $\tau$ leads to a shear $\gamma$. The principal stress and strain directions coincide due to the isotropic behavior in the plane. In the principal directions the relation between principal stresses and strains follow from the material stiffness matrix.

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \quad \rightarrow \ \det(\underline{\sigma} - \sigma \underline{I}) = 0 \ \rightarrow \ \begin{cases} \sigma_1 = \tau \\ \sigma_2 = -\tau \end{cases}$$

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 0 \end{bmatrix} \rightarrow \ \det(\underline{\varepsilon} - \varepsilon \underline{I}) = 0 \ \rightarrow \ \begin{cases} \varepsilon_1 = \frac{1}{2}\gamma \\ \varepsilon_2 = -\frac{1}{2}\gamma \end{cases}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix} = \begin{bmatrix} A & Q \\ Q & A \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \quad \rightarrow \quad \begin{array}{l} \sigma_1 = A\varepsilon_1 + Q\varepsilon_2 = \ \ \tau = \ \ K\gamma \\ \sigma_2 = Q\varepsilon_1 + A\varepsilon_2 = -\tau = -K\gamma \end{array} \quad \rightarrow$$

$$\left. \begin{array}{l} (A - Q)(\varepsilon_1 - \varepsilon_2) = 2K\gamma \\ \\ \varepsilon_1 = \frac{1}{2}\gamma \quad ; \quad \varepsilon_1 = -\frac{1}{2}\gamma \end{array} \right\} \quad \rightarrow \quad \boxed{K = \frac{1}{2}(A - Q)}$$

Examples of transversal isotropy are found in hexagonal crystals (CHP, Zn, Mg, Ti) and honeycomb composites. The material behavior of these materials can be described with 5 material parameters.

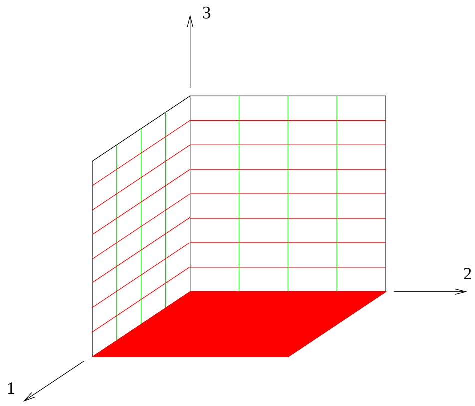



Fig. 5.50 : *Transversal material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & A & R & 0 & 0 & 0 \\ R & R & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$K = \frac{1}{2}(A - Q)$$

| 5 | material parameters |
|---|---|

### 5.1.6 Cubic

In the three perpendicular material directions the material properties are the same. In the symmetry planes there is no isotropic behavior. Only 3 material parameters remain.

Examples of cubic symmetry are found in BCC and FCC crystals (e.g. in Ag, Cu, Au, Fe, NaCl).

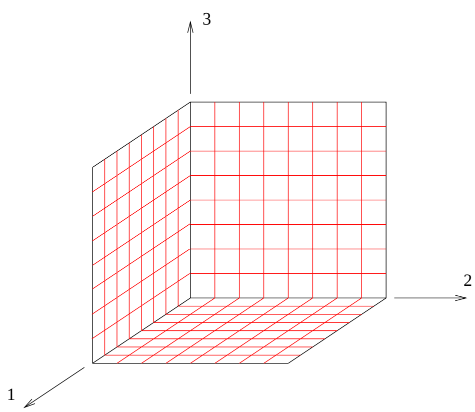



Fig. 5.51 : *Cubic material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L \neq \frac{1}{2}(A - Q)$$

| 3 | material parameters |
|---|---|

### 5.1.7 Isotropic

In all three directions the properties are the same and in each plane the properties are isotropic. Only 2 material parameters remain.

Isotropic material behavior is found for materials having a microstructure, which is sufficiently randomly oriented and distributed on a very small scale. This applies to metals with a randomly oriented polycrystalline structure, ceramics with a random granular structure and composites with random fiber/particle orientation.

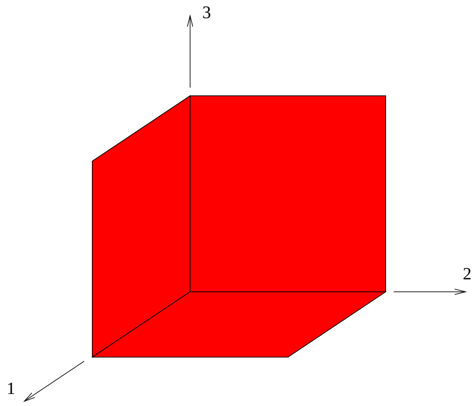



Fig. 5.52 : *Isotropic material*

$$\underline{\underline{C}} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix}$$

$$L = \tfrac{1}{2}(A - Q)$$

| 2 | material parameters |
|---|---|

**Engineering parameters**

In engineering practice the linear elastic material behavior is characterized by Young's moduli, shear moduli and Poisson ratios. They have to be measured in tensile and shear experiments. In this section these parameters are introduced for an isotropic material.

For orthotropic and transversal isotropic material, the stiffness and compliance matrices, expressed in engineering parameters, can be found in appendix A.

To express the material constants $A$, $Q$ and $L$ in the parameters $E$, $\nu$ and $G$, two simple tests are considered : a tensile test along the 1-axis and a shear test in the 13-plane.

In a tensile test the contraction strain $\varepsilon_d$ and the axial stress $\sigma$ are related to the axial strain $\varepsilon$. The expressions for $A$, $Q$ and $L$ result after some simple mathematics.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon & \varepsilon_d & \varepsilon_d & 0 & 0 & 0 \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} \sigma = A\varepsilon + 2Q\varepsilon_d \\ 0 = Q\varepsilon + (A+Q)\varepsilon_d \rightarrow \varepsilon_d = -\dfrac{Q}{A+Q}\,\varepsilon \end{array}\right\} \rightarrow \sigma = \dfrac{A^2 + AQ - 2Q^2}{A+Q}\,\varepsilon \\ \varepsilon_d = -\nu\varepsilon \qquad ; \qquad \sigma = E\varepsilon \end{array}\right\} \rightarrow$$

$$A = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \qquad Q = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad L = \frac{E}{2(1+\nu)}$$

When we analyze a shear test, the relation between the shear strain $\gamma$ and the shear stress $\tau$ is given by the shear modulus $G$. For isotropic material $G$ is a function of $E$ and $\nu$. For non-isotropic materials, the shear moduli are independent parameters.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \; ; \; \underset{\approx}{\sigma}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \tau \end{bmatrix}$$

$$\tau = L\gamma = \frac{E}{2(1+\nu)}\gamma = G\gamma$$

For an isotropic material, a hydrostatic stress will only result in volume change. The relation between the volume strain and the hydrostatic stress is given by the bulk modulus $K$, which is a function of $E$ and $\nu$.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & Q & 0 & 0 & 0 \\ Q & A & Q & 0 & 0 & 0 \\ Q & Q & A & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} \qquad \text{with} \quad L = \tfrac{1}{2}(A - Q)$$

$$\underset{\approx}{\varepsilon}^T = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} J - 1 \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} &= \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\ &= \frac{3(1-2\nu)}{E}\,\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \frac{1}{K}\,\tfrac{1}{3}\text{tr}(\underline{\sigma}) \end{aligned}$$

The compliance and stiffness matrices for isotropic material can now be fully written in terms of the Young's modulus ad the Poisson's ratio.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

Besides Young's modulus, shear modulus, bulk modulus and Poisson ratio in some formulations the so-called Lamé coefficients $\lambda$ and $\mu$ are used, where $\mu = G$ and $\lambda$ is a function of $E$ and $\nu$. The next tables list the relations between all these parameters.

| | $E,\nu$ | $\lambda,G$ | $K,G$ | $E,G$ | $E,K$ |
|---|---|---|---|---|---|
| $E$ | $E$ | $\frac{(2G+3\lambda)G}{\lambda+G}$ | $\frac{9KG}{3K+G}$ | $E$ | $E$ |
| $\nu$ | $\nu$ | $\frac{\lambda}{2(\lambda+G)}$ | $\frac{3K-2G}{2(3K+G)}$ | $\frac{E-2G}{2G}$ | $\frac{3K-E}{6K}$ |
| $G$ | $\frac{E}{2(1+\nu)}$ | $G$ | $G$ | $G$ | $\frac{3KE}{9K-E}$ |
| $K$ | $\frac{E}{3(1-2\nu)}$ | $\frac{3\lambda+2G}{3}$ | $K$ | $\frac{EG}{3(3G-E)}$ | $K$ |
| $\lambda$ | $\frac{E\nu}{(1+\nu)(1-2\nu)}$ | $\lambda$ | $\frac{3K-2G}{3}$ | $\frac{G(E-2G)}{3G-E}$ | $\frac{3K(3K-E)}{9K-E}$ |

| | $E,\lambda$ | $G,\nu$ | $\lambda,\nu$ | $\lambda K$ | $K,\nu$ |
|---|---|---|---|---|---|
| $E$ | $E$ | $2G(1+\nu)$ | $\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$ | $\frac{9K(K-\lambda)}{3K-\lambda}$ | $3K(1-2\nu)$ |
| $\nu$ | $\frac{-E-\lambda+\sqrt{(E+\lambda)^2+8\lambda^2}}{4\lambda}$ | $\nu$ | $\nu$ | $\frac{\lambda}{3K-\lambda}$ | $\nu$ |
| $G$ | $\frac{-3\lambda+E+\sqrt{(3\lambda-E)^2+8\lambda E}}{4}$ | $G$ | $\frac{\lambda(1-2\nu)}{2\nu}$ | $\frac{3(K-\lambda)}{2}$ | $\frac{3K(1-2\nu)}{2(1+\nu)}$ |
| $K$ | $\frac{E-3\lambda+\sqrt{(E-3\lambda)^2-12\lambda E}}{6}$ | $\frac{2G(1+\nu)}{3(1-2\nu)}$ | $\frac{\lambda(1+\nu)}{3\nu}$ | $K$ | $K$ |
| $\lambda$ | $\lambda$ | $\frac{2G\nu}{1-2\nu}$ | $\lambda$ | $\lambda$ | $\frac{3K\nu}{1+\nu}$ |

## 5.2 Isotropic material tensors

Isotropic linear elastic material behavior is characterized by only two independent material constants, for which we can choose Young's modulus $E$ and Poisson's ratio $\nu$. The isotropic material law can be written in tensorial form, where $\boldsymbol{\sigma}$ is related to $\boldsymbol{\varepsilon}$ with a fourth-order material stiffness tensor ${}^4\boldsymbol{C}$.

In column/matrix notation the strain components are related to the stress components by a $6 \times 6$ compliance matrix. Inversion leads to the $6 \times 6$ stiffness matrix, which relates strain components to stress components. It should be noted that shear strains are denoted as $\varepsilon_{ij}$ and not as $\gamma_{ij}$, as was done before.

The stiffness matrix is written as the sum of two matrices, which can then be written in tensorial form.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\text{with} \quad \alpha = \frac{E}{(1+\nu)(1-2\nu)}$$

The stiffness matrix is rewritten as the sum of two matrices, the second of which is a unit matrix. Also the first one can be reduced to a matrix with ones and zeros only.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \frac{E}{(1+\nu)} \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$= \frac{E}{(1+\nu)} \left[ \begin{bmatrix} \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & \frac{\nu}{(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu}{(1-2\nu)} & \frac{\nu}{(1-2\nu)} & \frac{(1-\nu)}{(1-2\nu)} - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \right.$$

$$
\left.\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

$$
= \left[ \frac{E\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{E}{(1+\nu)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}
$$

**Isotropic stiffness tensor**

The first matrix is the matrix representation of the fourth-order tensor $\boldsymbol{II}$. The second matrix is the representation of the symmetric fourth-order tensor ${}^4\boldsymbol{I}^s$. The resulting fourth-order material stiffness tensor ${}^4\boldsymbol{C}$ contains two material constants $c_0$ and $c_1$. It is observed that $c_0 = \lambda$ and $c_1 = 2\mu$, where $\lambda$ and $\mu$ are the Lamé coefficients introduced earlier.

$$
\begin{aligned}
\boldsymbol{\sigma} &= \left[\frac{E\nu}{(1+\nu)(1-2\nu)}\right] \boldsymbol{I}\,\mathrm{tr}(\boldsymbol{\varepsilon}) + \left[\frac{E}{(1+\nu)}\right] \boldsymbol{\varepsilon} \\
&= Q\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} + 2L\boldsymbol{\varepsilon} \\
&= c_0\,\mathrm{tr}(\boldsymbol{\varepsilon})\,\boldsymbol{I} + c_1\,\boldsymbol{\varepsilon} \\
&= \left[c_0\boldsymbol{II} + c_1\,{}^4\boldsymbol{I}^s\right] : \boldsymbol{\varepsilon} \qquad \text{with} \quad {}^4\boldsymbol{I}^s = \tfrac{1}{2}({}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc}) \\
&= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon}
\end{aligned}
$$

**Stiffness and compliance tensor**

The strain and stress tensors can both be written as the sum of an hydrostatic - $(.)^h$ - and a deviatoric - $(.)^d$ - part. Doing so, the stress-strain relation can be easily inverted.

$$
\begin{aligned}
\boldsymbol{\sigma} &= {}^4\boldsymbol{C} : \boldsymbol{\varepsilon} \\
&= \left[ c_0 \boldsymbol{I}\boldsymbol{I} + c_1\, {}^4\boldsymbol{I}^s \right] : \boldsymbol{\varepsilon} \\
&\qquad \text{with} \quad {}^4\boldsymbol{I}^s = \tfrac{1}{2}\left( {}^4\boldsymbol{I} + {}^4\boldsymbol{I}^{rc} \right) \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1 \boldsymbol{\varepsilon} \\
&= c_0 \mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1 \left\{ \boldsymbol{\varepsilon}^d + \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} \right\} \\
&= (c_0 + \tfrac{1}{3}c_1)\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + c_1 \boldsymbol{\varepsilon}^d \\
&= (3c_0 + c_1)\boldsymbol{\varepsilon}^h + c_1 \boldsymbol{\varepsilon}^d \\
&= \boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}^h + \boldsymbol{\varepsilon}^d \\
&= \frac{1}{3c_0 + c_1}\,\boldsymbol{\sigma}^h + \frac{1}{c_1}\,\boldsymbol{\sigma}^d \\
&= \frac{1}{3c_0 + c_1}\,\tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\left\{ \boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} \right\} \\
&= -\frac{c_0}{(3c_0 + c_1)c_1}\,\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} + \frac{1}{c_1}\,\boldsymbol{\sigma} \\
&= \left[ -\frac{c_0}{(3c_0 + c_1)c_1}\boldsymbol{I}\boldsymbol{I} + \frac{1}{c_1}\,{}^4\boldsymbol{I}^s \right] : \boldsymbol{\sigma} \\
&= \left[ \gamma_0 \boldsymbol{I}\boldsymbol{I} + \gamma_1\,{}^4\boldsymbol{I}^s \right] : \boldsymbol{\sigma} \\
&= {}^4\boldsymbol{S} : \boldsymbol{\sigma}
\end{aligned}
$$

$$
c_0 = \frac{\nu E}{(1+\nu)(1-2\nu)} = Q \quad ; \quad c_1 = \frac{E}{1+\nu} = 2L
$$

$$
\gamma_0 = -\frac{c_0}{(3c_0 + c_1)c_1} = -\frac{\nu}{E} = q \quad ; \quad \gamma_1 = \frac{1}{c_1} = \frac{1+\nu}{E} = \tfrac{1}{2}l
$$

The tensors can be written in components with respect to an orthonormal vector basis. This results in the relation between the stress and strain components, given below in index notation, where summation over equal indices is required (Einstein's convention).

$$
\begin{aligned}
\boldsymbol{\sigma} &= \left[ c_0 \boldsymbol{I}\boldsymbol{I} + c_1\,{}^4\boldsymbol{I}^s \right] : \boldsymbol{\varepsilon} \\
\sigma_{ij} &= \left[ c_0 \delta_{ij}\delta_{kl} + c_1 \tfrac{1}{2}\left( \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right) \right] \varepsilon_{lk} \\
&= c_0 \delta_{ij}\varepsilon_{kk} + c_1 \varepsilon_{ij} \\
&= c_1 \left( \varepsilon_{ij} + \frac{c_0}{c_1}\,\delta_{ij}\varepsilon_{kk} \right) \\
&= \frac{E}{1+\nu}\left( \varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\varepsilon_{kk} \right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= \left[ -\frac{c_0}{(3c_0 + c_1)c_1}\boldsymbol{I}\boldsymbol{I} + \frac{1}{c_1}\,{}^4\boldsymbol{I}^s \right] : \boldsymbol{\sigma} \\
\varepsilon_{ij} &= \left[ -\frac{c_0}{(3c_0 + c_1)c_1}\,\delta_{ij}\delta_{kl} + \right. \\
&\qquad \left. \frac{1}{c_1}\,\tfrac{1}{2}\left( \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right) \right] \sigma_{lk} \\
&= -\frac{c_0}{(3c_0 + c_1)c_1}\,\delta_{ij}\sigma_{kk} + \frac{1}{c_1}\,\sigma_{ij} \\
&= \frac{1}{c_1}\left( \sigma_{ij} - \frac{c_0}{3c_0 + c_1}\,\delta_{ij}\sigma_{kk} \right) \\
&= \frac{1+\nu}{E}\left( \sigma_{ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\sigma_{kk} \right)
\end{aligned}
$$

**Specific elastic energy**

The elastically stored energy per unit of volume (= the specific elastic energy) can be written as the sum of an hydrostatic and a deviatoric part. The hydrostatic part represents the specific energy associated with volume change. The deviatoric part indicates the specific energy needed for shape change.

$$W = \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\sigma} : {}^4\boldsymbol{S} : \boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : {}^4\boldsymbol{S} : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : \left(\gamma_0 \boldsymbol{I}\boldsymbol{I} + \gamma_1\, {}^4\boldsymbol{I}^s\right) : (\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d)$$

$$\gamma_0 \boldsymbol{I}\big[\boldsymbol{I} : \boldsymbol{\sigma}^h\big] = \gamma_0 \boldsymbol{I}\left[\boldsymbol{I} : \boldsymbol{I}\, \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\right] = \gamma_0 \boldsymbol{I}\left[\mathrm{tr}(\boldsymbol{\sigma})\right] = 3\gamma_0 \boldsymbol{\sigma}^h$$

$$\gamma_0 \boldsymbol{I}\big[\boldsymbol{I} : \boldsymbol{\sigma}^d\big] = \gamma_0\, \boldsymbol{I}\left[\mathrm{tr}(\boldsymbol{\sigma}^d)\right] = \gamma_0\, \boldsymbol{I}\big[0\big] = 0$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}^h + \boldsymbol{\sigma}^d) : (3\gamma_0 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^h + \gamma_1 \boldsymbol{\sigma}^d)$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^h = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} : \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \tfrac{1}{9}\mathrm{tr}^2(\boldsymbol{\sigma})(3) = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma})$$

$$\boldsymbol{\sigma}^h : \boldsymbol{\sigma}^d = \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I} : [\boldsymbol{\sigma} - \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})\boldsymbol{I}] = \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) - \tfrac{1}{3}\mathrm{tr}^2(\boldsymbol{\sigma}) = 0$$

$$= \left[\tfrac{1}{2}(\gamma_0 + \tfrac{1}{3}\gamma_1)\right] \mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\gamma_1\right] \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= \left[\tfrac{1}{2}\frac{1-2\nu}{3E}\right] \mathrm{tr}^2(\boldsymbol{\sigma}) + \left[\tfrac{1}{2}\frac{1+\nu}{E}\right] \boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{18K}\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d$$

$$= W^h + W^d$$

## 5.3 Thermo-elasticity

A temperature change $\Delta T$ of an unrestrained material invokes deformation. The total strain results from both mechanical and thermal effects and when deformations are small the total strain $\boldsymbol{\varepsilon}$ can be written as the sum of mechanical strains $\boldsymbol{\varepsilon}_m$ and thermal strains $\boldsymbol{\varepsilon}_T$. The thermal strains are related to the temperature change $\Delta T$ by the coefficient of thermal expansion tensor $\boldsymbol{A}$.

The stresses in terms of strains are derived by inversion of the compliance matrix $\underline{S}$. For thermally isotropic materials only the linear coefficient of thermal expansion $\alpha$ is relevant.

Anisotropic

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + \boldsymbol{\varepsilon}_T = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \boldsymbol{A}\Delta T \quad \rightarrow \quad \underset{\approx}{\varepsilon} = \underset{\approx}{\varepsilon}_m + \underset{\approx}{\varepsilon}_T = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \underline{\underline{A}}\,\Delta T$$

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \boldsymbol{A}\Delta T) \quad \rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}\left(\underset{\approx}{\varepsilon} - \underline{\underline{A}}\,\Delta T\right)$$

Isotropic

$$\boldsymbol{\varepsilon} = {}^4\boldsymbol{S} : \boldsymbol{\sigma} + \alpha\,\Delta T \boldsymbol{I} \quad \rightarrow \quad \underset{\approx}{\varepsilon} = \underline{\underline{S}}\,\underset{\approx}{\sigma} + \alpha\,\Delta T\,\underset{\approx}{I}$$

$$\boldsymbol{\sigma} = {}^4\boldsymbol{C} : (\boldsymbol{\varepsilon} - \alpha\Delta T \boldsymbol{I}) \quad \rightarrow \quad \underset{\approx}{\sigma} = \underline{\underline{C}}(\underset{\approx}{\varepsilon} - \alpha\,\Delta T\,\underset{\approx}{I})$$

For orthotropic material, this can be written in full matrix notation.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A+Q+R \\ Q+B+S \\ R+S+C \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## 5.4 Planar deformation

In many cases the state of strain or stress is planar. Both for plane strain and for plane stress, only strains and stresses in a plane are related by the material law. Here we assume that this plane is the 12-plane. For plane strain we than have $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, and for plane stress $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$. The material law for these planar situations can be derived from the general three-dimensional stress-strain relation, either from the stiffness matrix $\underline{\underline{C}}$ or from the compliance matrix $\underline{\underline{S}}$. Here the compliance and stiffness matrices are derived for the general orthotropic material. First the isothermal case is considered, subsequently planar relations are derived for thermo-elasticity. For cases with more material symmetry, the planar stress-strain relations can be simplified accordingly. The corresponding stiffness and compliance matrices can be found in appendix A, where they are specified in engineering constants.

### 5.4.1 Plane strain

For a plane strain state with $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, the stress $\sigma_{33}$ can be expressed in the planar strains $\varepsilon_{11}$ and $\varepsilon_{22}$. The material stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ can be extracted directly from $\underline{\underline{C}}$. The material compliance matrix $\underline{\underline{S}}_{\varepsilon}$ has to be derived by inversion.

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \rightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_{\varepsilon} \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_{\varepsilon} \underset{\sim}{\sigma}$$

We can derive by substitution :

$$\sigma_{33} = \frac{1}{AB^2 - Q^2}\left[(BR - QS)\sigma_{11} + (AS - QR)\sigma_{22}\right]$$

Because the components of the three-dimensional compliance matrix $\underline{\underline{S}}$ are most conveniently expressed in Young's moduli, Poisson's ratios and shear moduli, this matrix is a good starting point to derive the planar matrices for specific cases. The plane strain stiffness matrix $\underline{\underline{C}}_{\varepsilon}$ must then be determined by inversion.

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_{\varepsilon} & q_{\varepsilon} & 0 \\ q_{\varepsilon} & b_{\varepsilon} & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_{\varepsilon} \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_{\varepsilon} & q_{\varepsilon} & 0 \\ q_{\varepsilon} & b_{\varepsilon} & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \frac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_{\varepsilon} & Q_{\varepsilon} & 0 \\ Q_{\varepsilon} & B_{\varepsilon} & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_{\varepsilon} \underset{\sim}{\sigma}$$

We can now derive by substitution :

$$\sigma_{33} = -\frac{1}{\Delta_s}\left[(br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22}\right]$$

### 5.4.2 Plane stress

For the plane stress state, with $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, the two-dimensional material law can be easily derived from the three-dimensional compliance matrix $\underline{\underline{S}}_{\varepsilon}$. The strain $\varepsilon_{33}$ can be directly expressed in $\sigma_{11}$ and $\sigma_{22}$. The material stiffness matrix has to be derived by inversion.

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \rightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab - q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab - q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

We can derive by substitution :

$$\varepsilon_{33} = \frac{1}{ab - q^2} \left[ (br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

The same relations can be derived from the three-dimensional stiffness matrix $\underline{\underline{C}}$.

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{R}{C} & \dfrac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & 0 \\ -QC + RS & AC - R^2 & 0 \\ 0 & 0 & \frac{\Delta_c}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$$

$$= \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

### 5.4.3 Plane strain thermo-elastic

For thermo-elastic material behavior, the plane strain relations can be derived straightforwardly.

$$\begin{aligned} \sigma_{33} &= R\varepsilon_{11} + S\varepsilon_{22} - \alpha(R + S + C)\,\Delta T \qquad &(\text{from } \underline{\underline{C}}) \\ &= -\frac{r}{c}\,\sigma_{11} - \frac{s}{c}\,\sigma_{22} - \frac{\alpha}{c}\,\Delta T &(\text{from } \underline{\underline{S}}) \end{aligned}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A+Q+R \\ B+Q+S \\ 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} &= \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} A+Q+R \\ B+Q+S \\ 0 \end{bmatrix} \right\} \\ &= \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1+q_\varepsilon S + a_\varepsilon R \\ 1+q_\varepsilon R + b_\varepsilon S \\ 0 \end{bmatrix} \end{aligned}$$

### 5.4.4 Plane stress thermo-elastic

For plane stress the thermo-elastic stress-strain relations can be derived again.

$$\begin{aligned} \varepsilon_{33} &= r\sigma_{11} + s\sigma_{22} + \alpha \Delta T & \text{(from } \underline{\underline{S}}\text{)} \\ &= -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22} + \frac{1}{C}(R+S+C)\,\alpha\,\Delta T & \text{(from } \underline{\underline{C}}\text{)} \end{aligned}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} + \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} &= \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \\ &= \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \alpha \Delta T \begin{bmatrix} A_\sigma + Q_\sigma \\ B_\sigma + Q_\sigma \\ 0 \end{bmatrix} \end{aligned}$$

### 5.4.5 Plane strain/stress

In general we can write the stiffness and compliance matrix for planar deformation as a $3 \times 3$ matrix with components, which are specified for plane strain ($p = \varepsilon$) or plane stress ($p = \sigma$).

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} - \alpha \Delta T \begin{bmatrix} \Theta_{p1} \\ \Theta_{p2} \\ 0 \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix} + \alpha \Delta T \begin{bmatrix} \theta_{p1} \\ \theta_{p2} \\ 0 \end{bmatrix}$$

# 6 Elastic limit criteria

Loading of a material body causes deformation of the structure and, consequently, strains and stresses in the material. When either strains or stresses (or both combined) become too large, the material will be damaged, which means that irreversible microstructural changes will result. The structural and/or functional requirements of the structure or product will be hampered, which is referred to as failure.

Their are several failure modes, listed in the table below, each of them associated with a failure mechanism. In the following we will only consider plastic yielding. When the stress state exceeds the yield limit, the material behavior will not be elastic any longer. Irreversible microstructural changes (crystallographic slip in metals) will cause permanent (= plastic) deformation.

| **failure mode** | **mechanism** |
|---|---|
| plastic yielding | crystallographic slip (metals) |
| brittle fracture | (sudden) breakage of bonds |
| progressive damage | micro-cracks $\rightarrow$ growth $\rightarrow$ coalescence |
| fatigue | damage/fracture under cyclic loading |
| dynamic failure | vibration $\rightarrow$ resonance |
| thermal failure | creep / melting |
| elastic instabilities | buckling $\rightarrow$ plastic deformation |

## 6.1 Yield function

In a one-dimensional stress state (tensile test), yielding will occur when the absolute value of the stress $\sigma$ reaches the initial yield stress $\sigma_{y0}$. This can be tested with a yield criterion, where a yield function $f$ is used. When $f < 0$ the material behaves elastically and when $f = 0$ yielding occurs. Values $f > 0$ cannot be reached.

$$f(\sigma) = \sigma^2 - \sigma_{y0}^2 = 0 \quad \rightarrow \quad g(\sigma) = \sigma^2 = \sigma_{y0}^2 = g_t = \text{limit in tensile test}$$

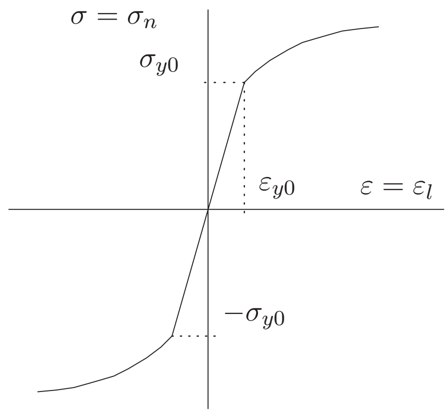



Fig. 6.53 : *Tensile curve with initial yield stress*

In a three-dimensional stress space, the yield criterion represents a yield surface. For elastic behavior ($f < 0$) the stress state is located inside the yield surface and for $f = 0$, the stress state is on the yield surface. Because $f > 0$ cannot be realized, stress states outside the yield surface can not exist. For isotropic material behavior, the yield function can be expressed in the principal stresses $\sigma_1$, $\sigma_2$ and $\sigma_3$. It can be visualized as a yield surface in the three-dimensional principal stress space.

$$
\begin{array}{llll}
f(\boldsymbol{\sigma}) = 0 & \rightarrow \quad g(\boldsymbol{\sigma}) = g_t & : & \text{yield surface in 6D stress space} \\
f(\sigma_1, \sigma_2, \sigma_3) = 0 & \rightarrow \quad g(\sigma_1, \sigma_2, \sigma_3) = g_t & : & \text{yield surface in 3D principal stress space}
\end{array}
$$

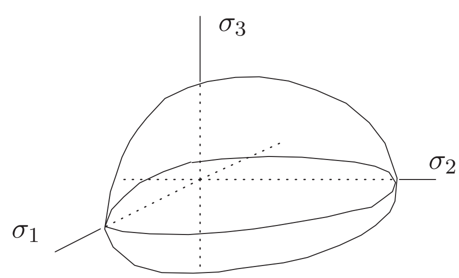



Fig. 6.54 : *Yield surface in three-dimensional principal stress space*

## 6.2 Principal stress space

The three-dimensional stress space is associated with a material point and has three axes, one for each principal stress value in that point. In the origin of the three-dimensional principal stress space, where $\sigma_1 = \sigma_2 = \sigma_3 = 0$, three orthonormal vectors $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ constitute a vector base. The stress state in the material point is characterized by the principal stresses and thus by a point in stress space with "coordinates" $\sigma_1$, $\sigma_2$ and $\sigma_3$. This point can also be identified with a vector $\vec{\sigma}$, having components $\sigma_1$, $\sigma_2$ and $\sigma_3$ with respect to the vector base $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.

The hydrostatic axis, where $\sigma_1 = \sigma_2 = \sigma_3$ can be identified with a unit vector $\vec{e}_p$. Perpendicular to $\vec{e}_p$ in the $\vec{e}_1\vec{e}_p$-plane a unit vector $\vec{e}_q$ can be defined. Subsequently the unit vector $\vec{e}_r$ is defined perpendicular to the $\vec{e}_p\vec{e}_q$-plane.

The vectors $\vec{e}_q$ and $\vec{e}_r$ span the so-called *Π*-plane perpendicular to the hydrostatic axis. Vectors $\vec{e}_p$, $\vec{e}_q$ and $\vec{e}_r$ constitute a orthonormal vector base. A random unit vector $\vec{e}_t(\phi)$ in the *Π*-plane can be expressed in $\vec{e}_q$ and $\vec{e}_r$.

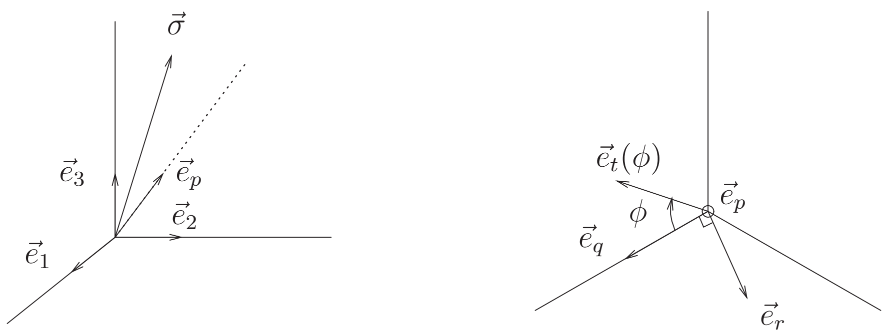



Fig. 6.55 : *Principal stress space*

hydrostatic axis $\quad \vec{e}_p = \frac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) \quad \text{with} \quad ||\vec{e}_p|| = 1$

plane ⊥ hydrostatic axis

$$\vec{e}_q^{\,*} = \vec{e}_1 - (\vec{e}_p \cdot \vec{e}_1)\vec{e}_p = \vec{e}_1 - \tfrac{1}{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \tfrac{1}{3}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$
$$\vec{e}_q = \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3)$$
$$\vec{e}_r = \vec{e}_p * \vec{e}_q = \tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) * \tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) = \tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)$$

vector in *Π*-plane $\quad \vec{e}_t(\phi) = \cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r$

A stress state can be represented by a vector in the principal stress space. This vector can be written as the sum of a vector along the hydrostatic axis and a vector in the *Π*-plane. These vectors are referred to as the hydrostatic and the deviatoric part of the stress vector.

$$\begin{aligned}
\vec{\sigma} &= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 = \vec{\sigma}^h + \vec{\sigma}^d \\
\vec{\sigma}^h &= (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p = \sigma^h\vec{e}_p = \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\vec{e}_p = \sqrt{3}\sigma_m\vec{e}_p \\
\sigma^h &= \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3) \\
\vec{\sigma}^d &= \vec{\sigma} - (\vec{\sigma} \cdot \vec{e}_p)\vec{e}_p \\
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 - \tfrac{1}{3}\sqrt{3}(\sigma_1 + \sigma_2 + \sigma_3)\tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3 - \tfrac{1}{3}(\sigma_1\vec{e}_1 + \sigma_2\vec{e}_1 + \sigma_3\vec{e}_1 + \sigma_1\vec{e}_2 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_2 + \sigma_1\vec{e}_3 + \sigma_2\vec{e}_3 + \sigma_3\vec{e}_3) \\
&= \tfrac{1}{3}\{(2\sigma_1 - \sigma_2 - \sigma_3)\vec{e}_1 + (-\sigma_1 + 2\sigma_2 - \sigma_3)\vec{e}_2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)\vec{e}_3\} \\
\sigma^d &= ||\vec{\sigma}^d|| = \sqrt{\vec{\sigma}^d \cdot \vec{\sigma}^d} \\
&= \tfrac{1}{3}\sqrt{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (-\sigma_1 + 2\sigma_2 - \sigma_3)^2 + (-\sigma_1 - \sigma_2 + 2\sigma_3)^2} \\
&= \sqrt{\tfrac{2}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1)} \\
&= \sqrt{\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d}
\end{aligned}
$$

Because the stress vector in the principal stress space can also be written as the sum of three vectors along the base vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, the principal stresses can be expressed in $\sigma^h$ and $\sigma^d$.

$$
\begin{aligned}
\vec{\sigma} &= \vec{\sigma}^h + \vec{\sigma}^d = \sigma^h\vec{e}_p + \sigma^d\vec{e}_t(\phi) \\
&= \sigma^h\vec{e}_p + \sigma^d\{\cos(\phi)\vec{e}_q - \sin(\phi)\vec{e}_r\} \\
&= \sigma^h\tfrac{1}{3}\sqrt{3}(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) + \sigma^d\{\cos(\phi)\tfrac{1}{6}\sqrt{6}(2\vec{e}_1 - \vec{e}_2 - \vec{e}_3) - \sin(\phi)\tfrac{1}{2}\sqrt{2}(\vec{e}_2 - \vec{e}_3)\} \\
&= \{\tfrac{1}{3}\sqrt{3}\,\sigma^h + \tfrac{1}{3}\sqrt{6}\,\sigma^d\cos(\phi)\}\vec{e}_1 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) - \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_2 + \\
&\quad \{\tfrac{1}{3}\sqrt{3}\,\sigma^h - \tfrac{1}{6}\sqrt{6}\,\sigma^d\cos(\phi) + \tfrac{1}{2}\sqrt{2}\,\sigma^d\sin(\phi)\}\vec{e}_3 \\
&= \sigma_1\vec{e}_1 + \sigma_2\vec{e}_2 + \sigma_3\vec{e}_3
\end{aligned}
$$

## 6.3 Yield criteria

In the following sections, various yield criteria are presented. Each of them starts from a hypothesis, stating when the material will yield. Such a hypothesis is based on experimental observation and is valid for a specific (class of) material(s). The yield criteria can be visualized in several stress spaces:

- the two-dimensional $(\sigma_1, \sigma_2)$-space for plane stress states with $\sigma_3 = 0$,
- the three-dimensional $(\sigma_1, \sigma_2, \sigma_3)$-space,
- the $\Pi$-plane and
- the $\sigma\tau$-plane, where Mohr's circles are used.

### 6.3.1 Maximum stress/strain

The maximum stress/strain criterion states that

| yielding occurs when one of the stress/strain components exceeds a limit value. |
|---|

This criterion is used for orthotropic materials.

$$\sigma_{ij} = \sigma_{max} \quad | \quad \varepsilon_{ij} = \varepsilon_{max} \quad ; \quad \{i,j\} = \{1,2,3\} \qquad \text{(orthotropic materials)}$$

### 6.3.2 Rankine

The maximum principal stress (or Rankine) criterion states that

| yielding occurs when the maximum principal stress reaches a limit value. |
| --- |

The Rankine criterion is used for brittle materials like cast iron. At failure these materials show *cleavage fracture*.

$$|\sigma_{max}| = max(|\sigma_i| \ ; \ i = 1,2,3) = \sigma_{max,t} = \sigma_{y0}(\text{brittle materials; cast iron})$$

The figure shows the yield surface in the principal stress space for a plane stress state with $\sigma_3 = 0$.

In the three-dimensional stress space the yield surface is a cube with side-length $2\sigma_{y0}$.

In the $(\sigma, \tau)$-space the Rankine criterion is visualized by to limits, which can not be exceeded by the absolute maximum of the principal stress.

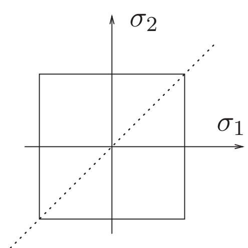



Fig. 6.56 : *Rankine yield surface in two-dimensional principal stress space*

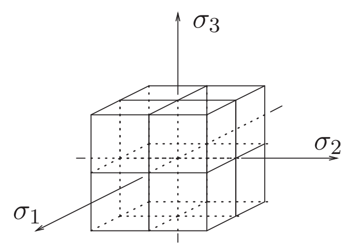



Fig. 6.57 : *Rankine yield surface in three-dimensional principal stress space*

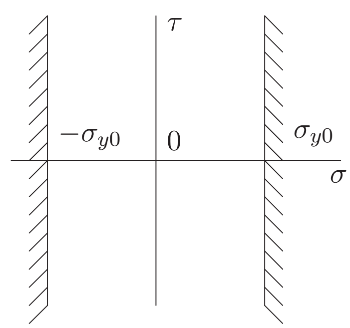



Fig. 6.58 : *Rankin yield limits in $(\sigma, \tau)$-space*

### 6.3.3 De Saint Venant

The maximum principal strain (or De Saint Venant) criterion states that

| yielding occurs when the maximum principal strain reaches a limit value. |
| --- |

From a tensile experiment this limit value appears to be the ratio of uni-axial yield stress and Young's modulus.

For $\sigma_1 > \sigma_2 > \sigma_3$, the maximum principal strain can be calculated from Hooke's law and its limit value can be expressed in the initial yield value $\sigma_{yo}$ and Young's modulus $E$.

$$\varepsilon_1 = \frac{1}{E}\sigma_1 - \frac{\nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_3 = \frac{\sigma_{y0}}{E} \quad \rightarrow \quad \sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma_{y0}$$

For other sequences of the principal stresses, relations are similar and can be used to construct the yield curve/surface in 2D/3D principal stress space.

$$\varepsilon_{max} = max(|\varepsilon_i| \ ; \ i = 1,2,3) = \varepsilon_{max_t} = \varepsilon_{y0} = \frac{\sigma_{y0}}{E}$$

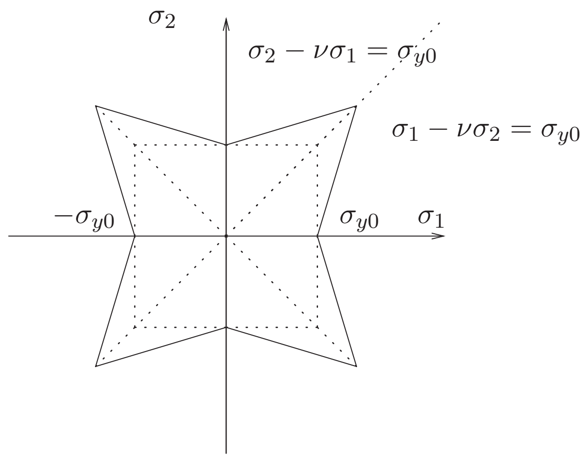



Fig. 6.59 : *Saint-Venant's yield curve in two-dimensional principal stress space*

### 6.3.4 Tresca

The Tresca criterion (Tresca, Coulomb, Mohr, Guest (1864)) states that

| yielding occurs when the maximum shear stress reaches a limit value. |
|---|

In a tensile test the limit value for the shear stress appears to be half the uni-axial yield stress.

$$\tau_{max} = \tfrac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max,t} = \tfrac{1}{2}\sigma_{y0} \rightarrow \bar{\sigma}_{TR} = \sigma_{max} - \sigma_{min} = \sigma_{y0}$$

Using Mohr's circles, it is easily seen how the maximum shear stress can be expressed in the maximum and minimum principal stresses.

For the plane stress case ($\sigma_3 = 0$) the yield curve in the $\sigma_1\sigma_2$-plane can be constructed using Mohr's circles. When both principal stresses are positive numbers, the yielding occurs when the largest reaches the one-dimensional yield stress $\sigma_{y0}$. When $\sigma_1$ is positive (= tensile stress), compression in the perpendicular direction, so a negative $\sigma_2$, implies that $\sigma_1$ must decrease to remain at the yield limit. Using Mohr's circles, this can easily be observed.

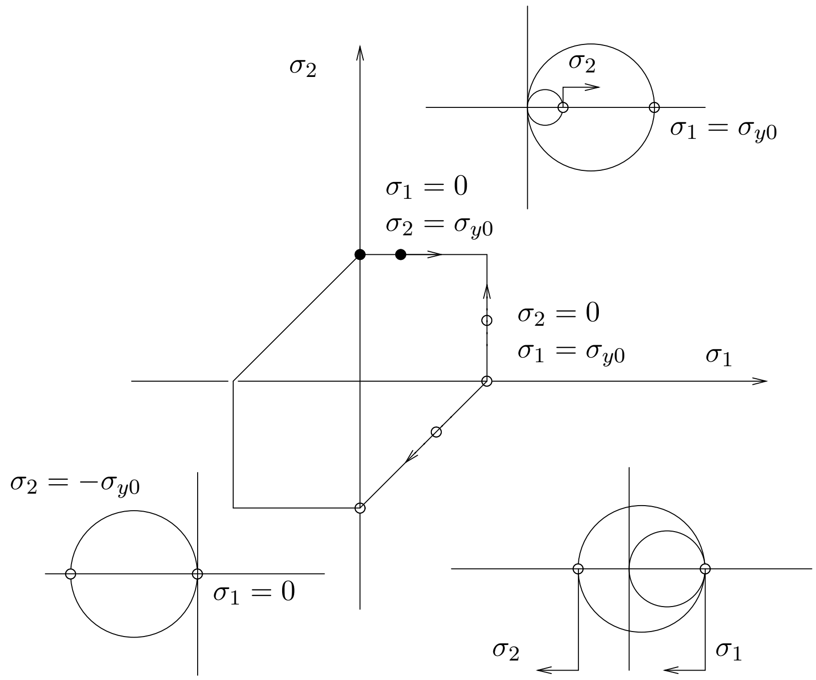



Fig. 6.60 : *Tresca yield curve in two-dimensional principal stress space*

Adding an extra hydrostatic stress state implies a translation in the three-dimensional principal stress space

$$\{\sigma_1, \sigma_2, \sigma_3\} \quad \rightarrow \quad \{\sigma_1 + c, \sigma_2 + c, \sigma_2 + c\}$$

i.e. a translation parallel to the hydrostatic axis where $\sigma_1 = \sigma_2 = \sigma_3$. This will never result in yielding or more plastic deformation, so the yield surface is a cylinder with its axis coinciding with (or parallel to) the hydrostatic axis.

In the $\Pi$-plane, the Tresca criterion is a regular 6-sided polygonal.

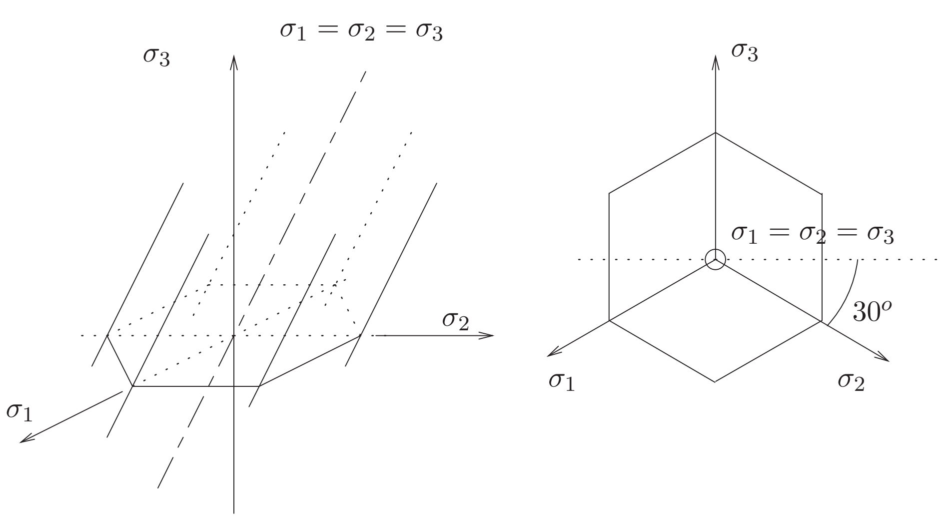


Fig. 6.61 : *Tresca yield surface in three-dimensional principal stress space and the Π-plane*

In the $\sigma\tau$-plane the Tresca yield criterion can be visualized with Mohr's circles.

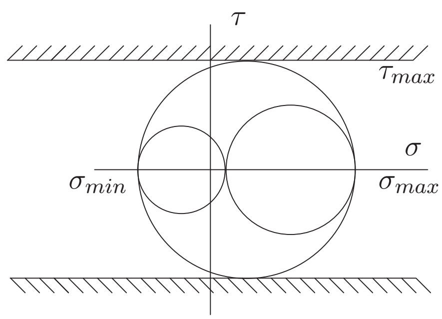



Fig. 6.62 : *Mohr's circles and Tresca yield limits in* $(\sigma, \tau)$*-space*

### 6.3.5 Von Mises

According to the Von Mises elastic limit criterion (Von Mises, Hubert, Hencky (1918)),

> yielding occurs when the specific shape deformation elastic energy reaches a critical value.

The specific *shape deformation energy* is also referred to as *distortional energy* or *deviatoric energy* or *shear strain energy*. It can be derived by splitting up the total specific elastic energy $W$ into a hydrostatic part $W^h$ and a deviatoric part $W^d$. The deviatoric $W^d$ can be expressed in $\boldsymbol{\sigma}^d$ and the hydrostatic $W^h$ can be expressed in the mean stress $\sigma_m = \frac{1}{3}\text{tr}(\boldsymbol{\sigma})$. The deviatoric part can be expressed in the second invariant $J_2$ of the deviatoric stress tensor and in the principal stresses.

For the tensile test the shape deformation energy $W_t^d$ can be expressed in the yield stress $\sigma_{y0}$. The Von Mises yield criterion $W^d = W_t^d$ can than be written as $\bar{\sigma}_{VM} = \sigma_{y0}$, where $\bar{\sigma}_{VM}$ is the equivalent or effective Von Mises stress, a function of all principal stresses. It is sometimes replaced by the octahedral shear stress $\tau_{oct} = \frac{1}{3}\sqrt{2}\bar{\sigma}_{VM}$.

$$W^d = W_t^d$$

$$\begin{aligned} W^d &= \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = \frac{1}{4G}\left\{\boldsymbol{\sigma} : \boldsymbol{\sigma} - \tfrac{1}{3}\text{tr}^2(\boldsymbol{\sigma})\right\} \quad \left( = \; -\frac{1}{2G}\,J_2(\boldsymbol{\sigma}^d)\right) \\ &= \frac{1}{4G}\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{12G}\,(\sigma_1 + \sigma_2 + \sigma_3)^2 \\ &= \frac{1}{4G}\,\tfrac{1}{3}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} \end{aligned}$$

$$W_t^d = \frac{1}{4G}\,\tfrac{1}{3}\,\left\{(\sigma - 0)^2 + (0 - 0)^2 + (0 - \sigma)^2\right\} = \frac{1}{4G}\,\tfrac{1}{3}2\sigma^2 = \frac{1}{4G}\,\tfrac{1}{3}\,2\sigma_{y0}^2$$

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right\}} = \sigma_{y0}$$

The Von Mises yield criterion can be expressed in Cartesian stress components.

$$\begin{aligned}
\bar{\sigma}_{VM}^2 &= \tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d = 3J_2 \\
&= \tfrac{3}{2}\mathrm{tr}(\underline{\sigma}^d\underline{\sigma}^d) \quad \text{with } \underline{\sigma}^d = \underline{\sigma} - \tfrac{1}{3}\mathrm{tr}(\underline{\sigma})\underline{I} \\
&= \tfrac{3}{2}\Big\{\left(\tfrac{2}{3}\sigma_{xx}-\tfrac{1}{3}\sigma_{yy}-\tfrac{1}{3}\sigma_{zz}\right)^2+\sigma_{xy}^2+\sigma_{xz}^2+ \\
&\qquad \left(\tfrac{2}{3}\sigma_{yy}-\tfrac{1}{3}\sigma_{zz}-\tfrac{1}{3}\sigma_{xx}\right)^2+\sigma_{yz}^2+\sigma_{yx}^2+ \\
&\qquad \left(\tfrac{2}{3}\sigma_{zz}-\tfrac{1}{3}\sigma_{xx}-\tfrac{1}{3}\sigma_{yy}\right)^2+\sigma_{zx}^2+\sigma_{zy}^2\Big\} \\
&= \left(\sigma_{xx}^2+\sigma_{yy}^2+\sigma_{zz}^2\right)-\left(\sigma_{xx}\sigma_{yy}+\sigma_{yy}\sigma_{zz}+\sigma_{zz}\sigma_{xx}\right)+2\left(\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2\right) \\
&= \sigma_{y0}^2
\end{aligned}$$

For plane stress ($\sigma_3 = 0$), the yield curve is an ellipse in the $\sigma_1\sigma_2$-plane. The length of the principal axes of the ellipse is $\sqrt{2}\sigma_{y0}$ and $\sqrt{\frac{1}{3}}\sigma_{y0}$.

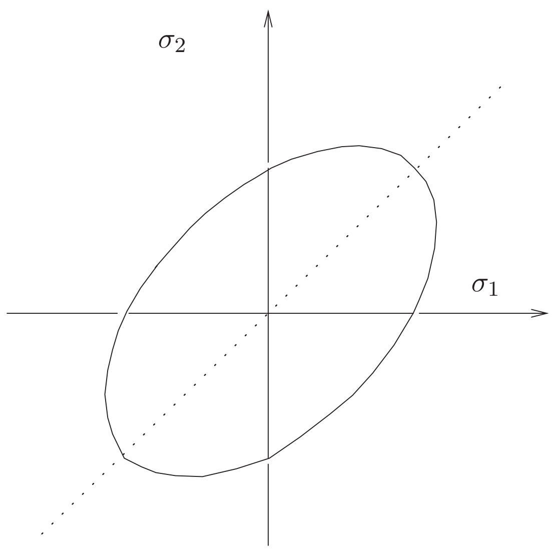



Fig. 6.63 : *Von Mises yield curve in two-dimensional principal stress space*

The three-dimensional Von Mises yield criterion is the equation of a cylindrical surface in three-dimensional principal stress space. Because hydrostatic stress does not influence yielding, the axis of the cylinder coincides with the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

In the $\Pi$-plane, the Von Mises criterion is a circle with radius $\sqrt{\frac{2}{3}}\sigma_{y0}$.

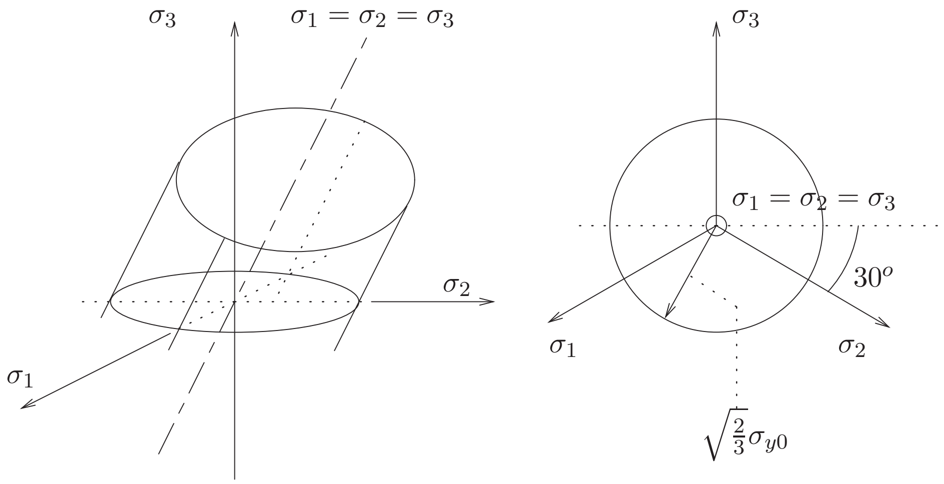



Fig. 6.64 : *Von Mises yield surface in three-dimensional principal stress space and the Π-plane*

### 6.3.6 Beltrami-Haigh

According to the elastic limit criterion of Beltrami-Haigh,

| yielding occurs when the total specific elastic energy $W$ reaches a critical value. |
|---|

$$W = W_t$$

$$\begin{aligned} W = W^h + W^d &= \frac{1}{18K}\,\mathrm{tr}^2(\boldsymbol{\sigma}) + \frac{1}{4G}\,\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d \\ &= \left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{1}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) \\ W_t &= \left(\frac{1}{18K} - \frac{1}{12G}\right)\sigma^2 + \frac{1}{4G}\,\sigma^2 = \frac{1}{2E}\,\sigma^2 = \frac{1}{2E}\,{\sigma_{y0}}^2 \end{aligned}$$

$$2E\left(\frac{1}{18K} - \frac{1}{12G}\right)(\sigma_1 + \sigma_2 + \sigma_3)^2 + \frac{2E}{4G}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) = \sigma_{y0}^2$$

The yield criterion contains elastic material parameters and thus depends on the elastic properties of the material. In three-dimensional principal stress space the yield surface is an ellipsoid. The longer axis coincides with (or is parallel to) the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$.

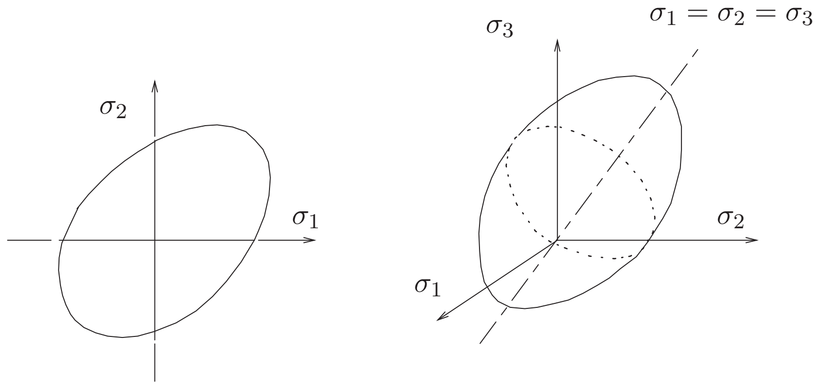



Fig. 6.65 : *Beltrami-Haigh yield curve and surface in principal stress space*

### 6.3.7 Mohr-Coulomb

A prominent difference in behavior under tensile and compression loading is seen in much materials, e.g. concrete, sand, soil and ceramics. In a tensile test such a material may have a yield stress $\sigma_{ut}$ and in compression a yield stress $\sigma_{uc}$ with $\sigma_{uc} > \sigma_{ut}$. The Mohr-Coulomb yield criterion states that

| yielding occurs when the shear stress reaches a limit value. |
|---|

For a plane stress state with $\sigma_3 = 0$ the yield contour in the $\sigma_1\sigma_2$-plane can be constructed in the same way as has been done for the Tresca criterion.

The yield surface in the three-dimensional principal stress space is a cone with axis along the hydrostatic axis.

The intersection with the plane $\sigma_3 = 0$ gives the yield contour for plane stress.

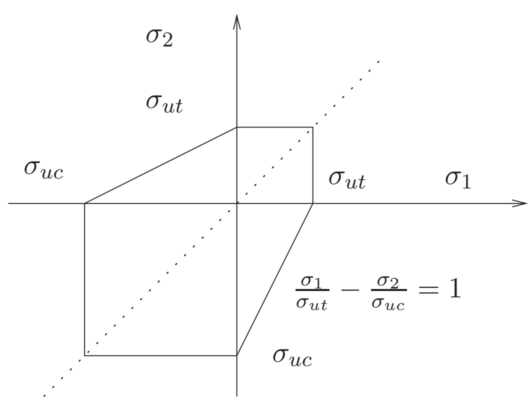



Fig. 6.66 : *Mohr-Coulomb yield curve in two-dimensional principal stress space*

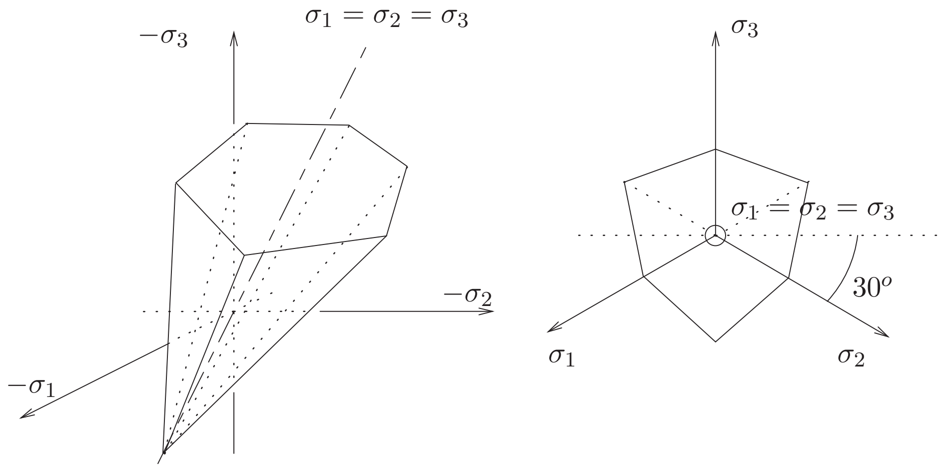



Fig. 6.67 : *Mohr-Coulomb yield surface in three-dimensional principal stress space and the Π-plane*

### 6.3.8 Drucker-Prager

For materials with internal friction and maximum adhesion, yielding can be described by the Drucker-Prager yield criterion. It relates to the Mohr-Coulomb criterion in the same way as the Von Mises criterion relates to the Tresca criterion.

For a plane stress state with $\sigma_3 = 0$ the Drucker-Prager yield contour in the $\sigma_1\sigma_2$-plane is a shifted ellipse.

In three-dimensional principal stress space the Drucker-Prager yield surface is a cone with circular cross-section.

$$\sqrt{\tfrac{2}{3}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} + \frac{6\sin(\phi)}{3-\sin(\phi)}\,p = \frac{6\cos(\phi)}{3-\sin(\phi)}\,C$$

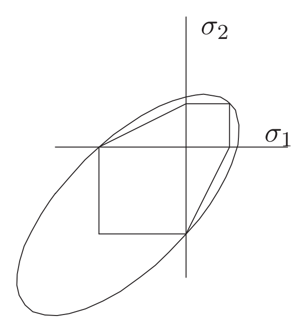



Fig. 6.68 : *Drucker-Prager yield curve in two-dimensional principal stress space*

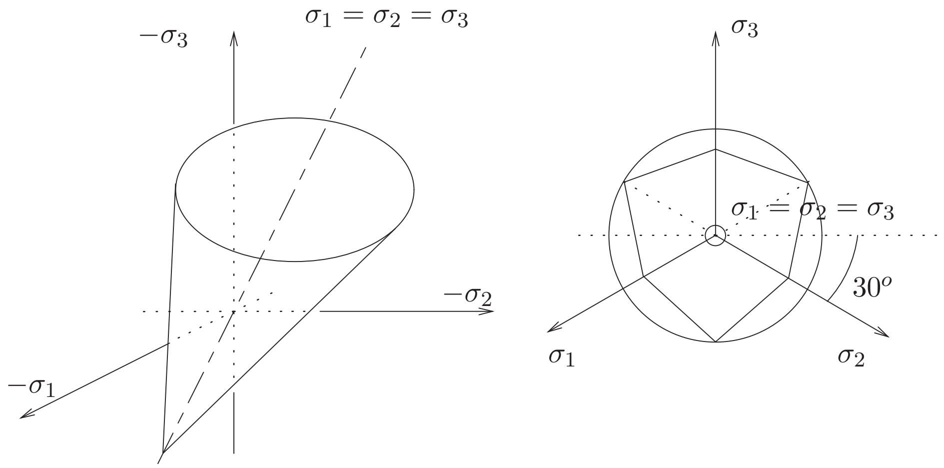



Fig. 6.69 : *Drucker-Prager yield surface in three-dimensional principal stress space and the Π-plane*

### 6.3.9 Other yield criteria

There are many more yield criteria, which are used for specific materials and loading conditions. The criteria of Hill, Hoffman and Tsai-Wu are used for orthotropic materials. In these criteria, there is a distinction between tensile and compressive stresses and their respective limit values.

parabolic Drucker-Prager $\quad \left(3J_2 + \sqrt{3}\,\beta\sigma_{y0}J_1\right)^{\frac{1}{2}} = \sigma_{y0}$

Buyokozturk $\quad \left(3J_2 + \sqrt{3}\,\beta\sigma_{y0}J_1 - 0.2J_1^2\right)^{\frac{1}{2}} = \sigma_{y0}$

Hill $\quad \dfrac{\sigma_{11}^2}{X^2} - \dfrac{\sigma_{11}\sigma_{22}}{XY} + \dfrac{\sigma_{22}^2}{Y^2} + \dfrac{\sigma_{12}^2}{S^2}$

Hoffman

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 +$$
$$\left(\frac{1}{S^2}\right)\sigma_{12}^2 - \left(\frac{1}{X_tX_c}\right)\sigma_{11}\sigma_{22} = 0$$

Tsai-Wu

$$\left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_{11} + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_{22} + \left(\frac{1}{X_tX_c}\right)\sigma_{11}^2 + \left(\frac{1}{Y_tY_c}\right)\sigma_{22}^2 +$$
$$\left(\frac{1}{S^2}\right)\sigma_{12}^2 + 2F_{12}\;\sigma_{11}\sigma_{22} = 0$$

with $\quad F_{12}^2 > \dfrac{1}{X_tX_c}\,\dfrac{1}{Y_tY_c}$

## 6.4 Examples

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**Equivalent Von Mises stress**

The stress state in a point is represented by the next Cauchy stress tensor :

$$\boldsymbol{\sigma} = 3\sigma\vec{e}_1\vec{e}_1 - \sigma\vec{e}_2\vec{e}_2 - 2\sigma\vec{e}_3\vec{e}_3 + \sigma(\vec{e}_1\vec{e}_2 + \vec{e}_2\vec{e}_1)$$

The Cauchy stress matrix is

$$\underline{\sigma} = \begin{bmatrix} 3\sigma & \sigma & 0 \\ \sigma & -\sigma & 0 \\ 0 & 0 & -2\sigma \end{bmatrix}$$

The Von Mises equivalent stress is defined as

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^d : \boldsymbol{\sigma}^d} = \sqrt{\tfrac{3}{2}\text{tr}(\boldsymbol{\sigma}^d \cdot \boldsymbol{\sigma}^d)} = \sqrt{\tfrac{3}{2}\text{tr}(\underline{\sigma}^d\underline{\sigma}^d)}$$

The trace of the matrix product is calculated first, using the average stress $\sigma_m = \frac{1}{3}\text{tr}(\underline{\sigma})$.

$$\begin{aligned}\text{tr}(\underline{\sigma}^d\underline{\sigma}^d) &= \text{tr}([\underline{\sigma} - \sigma_m\underline{I}]\,[\underline{\sigma} - \sigma_m\underline{I}]) = \text{tr}(\underline{\sigma}\,\underline{\sigma} - 2\sigma_m\underline{I} + \sigma_m^2\underline{I}) = \text{tr}(\underline{\sigma}\,\underline{\sigma}) - 6\sigma_m + 3\sigma_m^2 \\ &= \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{13}^2 - \tfrac{2}{3}\sigma_{11} - \tfrac{2}{3}\sigma_{22} - \tfrac{2}{3}\sigma_{33} + \\ &\quad \tfrac{1}{3}\sigma_{11}^2 + \tfrac{1}{3}\sigma_{22}^2 + \tfrac{1}{3}\sigma_{33}^2 + \tfrac{2}{3}\sigma_{11}\sigma_{22} + \tfrac{2}{3}\sigma_{22}\sigma_{33} + \tfrac{2}{3}\sigma_{33}\sigma_{11}\end{aligned}$$

Substitution of the given values for the stress components leads to

$$\text{tr}(\underline{\sigma}^d\underline{\sigma}^d) = 16\sigma^2 \quad\rightarrow\quad \bar{\sigma}_{VM}^2 = 24\sigma^2 \quad\rightarrow\quad \bar{\sigma}_{VM} = 2\sqrt{6}\,\sigma$$

---

**Equivalent Von Mises and Tresca stresses**

The Cauchy stress matrix for a stress state is

$$\underline{\sigma} = \begin{bmatrix} \sigma & \tau & 0 \\ \tau & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

with all component values positive.
The Tresca yield criterion states that yielding will occur when the maximum shear stress reaches a limit value, which is determined in a tensile experiment. The equivalent Tresca stress is two times this maximum shear stress.

$$\bar{\sigma}_{TR} = 2\tau_{max} = \sigma_{max} - \sigma_{min}$$

The limit value is the one-dimensional yield stress $\sigma_{y0}$. To calulate $\bar{\sigma}_{TR}$, we need the prinipal stresses, which can be determined by requiring the matrix $\underline{\sigma} - s\underline{I}$ to be singular.

$$\det(\underline{\sigma} - s\underline{I}) = \det \begin{bmatrix} \sigma - s & \tau & 0 \\ \tau & \sigma - s & 0 \\ 0 & 0 & \sigma - s \end{bmatrix} = 0 \quad \rightarrow$$
$$(\sigma - s)^3 - \tau^2(\sigma - s) = 0 \quad \rightarrow \quad (\sigma - s)\{(\sigma - s)^2 - \tau^2\} = 0 \quad \rightarrow$$
$$(\sigma - s)(\sigma - s + \tau)(\sigma - s - \tau) = 0 \quad \rightarrow$$
$$\sigma_1 = \sigma_{max} = \sigma + \tau \quad ; \quad \sigma_2 = \sigma \quad ; \quad \sigma_3 = \sigma_{min} = \sigma - \tau$$

The equivalent Tresca stress is

$$\bar{\sigma}_{TR} = 2\tau$$

so yielding according to Tresca will occur when

$$\tau = \tfrac{1}{2}\,\sigma_{y0}$$

The equivalent Von Mises stress is expressed in the principal stresses :

$$\bar{\sigma}_{VM} = \sqrt{\tfrac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}}$$

and can be calculated by substitution,

$$\bar{\sigma}_{VM} = \sqrt{3\tau^2} = \sqrt{3}\tau$$

Yielding according to Von Mises will occur when the equivalent stress reaches a limit value, the one-dimensional yield stress $\sigma_{y0}$, which results in

$$\tau = \tfrac{1}{3}\sqrt{3}\,\sigma_{y0}$$

# APPENDICES

# A Stiffness and compliance matrices

In chapter **??** the three-dimensional stiffness and compliance matrices have been derived for various materials. Increasing microstructural lattice symmetry gave rise to a reduction of the number of material constants. Starting from triclinic with no symmetry and characterized by 21 material constants, increased symmetry was seen for monoclinic (13 constants), orthotropic (9), quadratic (6), transversal isotropic (5), cubic (3) and finally, isotropic, with only 2 material constants.

In this appendix, we again present the material matrices for orthotropic, transversal isotropic and fully isotropic material. The material constants will be expressed in engineering constants, where we choose Young's moduli, Poisson's ratios and shear moduli.

In many engineering problems, the state of strain or stress is planar. Both for plane strain and plane stress, only the strain and stress components in a plane have to be related through a material law. Here we assume that this plane is the 12-plane. For plane strain we than have $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, and for plane stress $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$. The material law for these planar situations can be derived from the linear elastic three-dimensional stress-strain relation. This is done, first for the general orthotropic material law. The result is subsequently specified in engineering parameters for orthotropic, transversal isotropic and fully isotropic material.

## A.1 General orthotropic material law

The general orthotropic material law is expressed by the stiffness matrix $\underline{\underline{C}}$ and/or its inverse, the compliance matrix $\underline{\underline{S}}$.

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \begin{bmatrix} A & Q & R & 0 & 0 & 0 \\ Q & B & S & 0 & 0 & 0 \\ R & S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & M \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \underline{\underline{C}}\,\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} a & q & r & 0 & 0 & 0 \\ q & b & s & 0 & 0 & 0 \\ r & s & c & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 \\ 0 & 0 & 0 & 0 & 0 & m \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \underline{\underline{C}}^{-1}\,\underset{\sim}{\sigma} = \underline{\underline{S}}\,\underset{\sim}{\sigma}$$

The inverse of $\underline{\underline{C}}$ can be expresses in its components.

$$\underline{\underline{C}}^{-1} = \frac{1}{\Delta_c} \begin{bmatrix} BC - S^2 & -QC + RS & QS - BR & 0 & 0 & 0 \\ -QC + RS & AC - R^2 & -AS + QR & 0 & 0 & 0 \\ QS - BR & -AS + QR & AB - Q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_c(1/K) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_c(1/L) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_c(1/M) \end{bmatrix}$$

with $\quad \Delta_c = ABC - AS^2 - BR^2 - CQ^2 + 2QRS$

As will be clear later, it will mostly be easier to start with the compliance matrix and calculate the stiffness matrix by inversion.

$$\underline{\underline{S}}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & qs - br & 0 & 0 & 0 \\ -qc + rs & ac - r^2 & -as + qr & 0 & 0 & 0 \\ qs - br & -as + qr & ab - q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s(1/k) & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s(1/l) & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s(1/m) \end{bmatrix}$$

with $\quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$

Increasing material symmetry leads to a reduction in material parameters.

| | |
|---|---|
| quadratic | $B = A\ ;\ S = R\ ;\ M = L;$ |
| transversal isotropic | $B = A\ ;\ S = R\ ;\ M = L\ ;\ K = \frac{1}{2}(A - Q)$ |
| cubic | $C = B = A\ ;\ S = R = Q\ ;\ M = L = K \neq \frac{1}{2}(A - Q)$ |
| isotropic | $C = B = A\ ;\ S = R = Q\ ;\ M = L = K = \frac{1}{2}(A - Q)$ |

The planar stress-strain laws can be derived either from the stiffness matrix $\underline{\underline{C}}$ or from the compliance matrix $\underline{\underline{S}}$. The plane strain state will be denoted by the index $\varepsilon$ and the plane stress state will be indicated with the index $\sigma$.

### A.1.1 Plane strain

For a plane strain state with $\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0$, the stress $\sigma_{33}$ can be expressed in the planar strains $\varepsilon_{11}$ and $\varepsilon_{22}$. The material stiffness matrix $\underline{\underline{C}}_\varepsilon$ can be extracted directly from $\underline{\underline{C}}$. The material compliance matrix $\underline{\underline{S}}_\varepsilon$ has to be derived by inversion.

$$\varepsilon_{33} = \gamma_{23} = \gamma_{31} = 0 \quad \rightarrow \quad \sigma_{33} = R\varepsilon_{11} + S\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{AB - Q^2} \begin{bmatrix} B & -Q & 0 \\ -Q & A & 0 \\ 0 & 0 & \dfrac{AB - Q^2}{K} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

Because the components of the three-dimensional compliance matrix $\underline{\underline{S}}$ are most conveniently expressed in Young's moduli, Poisson's ratios and shear moduli, this matrix is a good starting point to derive the planar matrices for specific cases. The plane strain stiffness matrix $\underline{\underline{C}}_\varepsilon$ must then be determined by inversion.

$$\varepsilon_{33} = 0 = r\sigma_{11} + s\sigma_{22} + c\sigma_{33} \quad \rightarrow \quad \sigma_{33} = -\frac{r}{c}\sigma_{11} - \frac{s}{c}\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} - \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \begin{bmatrix} \frac{r}{c} & \frac{s}{c} & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$= \frac{1}{c} \begin{bmatrix} ac - r^2 & qc - rs & 0 \\ qc - sr & bc - s^2 & 0 \\ 0 & 0 & kc \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\varepsilon \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\varepsilon & q_\varepsilon & 0 \\ q_\varepsilon & b_\varepsilon & 0 \\ 0 & 0 & k \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \frac{1}{\Delta_s} \begin{bmatrix} bc - s^2 & -qc + rs & 0 \\ -qc + rs & ac - r^2 & 0 \\ 0 & 0 & \dfrac{\Delta_s}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\text{with} \quad \Delta_s = abc - as^2 - br^2 - cq^2 + 2qrs$$

$$= \begin{bmatrix} A_\varepsilon & Q_\varepsilon & 0 \\ Q_\varepsilon & B_\varepsilon & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \underline{\underline{C}}_\varepsilon \underset{\sim}{\sigma}$$

We can now derive by substitution :

$$\sigma_{33} = -\frac{1}{\Delta_s} \left[ (br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

### A.1.2 Plane stress

For the plane stress state, with $\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$, the two-dimensional material law can be easily derived from the three-dimensional compliance matrix $\underline{\underline{S}}_\varepsilon$. The strain $\varepsilon_{33}$ can be directly expressed in $\sigma_{11}$ and $\sigma_{22}$. The material stiffness matrix has to be derived by inversion.

$$\sigma_{33} = \sigma_{23} = \sigma_{31} = 0 \quad \rightarrow \quad \varepsilon_{33} = r\sigma_{11} + s\sigma_{22}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} a & q & 0 \\ q & b & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{1}{ab - q^2} \begin{bmatrix} b & -q & 0 \\ -q & a & 0 \\ 0 & 0 & \dfrac{ab - q^2}{k} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

We can derive by substitution :

$$\varepsilon_{33} = \frac{1}{ab - q^2} \left[ (br - qs)\varepsilon_{11} + (as - qr)\varepsilon_{22} \right]$$

The same relations can be derived from the three-dimensional stiffness matrix $\underline{\underline{C}}$.

$$\sigma_{33} = 0 = R\varepsilon_{11} + S\varepsilon_{22} + C\varepsilon_{33} \quad \rightarrow \quad \varepsilon_{33} = -\frac{R}{C}\varepsilon_{11} - \frac{S}{C}\varepsilon_{22}$$

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} A & Q & 0 \\ Q & B & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} - \begin{bmatrix} R \\ S \\ 0 \end{bmatrix} \begin{bmatrix} \dfrac{R}{C} & \dfrac{S}{C} & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix}$$

$$= \frac{1}{C} \begin{bmatrix} AC - R^2 & QC - RS & 0 \\ QC - SR & BC - S^2 & 0 \\ 0 & 0 & KC \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underline{\underline{C}}_\sigma \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} A_\sigma & Q_\sigma & 0 \\ Q_\sigma & B_\sigma & 0 \\ 0 & 0 & K \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_\sigma & q_\sigma & 0 \\ q_\sigma & b_\sigma & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \underline{\underline{S}}_\sigma \underset{\sim}{\sigma}$$

### A.1.3 Plane strain/stress

In general we can write the stiffness and compliance matrix for planar deformation as a $3 \times 3$ matrix with components, which are specified for plane strain ($p = \varepsilon$) or plane stress ($p = \sigma$).

$$\underline{\underline{C}}_p = \begin{bmatrix} A_p & Q_p & 0 \\ Q_p & B_p & 0 \\ 0 & 0 & K \end{bmatrix} \quad ; \quad \underline{\underline{S}}_p = \begin{bmatrix} a_p & q_p & 0 \\ q_p & b_p & 0 \\ 0 & 0 & k \end{bmatrix}$$

The general relations presented before can be used to calculate the components of $\underline{\underline{C}}_p$ and/or $\underline{\underline{S}}_p$ when components of the three-dimensional matrices $\underline{\underline{C}}$ and/or $\underline{\underline{S}}$ are known.

In the next sections the three-dimensional and planar material matrices are presented for orthonormal, transversal isotropic and fully isotropic material.

## A.2 Linear elastic orthotropic material

For an orthotropic material 9 material parameters are needed to characterize its mechanical behavior. Their names and formal definitions are :

$$\begin{array}{lll} \text{Young's moduli} & : & E_i = \dfrac{\partial \sigma_{ii}}{\partial \varepsilon_{ii}} \\ \text{Poisson's ratios} & : & \nu_{ij} = -\dfrac{\partial \varepsilon_{jj}}{\partial \varepsilon_{ii}} \\ \text{shear moduli} & : & G_{ij} = \dfrac{\partial \sigma_{ij}}{\partial \gamma_{ij}} \end{array}$$

The introduction of these parameters is easily accomplished in the compliance matrix $\underline{\underline{S}}$. Due to the symmetry of the compliance matrix $\underline{\underline{S}}$, the material parameters must obey the three Maxwell relations.

$$\underline{\underline{S}} = \begin{bmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} & -\nu_{31}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & -\nu_{32}E_3^{-1} & 0 & 0 & 0 \\ -\nu_{13}E_1^{-1} & -\nu_{23}E_2^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{23}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{31}^{-1} \end{bmatrix}$$

$$\text{with} \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \quad ; \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \quad ; \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \qquad \text{(Maxwell relations)}$$

The stiffness matrix $\underline{\underline{C}}$ can then be derived by inversion of $\underline{\underline{S}}$.

$$\underline{\underline{C}} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & \frac{\nu_{21}\nu_{32}+\nu_{31}}{E_2E_3} & 0 & 0 & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & \frac{\nu_{12}\nu_{31}+\nu_{32}}{E_1E_3} & 0 & 0 & 0 \\ \frac{\nu_{12}\nu_{23}+\nu_{13}}{E_1E_2} & \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s G_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s G_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s G_{31} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{12}\nu_{23}\nu_{31} - \nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

### A.2.1 Voigt notation

In composite mechanics the so-called Voigt notation is often used, where stress and strain components are simply numbered 1 to 6. Corresponding components of the compliance (and stiffness) matrix are numbered accordingly. However, there is more to it than that. The sequence of the shear components is changed. We will not use this changed sequence in the following.

$$\underset{\sim}{\sigma}^T = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}] = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_6\ \sigma_4\ \sigma_5]$$
$$\underset{\sim}{\varepsilon}^T = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \gamma_{12}\ \gamma_{23}\ \gamma_{31}] = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_6\ \varepsilon_4\ \varepsilon_5]$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

### A.2.2 Plane strain

For plane strain the stiffness matrix can be extracted from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\sigma_{33} = \nu_{13}\,\frac{E_3}{E_1}\,\sigma_{11} + \nu_{23}\,\frac{E_3}{E_2}\,\sigma_{22}$$

$$\underline{\underline{S}}_\varepsilon = \begin{bmatrix} \frac{1-\nu_{31}\nu_{13}}{E_1} & -\frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1} & \frac{1-\nu_{32}\nu_{23}}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}$$

$$\underline{\underline{C}}_\varepsilon = \underline{\underline{S}}_\varepsilon^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{32}\nu_{23}}{E_2E_3} & \frac{\nu_{31}\nu_{23}+\nu_{21}}{E_2E_3} & 0 \\ \frac{\nu_{13}\nu_{32}+\nu_{12}}{E_1E_3} & \frac{1-\nu_{31}\nu_{13}}{E_1E_3} & 0 \\ 0 & 0 & \Delta_s G_{12} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-\nu_{12}\nu_{23}\nu_{31}-\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s}\left\{\frac{\nu_{12}\nu_{32}+\nu_{13}}{E_1E_2}\,\varepsilon_{11} + \frac{\nu_{21}\nu_{13}+\nu_{23}}{E_1E_2}\,\varepsilon_{22}\right\}$$

### A.2.3 Plane stress

For plane stress the compliance matrix can be extracted from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\varepsilon_{33} = -\,\nu_{13}E_1^{-1}\sigma_{11} - \nu_{23}E_2^{-1}\sigma_{22}$$

$$\underline{\underline{S}}_\sigma = \begin{bmatrix} E_1^{-1} & -\nu_{21}E_2^{-1} & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & 0 \\ 0 & 0 & G_{12}^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{1}{1-\nu_{21}\nu_{12}} \begin{bmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1-\nu_{21}\nu_{12})G_{12} \end{bmatrix}$$

$$\varepsilon_{33} = -\,\frac{1}{1-\nu_{12}\nu_{21}}\,\{(\nu_{12}\nu_{23}+\nu_{13})\varepsilon_{11} + (\nu_{21}\nu_{13}+\nu_{23})\varepsilon_{22}\}$$

### A.3 Linear elastic transversal isotropic material

Considering an transversally isotropic material with the 12-plane isotropic, the Young's modulus $E_p$ and the Poisson's ratio $\nu_p$ in this plane can be measured. The associated shear modulus is related by $G_p = \dfrac{E_p}{2(1+\nu_p)}$. In the perpendicular direction we have the Young's modulus $E_3$, the shear moduli $G_{3p} = G_{p3}$ and two Poisson ratios, which are related by symmetry : $\nu_{p3}E_3 = \nu_{3p}E_p$.

$$\underline{\underline{S}} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & -\nu_{3p} E_3^{-1} & 0 & 0 & 0 \\ -\nu_{p3} E_p^{-1} & -\nu_{p3} E_p^{-1} & E_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_p^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{p3}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{3p}^{-1} \end{bmatrix}$$

$$\text{with} \qquad \frac{\nu_{p3}}{E_p} = \frac{\nu_{3p}}{E_3}$$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_p\nu_{3p}+\nu_{3p}}{E_pE_3} & 0 & 0 & 0 \\ \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{\nu_p\nu_{p3}+\nu_{p3}}{E_pE_p} & \frac{1-\nu_p\nu_p}{E_pE_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_s G_p & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta_s G_{p3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_s G_{3p} \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_p\nu_p-\nu_{p3}\nu_{3p}-\nu_{3p}\nu_{p3}-\nu_p\nu_{p3}\nu_{3p}-\nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$$

#### A.3.1 Plane strain

The plane strain stiffness matrix can be extracted from the three-dimensional stiffness matrix. The inverse of this 3x3 matrix is the plane strain compliance matrix.

$$\sigma_{33} = \frac{E_3\nu_{p3}}{E_p}(\sigma_{11}+\sigma_{22}) = \nu_{3p}(\sigma_{11}+\sigma_{22})$$

$$\underline{\underline{S}}_\varepsilon = \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_p} & -\frac{\nu_{3p}\nu_{p3}+\nu_p}{E_p} & 0 \\ -\frac{\nu_{p3}\nu_{3p}+\nu_p}{E_p} & \frac{1-\nu_{3p}\nu_{p3}}{E_p} & 0 \\ 0 & 0 & \frac{1}{G_p} \end{bmatrix}$$

$$\underline{\underline{C}}_\varepsilon = \underline{\underline{S}}_\varepsilon^{-1} = \frac{1}{\Delta_s} \begin{bmatrix} \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & \frac{\nu_{3p}\nu_{p3}+\nu_p}{E_pE_3} & 0 \\ \frac{\nu_{p3}\nu_{3p}+\nu_p}{E_pE_3} & \frac{1-\nu_{3p}\nu_{p3}}{E_pE_3} & 0 \\ 0 & 0 & \Delta_s G_p \end{bmatrix}$$

$$\text{with} \qquad \Delta_s = \frac{1-\nu_p\nu_p-\nu_{p3}\nu_{3p}-\nu_{3p}\nu_{p3}-\nu_p\nu_{p3}\nu_{3p}-\nu_p\nu_{3p}\nu_{p3}}{E_pE_pE_3}$$

$$\sigma_{33} = \frac{1}{\Delta_s} \frac{\nu_{p3}(\nu_p + 1)}{E_p^2} (\varepsilon_{11} + \varepsilon_{22})$$

### A.3.2 Plane stress

For plane stress the compliance matrix can be extracted directly from the three-dimensional compliance matrix. The inverse of this 3x3 matrix is the plane strain stiffness matrix.

$$\varepsilon_{33} = -\frac{\nu_{p3}}{E_p}(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_{\sigma} = \begin{bmatrix} E_p^{-1} & -\nu_p E_p^{-1} & 0 \\ -\nu_p E_p^{-1} & E_p^{-1} & 0 \\ 0 & 0 & G_p^{-1} \end{bmatrix}$$

$$\underline{\underline{C}}_{\sigma} = \underline{\underline{S}}_{\sigma}^{-1} = \frac{1}{1 - \nu_p \nu_p} \begin{bmatrix} E_p & \nu_p E_p & 0 \\ \nu_p E_p & E_p & 0 \\ 0 & 0 & (1 - \nu_p \nu_p) G_p \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu_{p3}}{1 - \nu_p} (\varepsilon_{11} + \varepsilon_{22})$$

## A.4 Linear elastic isotropic material

The linear elastic material behavior can be described with the material stiffness matrix $\underline{\underline{C}}$ or the material compliance matrix $\underline{\underline{S}}$. These matrices can be written in terms of the engineering elasticity parameters $E$ and $\nu$.

$$\underline{\underline{S}} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}} = \underline{\underline{S}}^{-1} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

### A.4.1 Plane strain

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_\varepsilon = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underline{\underline{C}}_\varepsilon = \underline{\underline{S}}_\varepsilon^{-1} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}\,\nu(\varepsilon_{11} + \varepsilon_{22})$$

It is immediately clear that problems will occur for $\nu = 0.5$, which is the value for incompressible material behavior.

### A.4.2 Plane stress

$$\varepsilon_{33} = -\frac{\nu}{E}\,(\sigma_{11} + \sigma_{22})$$

$$\underline{\underline{S}}_\sigma = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underline{\underline{C}}_\sigma = \underline{\underline{S}}_\sigma^{-1} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{1-\nu}\,(\varepsilon_{11} + \varepsilon_{22})$$