FRACTURE MECHANICS

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INTRODUCTION

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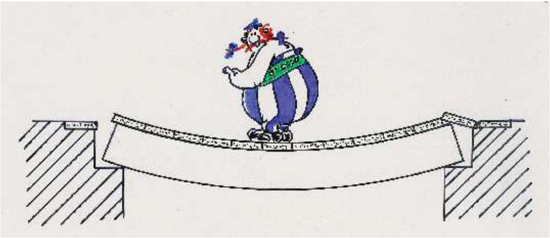
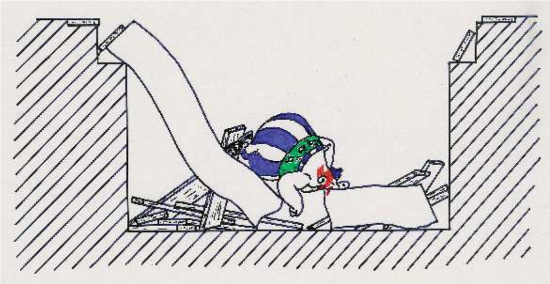
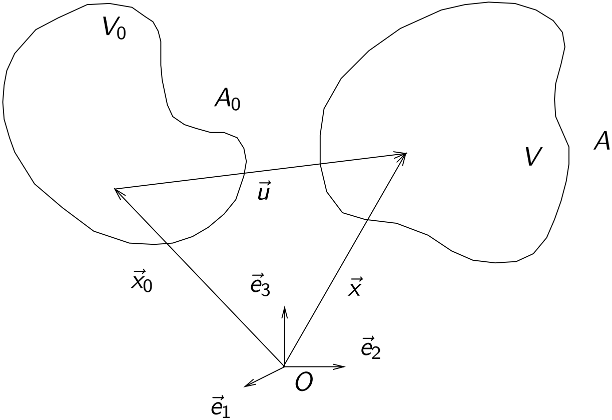
Introduction





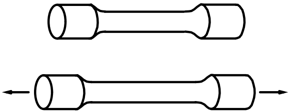
Continuum mechanics

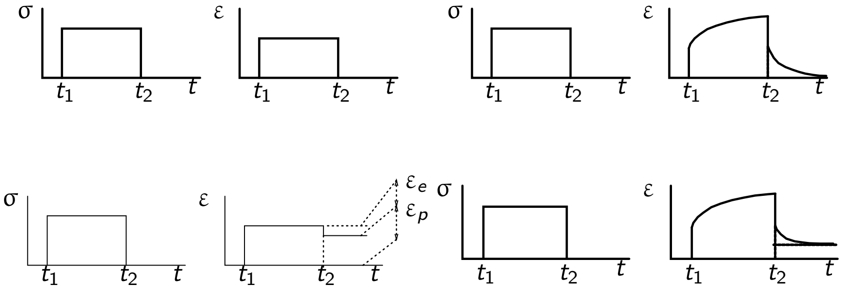


Continuum mechanics

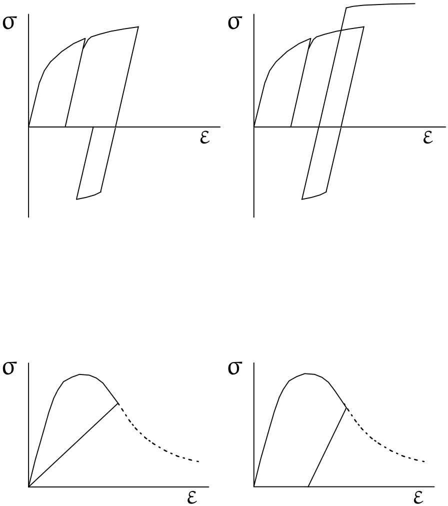
- volume / area	$V_0, V \;\; / \;\; A_0, A$
- base vectors	$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
- position vector	$\vec{x}_0, \vec{x}$
- displacement vector	$\vec{u}$
- strains	$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$
- compatibility relations	
- equilibrium equations	$\sigma_{ij,j} + \rho q_i = 0 \quad ; \quad \sigma_{ij} = \sigma_{ji}$
- density	ρ
- load/mass	q_i
- boundary conditions	$p_i = \sigma_{ij} n_j$
- material model	$\sigma_{ij} = N_{ij}(\varepsilon_{kl})$

Material behavior

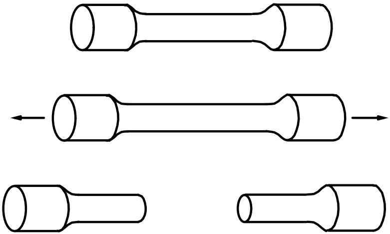




Stress-strain curves



Fracture



Fracture mechanics

questions :

- when crack growth ? ($\rightarrow$ crack growth criteria)
- crack growth rate ?
- residual strength ?
- life time ?
- inspection frequency ?
- repair required ?

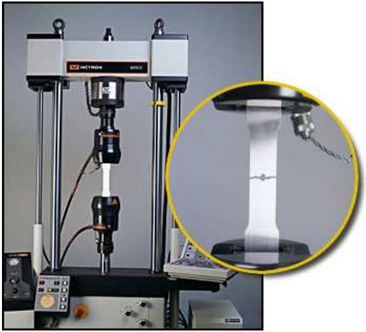
fields of science :

- material science and chemistry
- theoretical and numerical mathematics
- experimental and theoretical mechanics

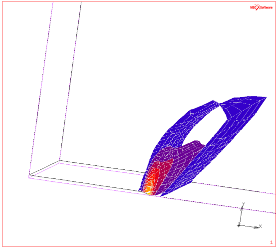
Overview of fracture mechanics

- LEFM (Linear Elastic Fracture Mechanics)
  - energy balance
  - crack tip stresses
  - SSY (Small Scale Yielding)
- DFM (Dynamic Fracture Mechanics)
- NLFM (Non-Linear Fracture Mechanics)
  EPFM (Elasto-Plastic Fracture Mechanics)
- Numerical methods : EEM / BEM
- Fatigue (HCF / LCF)
- CDM (Continuum Damage Mechanics)
- Micro mechanics
  - micro-cracks (intra grain)
  - voids (intra grain)
  - cavities at grain boundaries
  - rupture & disentangling of molecules
  - rupture of atomic bonds
  - dislocation slip

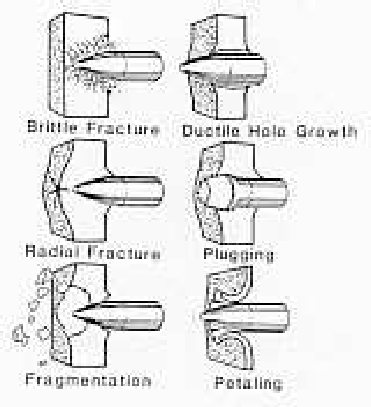
Experimental fracture mechanics



Linear elastic fracture mechanics



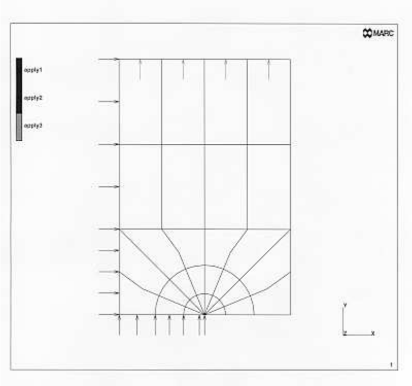
Dynamic fracture mechanics

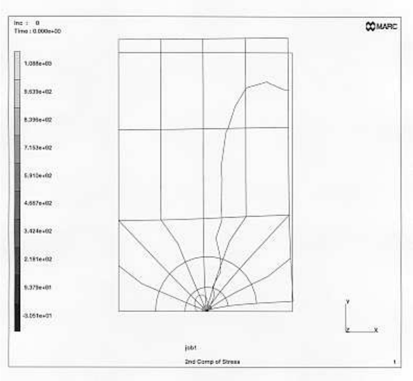


Nonlinear fracture mechanics

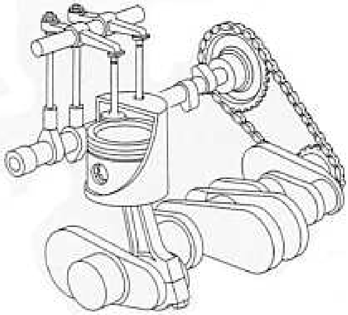
- CTOD
- J-integral

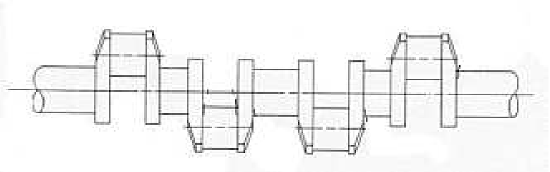
Numerical techniques





Fatigue





Objectives

Insight in :

- crack growth mechanisms
- brittle / ductile
- energy balance
- crack tip stresses
- crack growth direction
- plastic crack tip zone
- crack growth speed
- nonlinear fracture mechanics
- numerical methods
- fatigue

FRACTURE MECHANISMS

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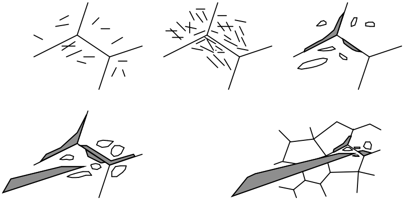
Fracture mechanisms

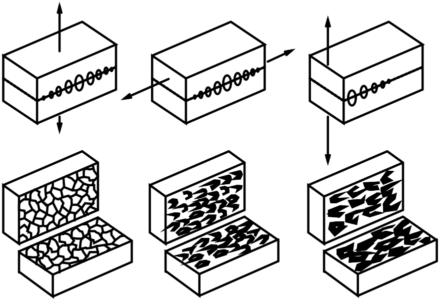
- shear fracture
- cleavage fracture
- fatigue fracture
- crazing
- de-adhesion

Shearing

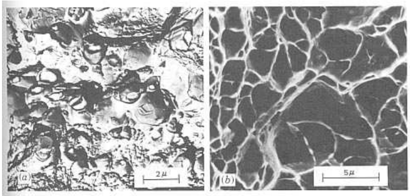
dislocations $\longrightarrow$ voids $\longrightarrow$ crack



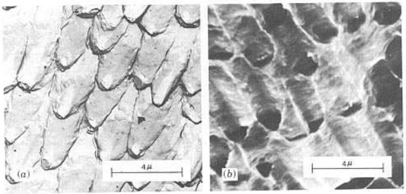
dimples $\longrightarrow$ load direction



Dimples

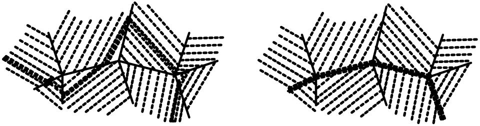


TEM **SEM**



TEM **SEM**

Cleavage



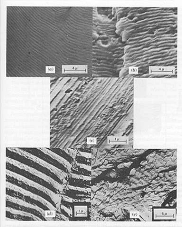
intra-granulair inter-granulair

- intra-granular
  - HCP-, BCC-crystal
  - T low
  - $\dot{\varepsilon}$ high
  - 3D-stress state
- inter-granular
  - weak grain boundary
  - environment (H_2)
  - T high

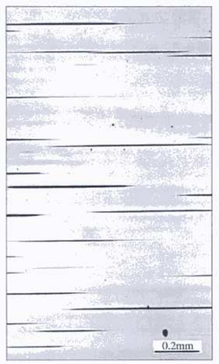
Fatigue

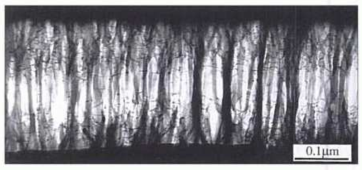
clam shell pattern

striations



Crazing





- stress whitening
- crazing materials : PS, PMMA

DUCTILE/BRITTLE

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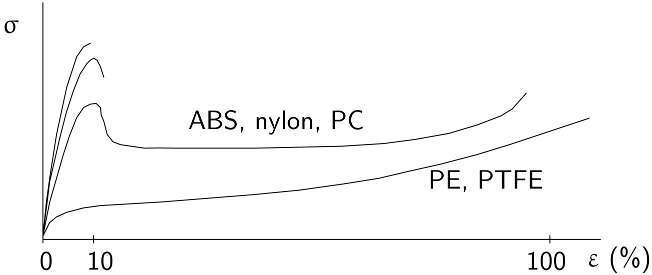
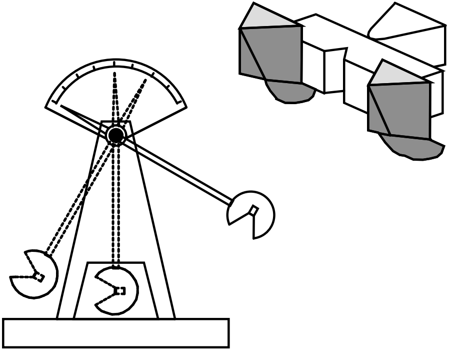
Ductile - brittle behavior

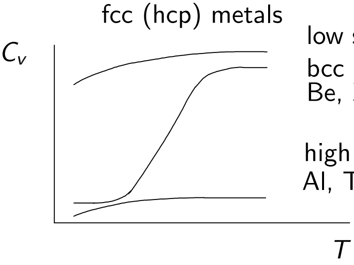


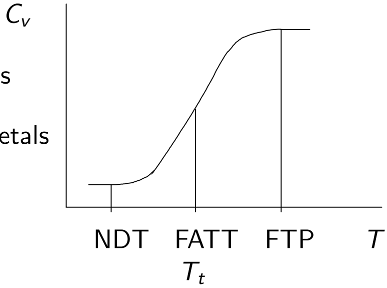
- surface energy : γ $[Jm^{-2}]$
  solids : $\gamma \approx 1$ $[Jm^{-2}]$
- independent from cleavage/shearing
- ex.: alloyed steels; rubber

Charpy v-notch test



Charpy Cv-value





-	Impact Toughness	C_v
-	Nil Ductility Temperature	NDT
-	Nil Fracture Appearance Transition Temperature	$FATT\,(T_t)$
-	Nil Fracture Transition Plastic	FTP

THEORETICAL STRENGTH

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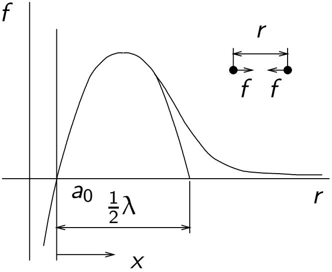
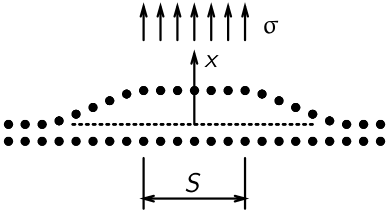
Theoretical strength





$$f(x) = f_{max} \sin\left(\frac{2\pi x}{\lambda}\right) \quad ; \quad x = r - a_0$$

$$\sigma(x) = \frac{1}{S}\sum f(x) = \sigma_{max} \sin\left(\frac{2\pi x}{\lambda}\right)$$

Energy balance

available elastic energy per surface-unity [N m^{-1}]

$$\begin{aligned} U_i &= \frac{1}{S}\int_{x=0}^{x=\lambda/2} \sum f(x)\, dx \\ &= \int_{x=0}^{x=\lambda/2} \sigma_{max} \sin\left(\frac{2\pi x}{\lambda}\right) dx \\ &= \sigma_{max}\frac{\lambda}{\pi} \qquad [\mathrm{Nm}^{-1}] \end{aligned}$$

required surface energy

$$U_a = 2\gamma \qquad [\mathrm{Nm}^{-1}]$$

energy balance at fracture

$$U_i = U_a \qquad \rightarrow \qquad \lambda = \frac{2\pi\gamma}{\sigma_{max}} \qquad \rightarrow$$

$$\sigma = \sigma_{max} \sin\left(\frac{x}{\gamma}\sigma_{max}\right)$$

Approximations

linearization

$$\sigma = \sigma_{max} \sin\left(\frac{x}{\gamma}\sigma_{max}\right) \approx \frac{x}{\gamma}\,\sigma^2_{max}$$

linear strain of atomic bond

$$\varepsilon = \frac{x}{a_0} \qquad \rightarrow \qquad x = \varepsilon a_0 \quad \rightarrow \quad \sigma = \frac{\varepsilon a_0}{\gamma}\sigma^2_{max}$$

elastic modulus

$$E = \left(\frac{d\sigma}{d\varepsilon}\right)\bigg|_{x=0} = \left(\frac{d\sigma}{dx}\,a_0\right)\bigg|_{x=0} = \sigma^2_{max}\,\frac{a_0}{\gamma} \qquad \rightarrow$$

$$\sigma_{max} = \sqrt{\frac{E\gamma}{a_0}}$$

theoretical strength $\qquad \sigma_{th} = \sqrt{\dfrac{E\gamma}{a_0}}$

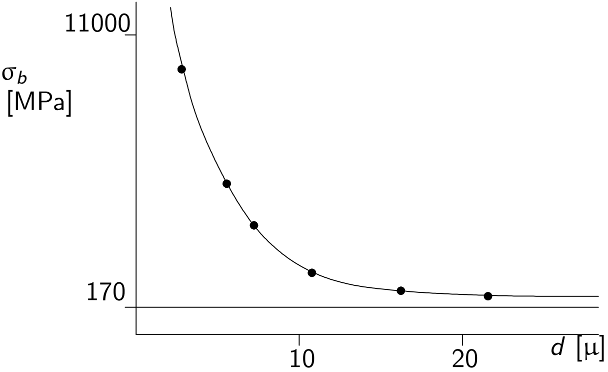
Discrepancy with experimental observations

	a_0 [m]	E [GPa]	σ_{th} [GPa]	σ_b [MPa]	σ_{th}/σ_b
glass	$3 * 10^{-10}$	60	14	170	82
steel	10^{-10}	210	45	250	180
silica fibers	10^{-10}	100	31	25000	1.3
iron whiskers	10^{-10}	295	54	13000	4.2
silicon whiskers	10^{-10}	165	41	6500	6.3
alumina whiskers	10^{-10}	495	70	15000	4.7
ausformed steel	10^{-10}	200	45	3000	15
piano wire	10^{-10}	200	45	2750	16.4

discrepancy with experiments

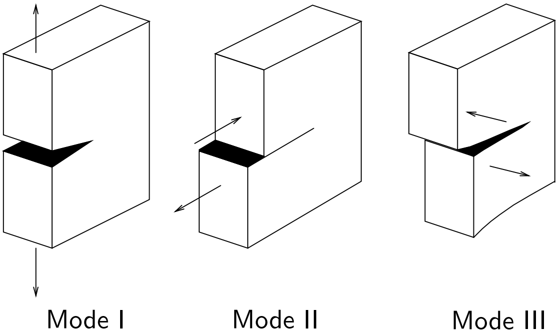
$$\boxed{\sigma_{th} \gg \sigma_b}$$

Griffith's experiments



DEFECTS $\rightarrow$ FRACTURE MECHANICS

Crack loading modes



Mode I	=	opening mode
Mode II	=	sliding mode
Mode III	=	tearing mode

EXPERIMENTAL TECHNIQUES

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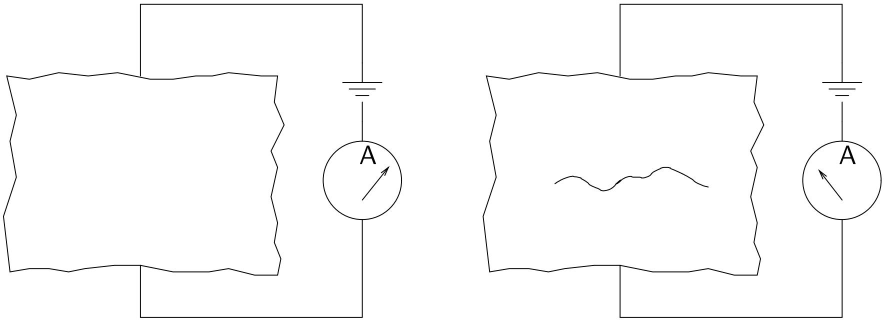
Surface cracks

- dye penetration
  - small surface cracks
  - fast and cheap
  - on-site
- magnetic particles
  - cracks $\longrightarrow$ disturbance of magnetic field
  - surface cracks
  - for magnetic materials only
- eddy currents
  - impedance change of a coil
  - penetration depth : a few mm's
  - difficult interpretation

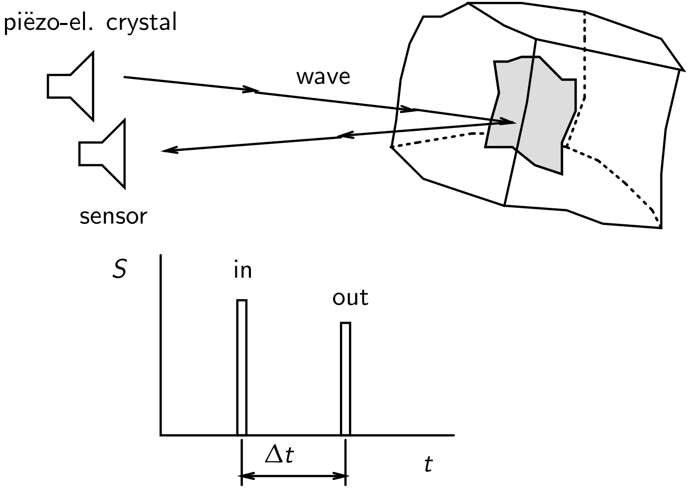
Electrical resistance



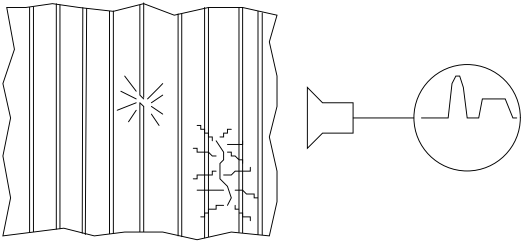
X-ray

orientation dependency

Ultrasound

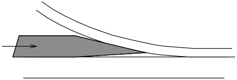


Acoustic emission

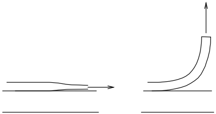


registration "intern" sounds (hits)

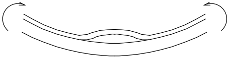
Adhesion tests



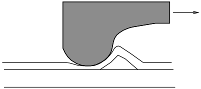
blade wedge test



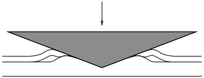
peel test (0^o and 90^o)



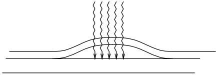
bending test



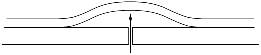
scratch test



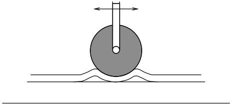
indentation test



laser blister test



pressure blister test



fatigue friction test

ENERGY BALANCE

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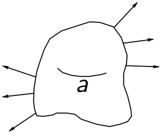
Energy balance



$B =$ thickness
$A = Ba$

$\dot{B} = 0$

$$\dot{U}_e = \dot{U}_i + \dot{U}_a + \dot{U}_d + \dot{U}_k \qquad [\mathrm{Js}^{-1}]$$

$$\frac{d}{dt}(\ ) = \frac{dA}{dt}\frac{d}{dA}(\ ) = \dot{A}\frac{d}{dA}(\ ) = \dot{a}\frac{d}{da}(\ )$$

$$\frac{dU_e}{da} = \frac{dU_i}{da} + \frac{dU_a}{da} + \frac{dU_d}{da} + \frac{dU_k}{da} \qquad [\mathrm{Jm}^{-1}]$$

$$\boxed{\frac{dU_e}{da} - \frac{dU_i}{da} = \frac{dU_a}{da} + \frac{dU_d}{da} + \frac{dU_k}{da}} \qquad [\mathrm{Jm}^{-1}]$$

Griffith's energy balance

- no dissipation
- no kinetic energy

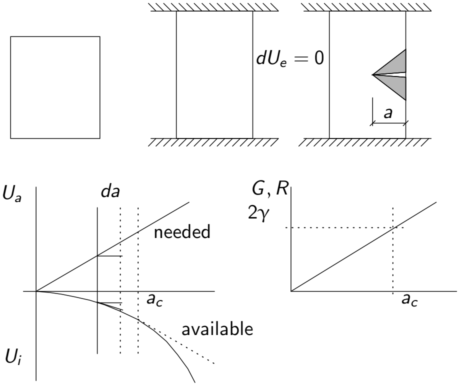
energy balance
$$\frac{dU_e}{da} - \frac{dU_i}{da} = \frac{dU_a}{da}$$

energy release rate
$$G = \frac{1}{B}\left(\frac{dU_e}{da} - \frac{dU_i}{da}\right) \qquad [\mathrm{Jm}^{-2}]$$

crack resistance force
$$R = \frac{1}{B}\left(\frac{dU_a}{da}\right) = 2\gamma \qquad [\mathrm{Jm}^{-2}]$$

Griffith's crack criterion
$$\boxed{G = R = 2\gamma} \qquad [\mathrm{Jm}^{-2}]$$

Griffith's energy balance

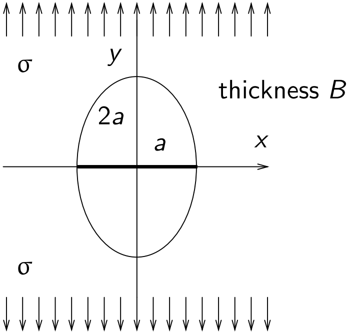


$$-\frac{dU_i}{da} < \frac{dU_a}{da} \quad \rightarrow \quad \text{no crack growth}$$

$$-\frac{dU_i}{da} > \frac{dU_a}{da} \quad \rightarrow \quad \text{unstable crack growth}$$

$$-\frac{dU_i}{da} = \frac{dU_a}{da} \quad \rightarrow \quad \text{critical crack length}$$

Griffith stress



$$U_i = 2\pi a^2 B\, \tfrac{1}{2} \frac{\sigma^2}{E} \quad ; \quad U_a = 4aB\,\gamma \qquad [\mathrm{Nm} = \mathrm{J}]$$

$$G = -\frac{1}{B}\left(\frac{dU_i}{da}\right) = \frac{1}{B}\left(\frac{dU_a}{da}\right) = R \quad \rightarrow \quad 2\pi a \frac{\sigma^2}{E} = 4\gamma \qquad [\mathrm{Jm}^{-2}]$$

Griffith stress $\sigma_{gr} = \sqrt{\frac{2\gamma E}{\pi a}}$; critical crack length $a_c = \frac{2\gamma E}{\pi \sigma^2}$

Griffith stress: plane stress

$$\sigma_{gr} = \sqrt{\frac{2\gamma E}{(1-\nu^2)\pi a}}$$

Discrepancy with experimental observations

$$\boxed{\sigma_{gr} \ll \sigma_c}$$

reason	neglection of dissipation
remedy	measure critical energy release rate G_c

glass	$G_c = 6$	$[\mathrm{Jm}^{-2}]$
wood	$G_c = 10^4$	$[\mathrm{Jm}^{-2}]$
steel	$G_c = 10^5$	$[\mathrm{Jm}^{-2}]$
composite		

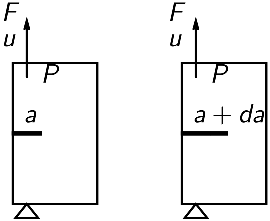
design problem / high alloyed steel / bone (elephant and mouse)

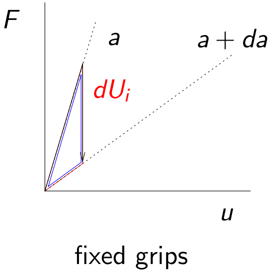
energy balance $$G = \frac{1}{B}\left(\frac{dU_e}{da} - \frac{dU_i}{da}\right) = R = G_c$$

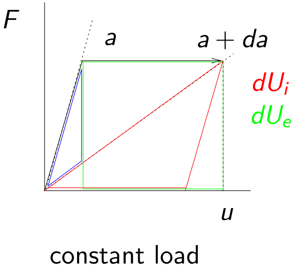
critical crack length $a_c = \dfrac{G_c E}{2\pi\sigma^2}$; Griffith's crack criterion $G = G_c$

Compliance change

compliance : $C = u/F$







Compliance change : Fixed grips

fixed grips : $$dU_e = 0$$

$$\begin{aligned} dU_i &= U_i(a+da) - U_i(a) \qquad\qquad (<0) \\ &= \tfrac{1}{2}(F+dF)u - \tfrac{1}{2}Fu \\ &= \tfrac{1}{2}udF \end{aligned}$$

Griffith's energy balance

$$\begin{aligned} G &= -\frac{1}{2B}u\frac{dF}{da} = \frac{1}{2B}\,\frac{u^2}{C^2}\,\frac{dC}{da} \\ &= \frac{1}{2B}\,F^2\,\frac{dC}{da} \end{aligned}$$

Compliance change : Constant load

constant load

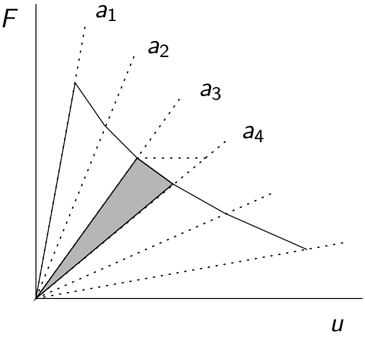
$$dU_e = U_e(a + da) - U_e(a) = Fdu$$

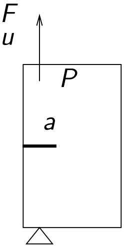
$$\begin{aligned} dU_i &= U_i(a + da) - U_i(a) \qquad\qquad (> 0) \\ &= \tfrac{1}{2}F(u + du) - \tfrac{1}{2}Fu \\ &= \tfrac{1}{2}Fdu \end{aligned}$$

Griffith's energy balance

$$\begin{aligned} G &= \frac{1}{2B}\,F\,\frac{du}{da} \\ &= \frac{1}{2B}\,F^2\,\frac{dC}{da} \end{aligned}$$

Compliance change : Experiment

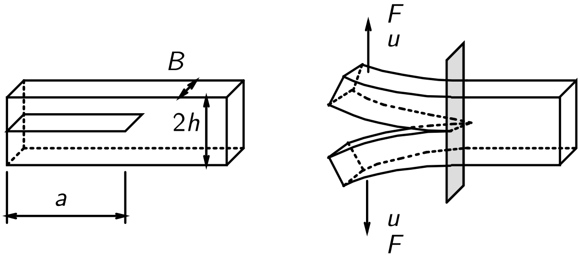




$$G = \frac{\text{shaded area}}{a_4 - a_3} \frac{1}{B}$$

no *fixed grips* AND no *constant load* BUT small deviation !!

Example

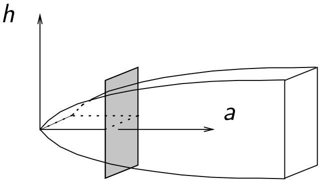


$$u = \frac{Fa^3}{3EI} = \frac{4Fa^3}{EBh^3} \qquad \rightarrow \qquad C = \frac{\Delta u}{F} = \frac{2u}{F} = \frac{8a^3}{EBh^3} \qquad \rightarrow \qquad \frac{dC}{da} = \frac{24a^2}{EBh^3}$$

$$G = \frac{1}{B}\left[\frac{1}{2}F^2\frac{dC}{da}\right] = \frac{12F^2a^2}{EB^2h^3} \qquad [\mathrm{J\ m^{-2}}]$$

$$G_c = 2\gamma \qquad \rightarrow \qquad F_c = \frac{B}{a}\sqrt{\tfrac{1}{6}\gamma Eh^3}$$

Example



question : which $h(a)$ makes $\frac{dC}{da}$ independent from a ?

$$C = \frac{\Delta u}{F} = \frac{2u}{F} = \frac{8a^3}{EBh^3} \qquad \rightarrow \qquad \frac{dC}{da} = \frac{24a^2}{EBh^3}$$

$$\text{choice} \quad : \quad h = h_0 a^n \quad \rightarrow$$

$$u = \frac{Fa^3}{3(1-n)EI} = \frac{4Fa^3}{(1-n)EBh^3} = \frac{4Fa^{3(1-n)}}{(1-n)EBh_0^3}$$

$$C = \frac{2u}{F} = \frac{8a^{3(1-n)}}{(1-n)EBh_0^3} \qquad \rightarrow \qquad \frac{dC}{da} = \frac{24a^{(2-3n)}}{EBh_0^3}$$

$$\frac{dC}{da} \quad \text{constant for} \quad n = \tfrac{2}{3} \qquad \rightarrow \qquad h = h_0 a^{\frac{2}{3}}$$

LINEAR ELASTIC STRESS ANALYSIS

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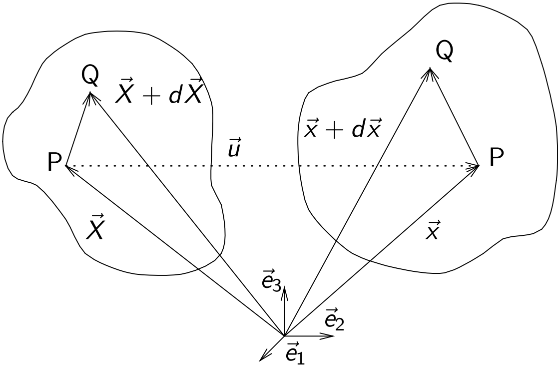
Deformation



$$
\begin{aligned}
x_i &= X_i + u_i(X_i) \\
x_i + dx_i &= X_i + dX_i + u_i(X_i + dX_i) = X_i + dX_i + u_i(X_i) + u_{i,j} dX_j dx_i \\
&= dX_i + u_{i,j} dX_j = (\delta_{ij} + u_{i,j}) dX_j \\
ds &= \|d\vec{x}\| = \sqrt{dx_i dx_i} \qquad ; \qquad dS = \|d\vec{X}\| = \sqrt{dX_i dX_i}
\end{aligned}
$$

Strains

$$
\begin{aligned}
ds^2 &= dx_i dx_i = [(\delta_{ij} + u_{i,j})dX_j][(\delta_{ik} + u_{i,k})dX_k] \\
&= (\delta_{ij}\delta_{ik} + \delta_{ij}u_{i,k} + u_{i,j}\delta_{ik} + u_{i,j}u_{i,k})dX_j dX_k \\
&= (\delta_{jk} + u_{j,k} + u_{k,j} + u_{i,j}u_{i,k})dX_j dX_k \\
&= (\delta_{ij} + u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dX_i dX_j \\
&= dX_i dX_i + (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dX_i dX_j \\
&= dS^2 + (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dX_i dX_j
\end{aligned}
$$

$$
\begin{aligned}
ds^2 - dS^2 &= (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})dX_i dX_j \\
&= 2\gamma_{ij} dX_i dX_j
\end{aligned}
$$

Green-Lagrange strains $\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$

linear strains $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

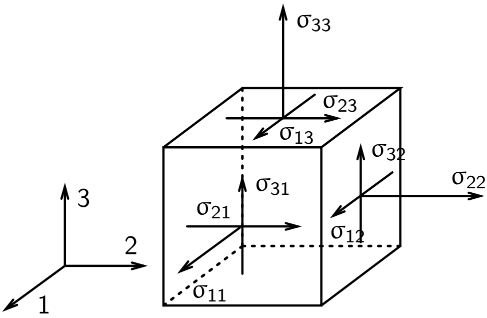
Compatibility

3 displacement components $\longrightarrow$ 9 strain components $\longrightarrow$ 6 dependencies $\longrightarrow$ 6 compatibility equations

$$
\begin{aligned}
&2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0 \\
&2\varepsilon_{23,23} - \varepsilon_{22,33} - \varepsilon_{33,22} = 0 \\
&2\varepsilon_{31,31} - \varepsilon_{33,11} - \varepsilon_{11,33} = 0 \\
&\varepsilon_{11,23} + \varepsilon_{23,11} - \varepsilon_{31,12} - \varepsilon_{12,13} = 0 \\
&\varepsilon_{22,31} + \varepsilon_{31,22} - \varepsilon_{12,23} - \varepsilon_{23,21} = 0 \\
&\varepsilon_{33,12} + \varepsilon_{12,33} - \varepsilon_{23,31} - \varepsilon_{31,32} = 0
\end{aligned}
$$

Stress

unity normal vector	$\vec{n} = n_i \vec{e}_i$
stress vector	$\vec{p} = p_i \vec{e}_i$
Cauchy stress components	$p_i = \sigma_{ij} n_j$
stress cube	



Linear elastic material behavior

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{lk}$$

material symmetry $\quad\rightarrow\quad$ isotropic material $\quad\rightarrow\quad$ 2 mat.pars

Hooke's law for isotropic materials

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\varepsilon_{kk}\right) \qquad i = 1, 2, 3$$

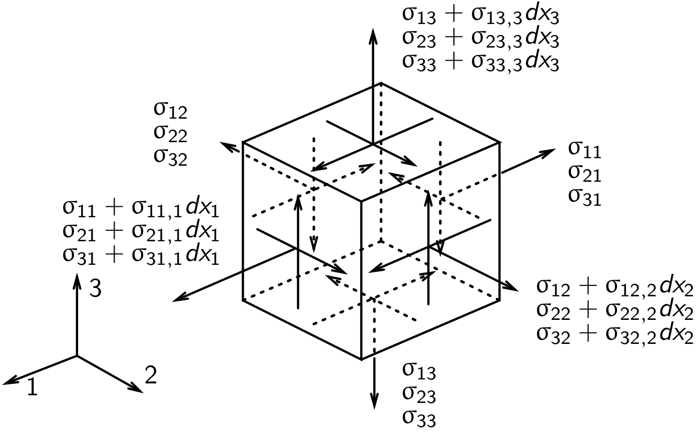
$$\varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\sigma_{kk}\right) \qquad i = 1, 2, 3$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix} = \alpha \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix}$$

$$\alpha = E/[(1+\nu)(1-2\nu)]$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

Equilibrium equations



volume load	ρq_i	
force equilibrium	$\sigma_{ij,j} + \rho q_i = 0$	$i = 1, 2, 3$
moment equilibrium	$\sigma_{ij} = \sigma_{ji}$	

Plane stress $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$

equilibrium ($q_i = 0$)	$\sigma_{11,1} + \sigma_{12,2} = 0 \quad ; \quad \sigma_{21,1} + \sigma_{22,2} = 0$
compatibility	$2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0$
Hooke's law	

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-\nu}\,\delta_{ij}\varepsilon_{kk}\right) \; ; \; \varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\,\delta_{ij}\sigma_{kk}\right) \qquad i = 1,2$$

Hooke's law in matrix notation

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix}\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{E}\left(\sigma_{11} + \sigma_{22}\right) = -\frac{\nu}{1-\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right)$$

$$\varepsilon_{13} = \varepsilon_{23} = 0$$

Plane strain $\quad \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$

equilibrium ($q_i = 0$) $\quad \sigma_{11,1} + \sigma_{12,2} = 0 \quad ; \quad \sigma_{21,1} + \sigma_{22,2} = 0$

compatibility $\quad 2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0$

Hooke's law

$$\varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) \;;\; \sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \qquad i = 1,2$$

Hooke's law in matrix notation

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \frac{1+\nu}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

$$\sigma_{33} = \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11} + \varepsilon_{22}\right) = \nu\left(\sigma_{11} + \sigma_{22}\right)$$

$$\sigma_{13} = \sigma_{23} = 0$$

Displacement method

$$\left.\begin{aligned} &\sigma_{ij,j} = 0 \\ &\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\,\delta_{ij}\,\varepsilon_{kk}\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} &\frac{E}{1+\nu}\left(\varepsilon_{ij,j} + \frac{\nu}{1-2\nu}\,\delta_{ij}\varepsilon_{kk,j}\right) = 0 \\ &\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} &\frac{E}{1+\nu}\,\tfrac{1}{2}\,\left(u_{i,jj} + u_{j,ij}\right) + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}u_{k,kj} = 0 \\ &\text{BC's} \end{aligned}\right\}$$

$$u_i \quad \longrightarrow \quad \varepsilon_{ij} \quad \longrightarrow \quad \sigma_{ij}$$

Stress function method

$$\left.\begin{array}{l} \psi(x_1,x_2) \quad \rightarrow \quad \sigma_{ij} = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \quad \rightarrow \quad \sigma_{ij,j} = 0 \\ \varepsilon_{ij} = \dfrac{1+\nu}{E}\left(\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right) \end{array}\right\}$$

$$\left.\begin{array}{l} \varepsilon_{ij} = \dfrac{1+\nu}{E}\{-\psi_{,ij} + (1-\nu)\delta_{ij}\psi_{,kk}\} \\ 2\varepsilon_{12,12} - \varepsilon_{11,22} - \varepsilon_{22,11} = 0 \end{array}\right\}$$

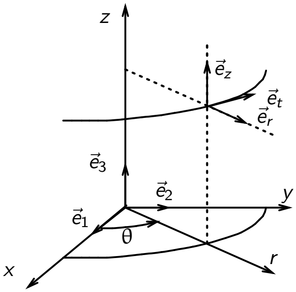
$$2\psi_{,1122} + \psi_{,2222} + \psi_{,1111} = 0 \qquad \rightarrow$$

$$\left.\begin{array}{l} (\psi_{,11} + \psi_{,22})_{,11} + (\psi_{,11} + \psi_{,22})_{,22} = 0 \\ \text{Laplace operator} \quad : \quad \nabla^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} = (\ )_{11} + (\ )_{22} \end{array}\right\} \quad \rightarrow$$

$$\left.\begin{array}{ll} \text{bi-harmonic equation} & \nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \\ \text{BC's} & \end{array}\right\}$$

$$\psi \quad \rightarrow \quad \sigma_{ij} \quad \rightarrow \quad \varepsilon_{ij} \quad \rightarrow \quad u_i$$

Cylindrical coordinates

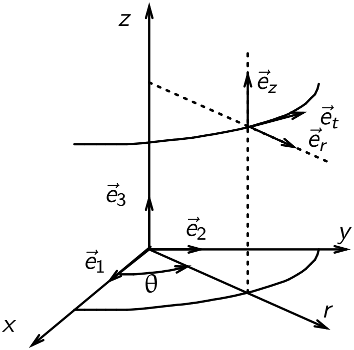


vector bases $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \quad \rightarrow \quad \{\vec{e}_r, \vec{e}_t, \vec{e}_z\}$

$$\begin{aligned} \vec{e}_r &= \vec{e}_r(\theta) = \vec{e}_1 \cos\theta + \vec{e}_2 \sin\theta \\ \vec{e}_t &= \vec{e}_t(\theta) = -\vec{e}_1 \sin\theta + \vec{e}_2 \cos\theta \end{aligned}$$

$$\frac{\partial}{\partial\theta}\{\vec{e}_r(\theta)\} = \vec{e}_t(\theta) \qquad ; \qquad \frac{\partial}{\partial\theta}\{\vec{e}_t(\theta)\} = -\vec{e}_r(\theta)$$

Laplace operator



gradient operator
$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_t \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z}$$

Laplace operator
$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

two-dimensional
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Bi-harmonic equation

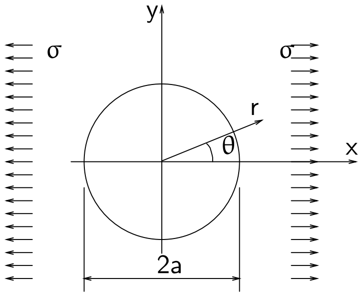
bi-harmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right) = 0$$

stress components

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \\
\sigma_{tt} &= \frac{\partial^2 \psi}{\partial r^2} \\
\sigma_{rt} &= \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial r \partial \theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)
\end{aligned}$$

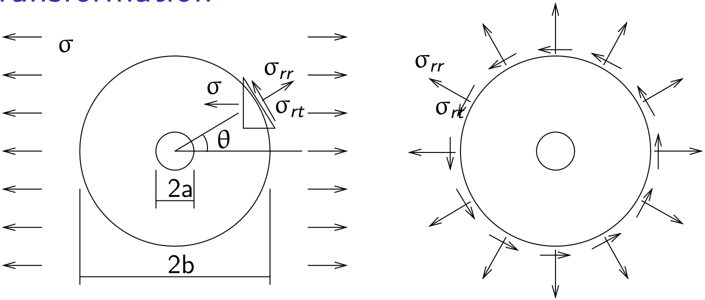
Circular hole in 'infinite' plate



$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \theta^2}\right)=0$$

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\;;\quad \sigma_{tt}=\frac{\partial^2\psi}{\partial r^2}\;;\quad \sigma_{rt}=\frac{1}{r^2}\frac{\partial\psi}{\partial\theta}-\frac{1}{r}\frac{\partial\psi}{\partial r\partial\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)$$

Load transformation



equilibrium

$$\sigma_{rr}(r=b,\theta) = \tfrac{1}{2}\sigma + \tfrac{1}{2}\sigma\cos(2\theta)$$
$$\sigma_{rt}(r=b,\theta) = -\tfrac{1}{2}\sigma\sin(2\theta)$$

two load cases

$$\begin{array}{ll} I. & \sigma_{rr}(r=a) = \sigma_{rt}(r=a) = 0 \\ & \sigma_{rr}(r=b) = \tfrac{1}{2}\sigma \quad ; \quad \sigma_{rt}(r=b) = 0 \\ II. & \sigma_{rr}(r=a) = \sigma_{rt}(r=a) = 0 \\ & \sigma_{rr}(r=b) = \tfrac{1}{2}\sigma\cos(2\theta) \quad ; \quad \sigma_{rt}(r=b) = -\tfrac{1}{2}\sigma\sin(2\theta) \end{array}$$

Load case I

$$\sigma_{rr}(r=a) = \sigma_{rt}(r=a) = 0$$
$$\sigma_{rr}(r=b) = \tfrac{1}{2}\sigma \quad ; \quad \sigma_{rt}(r=b) = 0$$

Airy function $\qquad \psi = f(r)$

stress components

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = \frac{1}{r}\frac{df}{dr} \; ; \quad \sigma_{tt} = \frac{\partial^2\psi}{\partial r^2} = \frac{d^2 f}{dr^2} \; ; \quad \sigma_{rt} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right) = 0$$

bi-harmonic equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) = 0$$

Solution

general solution $\quad \psi(r) = A \ln r + Br^2 \ln r + Cr^2 + D$

stresses

$$\sigma_{rr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C$$
$$\sigma_{tt} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C$$

strains (from Hooke's law for plane stress)

$$\varepsilon_{rr} = \frac{1}{E}\left[\frac{A}{r^2}(1+\nu) + B\{(1-3\nu) + 2(1-\nu)\ln r\} + 2C(1-\nu)\right]$$
$$\varepsilon_{tt} = \frac{1}{E}\frac{1}{r}\left[-\frac{A}{r}(1+\nu) + B\{(3-\nu)r + 2(1-\nu)r\ln r\} + 2C(1-\nu)r\right]$$

compatibility $\quad \varepsilon_{rr} = \frac{du}{dr} = \frac{d(r\,\varepsilon_{tt})}{dr} \quad \rightarrow \quad B = 0$

2 BC's and $b \gg a \quad \rightarrow \quad A$ and $C \quad \rightarrow$

$$\sigma_{rr} = \tfrac{1}{2}\sigma(1 - \frac{a^2}{r^2}) \quad ; \quad \sigma_{tt} = \tfrac{1}{2}\sigma(1 + \frac{a^2}{r^2}) \quad ; \quad \sigma_{rt} = 0$$

Load case II

$$\sigma_{rr}(r=a) = \sigma_{rt}(r=a) = 0$$
$$\sigma_{rr}(r=b) = \tfrac{1}{2}\sigma\cos(2\theta) \quad ; \quad \sigma_{rt}(r=b) = -\tfrac{1}{2}\sigma\sin(2\theta)$$

Airy function $\qquad \psi(r,\theta) = g(r)\cos(2\theta)$

stress components

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \quad ; \quad \sigma_{tt} = \frac{\partial^2\psi}{\partial r^2}$$
$$\sigma_{rt} = \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)$$

bi-harmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left\{\left(\frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{4}{r^2}g\right)\cos(2\theta)\right\} = 0 \qquad \rightarrow$$
$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{4}{r^2}g\right)\cos(2\theta) = 0$$

Solution

general solution
$$g = Ar^2 + Br^4 + C\frac{1}{r^2} + D \quad \rightarrow$$

$$\psi = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos(2\theta)$$

stresses
$$\sigma_{rr} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos(2\theta)$$

$$\sigma_{tt} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos(2\theta)$$

$$\sigma_{rt} = \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin(2\theta)$$

4 BC's and $b \gg a \quad \rightarrow \quad A, B, C$ and $D \quad \rightarrow$

$$\sigma_{rr} = \tfrac{1}{2}\sigma\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos(2\theta)$$

$$\sigma_{tt} = -\tfrac{1}{2}\sigma\left(1 + \frac{3a^4}{r^4}\right)\cos(2\theta)$$

$$\sigma_{rt} = -\tfrac{1}{2}\sigma\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin(2\theta)$$

Stresses for total load

$$\sigma_{rr} = \frac{\sigma}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos(2\theta)\right]$$

$$\sigma_{tt} = \frac{\sigma}{2}\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos(2\theta)\right]$$

$$\sigma_{rt} = -\frac{\sigma}{2}\left[1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right]\sin(2\theta)$$

Special points

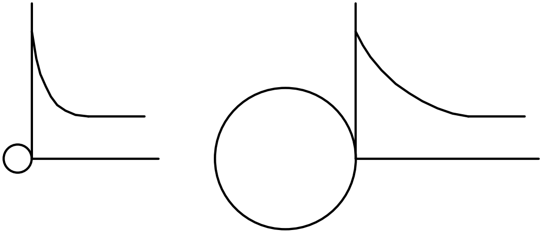
$$\sigma_{rr}(r=a,\theta) = \sigma_{rt}(r=a,\theta) = \sigma_{rt}(r,\theta=0) = 0$$
$$\sigma_{tt}(r=a,\theta=\tfrac{\pi}{2}) = 3\sigma$$
$$\sigma_{tt}(r=a,\theta=0) = -\sigma$$

stress concentration factor

$$K_t = \frac{\sigma_{max}}{\sigma} = 3 \qquad [-]$$

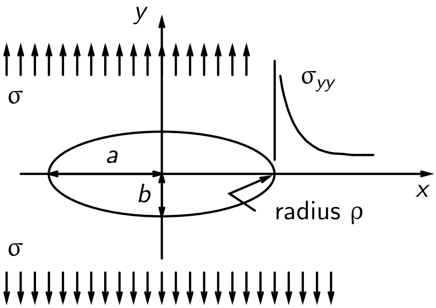
K_t is independent of hole diameter !

Stress gradients



large hole : smaller stress gradient $\longrightarrow$
larger area with higher stress $\longrightarrow$
higher chance for critical defect in high stress area

Elliptical hole



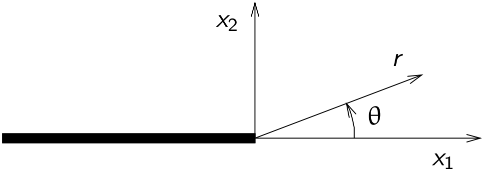
$$\sigma_{yy}(x=a, y=0) = \sigma\left(1+2\frac{a}{b}\right) = \sigma\left(1+2\sqrt{a/\rho}\right)$$
$$\approx 2\sigma\sqrt{a/\rho}$$

stress concentration factor $\quad K_t = 2\sqrt{a/\rho} \quad$ [-]

CRACK TIP STRESS

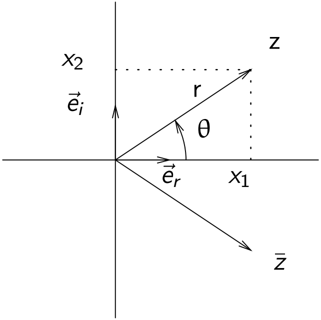
back to index

Complex plane



crack tip = singular point $\quad \longrightarrow$
complex function theory $\quad \longrightarrow$
complex Airy function $\qquad$ (Westergaard, 1939)

Complex variables



$$z = x_1 + ix_2 = re^{i\theta} \qquad ; \qquad \bar{z} = x_1 - ix_2 = re^{-i\theta}$$

$$x_1 = \frac{1}{2}(z + \bar{z}) \qquad ; \qquad x_2 = \frac{1}{2i}(z - \bar{z}) = -\frac{1}{2}i(z - \bar{z})$$

$$\vec{z} = x_1\vec{e}_r + x_2\vec{e}_i = x_1\vec{e}_r + x_2 i\vec{e}_r = (x_1 + ix_2)\vec{e}_r$$

Complex functions

complex function

$$\left.\begin{array}{l} f(z) = \phi + i\zeta = \phi(x_1, x_2) + i\zeta(x_1, x_2) = f \\[2ex] f(\bar{z}) = \phi(x_1, x_2) - i\zeta(x_1, x_2) = \bar{f} \end{array}\right\} \quad \rightarrow$$

$$\phi = \tfrac{1}{2}\{f + \bar{f}\} \qquad ; \qquad \zeta = -\tfrac{1}{2}i\{f - \bar{f}\}$$

$$\nabla^2\phi = \nabla^2\zeta = 0$$

appendix !!

Laplace operator

complex function $\qquad g(x_1, x_2) = g(z, \bar{z})$

Laplacian $\qquad \nabla^2 g = \dfrac{\partial^2 g}{\partial x_1^2} + \dfrac{\partial^2 g}{\partial x_2^2}$

derivatives $\qquad$ (see App. A)

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial z}\frac{\partial z}{\partial x_1} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_1} = \frac{\partial g}{\partial z} + \frac{\partial g}{\partial \bar{z}} \quad ; \quad \frac{\partial^2 g}{\partial x_1^2} = \frac{\partial^2 g}{\partial z^2} + 2\frac{\partial g}{\partial z \partial \bar{z}} + \frac{\partial^2 g}{\partial \bar{z}^2}$$

$$\frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial z}\frac{\partial z}{\partial x_2} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{z}}{\partial x_2} = i\frac{\partial g}{\partial z} - i\frac{\partial g}{\partial \bar{z}} \quad ; \quad \frac{\partial^2 g}{\partial x_2^2} = -\frac{\partial^2 g}{\partial z^2} + 2\frac{\partial g}{\partial z \partial \bar{z}} - \frac{\partial^2 g}{\partial \bar{z}^2}$$

Laplacian $\qquad \nabla^2 g = \dfrac{\partial^2 g}{\partial x_1^2} + \dfrac{\partial^2 g}{\partial x_2^2} = 4\dfrac{\partial g}{\partial z \partial \bar{z}} \qquad \rightarrow$

$$\nabla^2 = 4\frac{\partial}{\partial z \partial \bar{z}}$$

Bi-harmonic equation

Airy function $\psi(z, \bar{z})$

bi-harmonic equation $\nabla^2 \left(\nabla^2 \psi(z, \bar{z})\right) = 0$

Solution of bi-harmonic equation

real part of complex function f satisfies Laplace eqn.

$$\nabla^2 \left(\nabla^2 \psi(z,\bar{z})\right) = \nabla^2 \left(\phi(z,\bar{z})\right) = 0 \quad \rightarrow \quad \phi = f + \bar{f}$$

choice Airy function

$$\nabla^2 \psi = 4 \frac{\partial \psi}{\partial z \partial \bar{z}} = \phi = f + \bar{f}$$

integration

$$\boxed{\psi = \tfrac{1}{2}\left[\bar{z}\Omega + z\bar{\Omega} + \omega + \bar{\omega}\right]}$$

unknown functions : $\Omega \quad ; \quad \bar{\Omega} \quad ; \quad \omega \quad ; \quad \bar{\omega}$

Stresses

Airy function

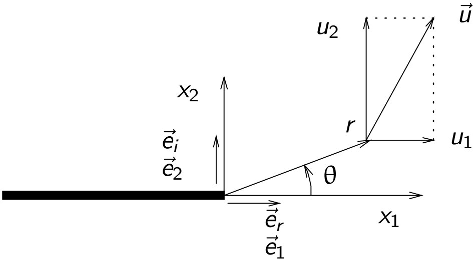
$$\psi = \frac{1}{2}\left[\bar{z}\Omega + z\bar{\Omega} + \omega + \bar{\omega}\right]$$

stress components

$$\sigma_{ij} = \sigma_{ij}(z,\bar{z}) = -\psi_{,ij} + \delta_{ij}\psi_{,kk} \qquad \rightarrow$$

$$\begin{aligned}
\sigma_{11} &= -\psi_{,11} + \psi_{,\gamma\gamma} = \psi_{,22} \\
&= \Omega' + \bar{\Omega}' - \frac{1}{2}\left\{\bar{z}\Omega'' + \omega'' + z\bar{\Omega}'' + \bar{\omega}''\right\} \\
\sigma_{22} &= -\psi_{,22} + \psi_{,\gamma\gamma} = \psi_{,11} \\
&= \Omega' + \bar{\Omega}' + \frac{1}{2}\left\{\bar{z}\Omega'' + \omega'' + z\bar{\Omega}'' + \bar{\omega}''\right\} \\
\sigma_{12} &= -\psi_{,12} \\
&= -\frac{1}{2}i\left\{\bar{z}\Omega'' + \omega'' - z\bar{\Omega}'' - \bar{\omega}''\right\}
\end{aligned}$$

Displacement



definition of complex displacement

$$
\begin{aligned}
\vec{u} &= u_1\vec{e}_1 + u_2\vec{e}_2 = u_1\vec{e}_r + u_2\vec{e}_i \\
&= u_1\vec{e}_r + u_2 i\vec{e}_r = (u_1 + iu_2)\vec{e}_r \\
&= u\vec{e}_r \qquad \longrightarrow \\
u &= u_1 + iu_2 = u_1(x_1, x_2) + iu_2(x_1, x_2) = u(z, \bar{z}) \\
\bar{u} &= u_1 - iu_2 = \bar{u}(z, \bar{z})
\end{aligned}
$$

Schematic

$$\frac{\partial u}{\partial \bar{z}} \quad \to \quad u(z,\bar{z}) \qquad \text{with int."const."} \qquad M(z) \quad \to$$

$$\left.\begin{array}{l} \left.\begin{array}{ll} \to & \dfrac{\partial u}{\partial z} \\[2em] \to & \dfrac{\partial \bar{u}}{\partial \bar{z}} \end{array}\right\} \quad \to \quad \dfrac{\partial u}{\partial z} + \dfrac{\partial \bar{u}}{\partial \bar{z}} \sim M(z)\ ,\ \bar{M}(\bar{z}) \\[3em] \dfrac{\partial u}{\partial z} + \dfrac{\partial \bar{u}}{\partial \bar{z}} = \varepsilon_{11} + \varepsilon_{22} \sim \sigma_{11}\ ,\ \sigma_{22} \end{array}\right\} \quad \to$$

$$M(z) \quad \to \quad u(z,\bar{z})$$

Displacement derivatives

$$
\begin{aligned}
\frac{\partial u}{\partial \bar{z}} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial \bar{z}} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial \bar{z}} = \tfrac{1}{2}\left\{\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2}\right\} \\
&= \tfrac{1}{2}\left\{\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} + i\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right\} = \tfrac{1}{2}\left(\varepsilon_{11} - \varepsilon_{22} + 2i\varepsilon_{12}\right) \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial z} = \tfrac{1}{2}\left\{\frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2}\right\} \\
&= \tfrac{1}{2}\left\{\frac{\partial u_1}{\partial x_1} + i\frac{\partial u_2}{\partial x_1} - i\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right\} = \tfrac{1}{2}\left\{\varepsilon_{11} + \varepsilon_{22} + i\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right\} \\
\frac{\partial \bar{u}}{\partial z} &= \frac{\partial \bar{u}}{\partial x_1}\frac{\partial x_1}{\partial z} + \frac{\partial \bar{u}}{\partial x_2}\frac{\partial x_2}{\partial z} = \tfrac{1}{2}\left\{\frac{\partial \bar{u}}{\partial x_1} - i\frac{\partial \bar{u}}{\partial x_2}\right\} \\
&= \tfrac{1}{2}\left\{\frac{\partial u_1}{\partial x_1} - i\frac{\partial u_2}{\partial x_1} - i\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right\} = \tfrac{1}{2}\left(\varepsilon_{11} - \varepsilon_{22} - 2i\varepsilon_{12}\right) \\
\frac{\partial \bar{u}}{\partial \bar{z}} &= \frac{\partial \bar{u}}{\partial x_1}\frac{\partial x_1}{\partial \bar{z}} + \frac{\partial \bar{u}}{\partial x_2}\frac{\partial x_2}{\partial \bar{z}} = \tfrac{1}{2}\left\{\frac{\partial \bar{u}}{\partial x_1} + i\frac{\partial \bar{u}}{\partial x_2}\right\} \\
&= \tfrac{1}{2}\left\{\frac{\partial u_1}{\partial x_1} - i\frac{\partial u_2}{\partial x_1} + i\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right\} = \tfrac{1}{2}\left\{\varepsilon_{11} + \varepsilon_{22} - i\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right\}
\end{aligned}
$$

General solution

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\varepsilon_{11} - \varepsilon_{22} + 2i\varepsilon_{12}\right) \\ \\ \text{Hooke's law (pl.strain)} \end{array}\right\} \quad \rightarrow$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\,\frac{1+\nu}{E}\,\left[\sigma_{11} - \sigma_{22} + 2i\sigma_{12}\right]$$
$$= -\,\frac{1+\nu}{E}\,\left[z\bar{\Omega}'' + \bar{\omega}''\right]$$

Integration $\quad \rightarrow$

$$u = -\frac{1+\nu}{E}\left[z\bar{\Omega}' + \bar{\omega}' + M\right]$$

Integration function

$$u = -\frac{1+\nu}{E}\left[z\bar{\Omega}' + \bar{\omega}' + M\right] \quad \rightarrow \quad \frac{\partial u}{\partial z} = -\frac{1+\nu}{E}\left[\bar{\Omega}' + M'\right]$$

$$\bar{u} = -\frac{1+\nu}{E}\left[\bar{z}\Omega' + \omega' + \bar{M}\right] \quad \rightarrow \quad \frac{\partial \bar{u}}{\partial \bar{z}} = -\frac{1+\nu}{E}\left[\Omega' + \bar{M}'\right]$$

$$\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}} = -\frac{1+\nu}{E}\left[\bar{\Omega}' + \Omega' + M' + \bar{M}'\right]$$

$$\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}} = \varepsilon_{11} + \varepsilon_{22} = \frac{1+\nu}{E}\left[(1-2\nu)(\sigma_{11}+\sigma_{22})\right]$$

$$= \frac{(1+\nu)(1-2\nu)}{E}\,2\left[\Omega' + \bar{\Omega}'\right]$$

$$M' + \bar{M}' = -(3-4\nu)\left[\bar{\Omega}' + \Omega'\right] \quad \rightarrow \quad M = -(3-4\nu)\Omega = -\kappa\Omega$$

$$u = -\frac{1+\nu}{E}\left[z\bar{\Omega}' + \bar{\omega}' - \kappa\Omega\right]$$

Choice of complex functions

$$\left.\begin{aligned}\Omega &= (\alpha + i\beta) z^{\lambda+1} = (\alpha + i\beta) r^{\lambda+1} e^{i\theta(\lambda+1)} \\ \omega' &= (\gamma + i\delta) z^{\lambda+1} = (\gamma + i\delta) r^{\lambda+1} e^{i\theta(\lambda+1)}\end{aligned}\right\} \quad \rightarrow$$

$$\left.\begin{aligned}\bar{\Omega} &= (\alpha - i\beta) \bar{z}^{\lambda+1} = (\alpha - i\beta) r^{\lambda+1} e^{-i\theta(\lambda+1)} \\ \bar{\Omega}' &= (\alpha - i\beta)(\lambda+1) \bar{z}^{\lambda} = (\alpha - i\beta)(\lambda+1) r^{\lambda} e^{-i\theta\lambda} \\ \bar{\omega}' &= (\gamma - i\delta) \bar{z}^{\lambda+1} = (\gamma - i\delta) r^{\lambda+1} e^{-i\theta(\lambda+1)}\end{aligned}\right\} \quad \rightarrow$$

$$u = \frac{1}{2\mu} r^{\lambda+1} \left[\kappa(\alpha + i\beta) e^{i\theta(\lambda+1)} - (\alpha - i\beta)(\lambda+1) e^{i\theta(1-\lambda)} - (\gamma - i\delta) e^{-i\theta(\lambda+1)}\right]$$

with $\qquad \mu = \dfrac{E}{2(1+\nu)}$

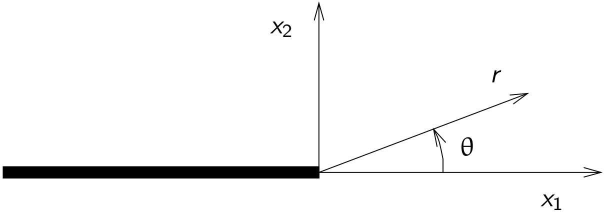
displacement finite $\quad \rightarrow \quad \boxed{\lambda > -1}$

Displacement components

$$\left.\begin{aligned}
&u = \frac{1}{2\mu} r^{\lambda+1} \left[\kappa(\alpha + i\beta) e^{i\theta(\lambda+1)} - (\alpha - i\beta)(\lambda+1) e^{i\theta(1-\lambda)} - (\gamma - i\delta) e^{-i\theta(\lambda+1)} \right] \\
&e^{i\theta} = \cos(\theta) + i\sin(\theta)
\end{aligned}\right\}$$

$$\begin{aligned}
u = {} & \frac{1}{2\mu} r^{\lambda+1} \\
& \Big[\quad \Big\{ \quad \kappa\alpha\cos(\theta(\lambda+1)) - \kappa\beta\sin(\theta(\lambda+1)) - \\
& \alpha(\lambda+1)\cos(\theta(1-\lambda)) - \beta(\lambda+1)\sin(\theta(1-\lambda)) - \\
& \gamma\cos(\theta(\lambda+1)) + \delta\sin(\theta(\lambda+1)) \quad \Big\} + \\
& i \quad \Big\{ \quad \kappa\alpha\sin(\theta(\lambda+1)) + \kappa\beta\cos(\theta(\lambda+1)) - \\
& \alpha(\lambda+1)\sin(\theta(1-\lambda)) + \beta(\lambda+1)\cos(\theta(1-\lambda)) + \\
& \gamma\sin(\theta(\lambda+1)) + \delta\cos(\theta(\lambda+1)) \quad \Big\} \quad \Big] \\
= {} & u_1 + iu_2
\end{aligned}$$

Mode I : displacement



displacement for Mode I

$$\left.\begin{array}{l} u_1(\theta > 0) = u_1(\theta < 0) \\ u_2(\theta > 0) = -u_2(\theta < 0) \end{array}\right\} \quad \rightarrow \quad \boxed{\beta = \delta = 0} \quad \longrightarrow$$

$$\Omega = \alpha z^{\lambda+1} = \alpha r^{\lambda+1} e^{i(\lambda+1)\theta}$$

$$\omega' = \gamma z^{\lambda+1} = \gamma r^{\lambda+1} e^{i(\lambda+1)\theta}$$

Mode I : stress components

$$
\begin{aligned}
\sigma_{11} &= (\lambda+1)\left[\alpha z^{\lambda} + \alpha\bar{z}^{\lambda} - \tfrac{1}{2}\left\{\alpha\lambda\bar{z}z^{\lambda-1} + \gamma z^{\lambda} + \alpha\lambda z\bar{z}^{\lambda-1} + \gamma\bar{z}^{\lambda}\right\}\right] \\
\sigma_{22} &= (\lambda+1)\left[\alpha z^{\lambda} + \alpha\bar{z}^{\lambda} + \tfrac{1}{2}\left\{\alpha\lambda\bar{z}z^{\lambda-1} + \gamma z^{\lambda} + \alpha\lambda z\bar{z}^{\lambda-1} + \gamma\bar{z}^{\lambda}\right\}\right] \\
\sigma_{12} &= -\tfrac{1}{2}i(\lambda+1)\left[\alpha\lambda\bar{z}z^{\lambda-1} + \gamma z^{\lambda} - \alpha\lambda z\bar{z}^{\lambda-1} - \gamma\bar{z}^{\lambda}\right]
\end{aligned}
$$

with $\quad z = re^{i\theta} \quad ; \quad \bar{z} = re^{-i\theta} \quad \rightarrow$

$$
\begin{aligned}
\sigma_{11} &= (\lambda+1)r^{\lambda}\left[\alpha e^{i\lambda\theta} + \alpha e^{-i\lambda\theta} - \right. \\
&\qquad \left. \tfrac{1}{2}\left\{\alpha\lambda e^{i(\lambda-2)\theta} + \gamma e^{i\lambda\theta} + \alpha\lambda e^{-i(\lambda-2)\theta} + \gamma e^{-i\lambda\theta}\right\}\right] \\
\sigma_{22} &= (\lambda+1)r^{\lambda}\left[\alpha e^{i\lambda\theta} + \alpha e^{-i\lambda\theta} + \right. \\
&\qquad \left. \tfrac{1}{2}\left\{\alpha\lambda e^{i(\lambda-2)\theta} + \gamma e^{i\lambda\theta} + \alpha\lambda e^{-i(\lambda-2)\theta} + \gamma e^{-i\lambda\theta}\right\}\right] \\
\sigma_{12} &= -\tfrac{1}{2}i(\lambda+1)r^{\lambda}\left[\alpha\lambda e^{i(\lambda-2)\theta} + \gamma e^{i\lambda\theta} - \alpha\lambda e^{-i(\lambda-2)\theta} - \gamma e^{-i\lambda\theta}\right]
\end{aligned}
$$

Mode I : stress components

$$\text{with} \qquad e^{i\theta} + e^{-i\theta} = 2\cos(\theta) \quad ; \quad e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \quad \rightarrow$$

$$\begin{aligned}
\sigma_{11} &= 2(\lambda+1)r^{\lambda}\left[\alpha\cos(\lambda\theta) + \tfrac{1}{2}\left\{\alpha\lambda\cos((\lambda-2)\theta) + \gamma\cos(\lambda\theta)\right\}\right] \\
\sigma_{22} &= 2(\lambda+1)r^{\lambda}\left[\alpha\cos(\lambda\theta) - \tfrac{1}{2}\left\{\alpha\lambda\cos((\lambda-2)\theta) + \gamma\cos(\lambda\theta)\right\}\right] \\
\sigma_{12} &= (\lambda+1)r^{\lambda}\left[\alpha\lambda\sin((\lambda-2)\theta) + \gamma\sin(\lambda\theta)\right]
\end{aligned}$$

Stress boundary conditions

$$\sigma_{11} = 2(\lambda+1)r^{\lambda}\left[\alpha\cos(\lambda\theta) + \tfrac{1}{2}\left\{\alpha\lambda\cos((\lambda-2)\theta) + \gamma\cos(\lambda\theta)\right\}\right]$$
$$\sigma_{22} = 2(\lambda+1)r^{\lambda}\left[\alpha\cos(\lambda\theta) - \tfrac{1}{2}\left\{\alpha\lambda\cos((\lambda-2)\theta) + \gamma\cos(\lambda\theta)\right\}\right]$$
$$\sigma_{12} = (\lambda+1)r^{\lambda}\left[\alpha\lambda\sin((\lambda-2)\theta) + \gamma\sin(\lambda\theta)\right]$$

crack surfaces are stress free $\longrightarrow$

$$\sigma_{22}(\theta=\pm\pi) = \sigma_{12}(\theta=\pm\pi) = 0 \quad \longrightarrow$$

$$\begin{bmatrix} (\lambda-2)\cos(\lambda\pi) & \cos(\lambda\pi) \\ \lambda\sin(\lambda\pi) & \sin(\lambda\pi) \end{bmatrix} \begin{bmatrix} \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow$$

$$det\begin{bmatrix} (\lambda-2)\cos(\lambda\pi) & \cos(\lambda\pi) \\ \lambda\sin(\lambda\pi) & \sin(\lambda\pi) \end{bmatrix} = -\sin(2\lambda\pi) = 0 \rightarrow 2\pi\lambda = n\pi \rightarrow$$

$\lambda = -\frac{1}{2}, \frac{n}{2},$ with $n = 0, 1, 2, ..$

Stress field

$$\begin{array}{llllllll} \lambda = -\frac{1}{2} & \rightarrow & \alpha = 2\gamma & ; & \lambda = 0 & \rightarrow & \alpha = \frac{1}{2}\gamma \\ \lambda = \frac{1}{2} & \rightarrow & \alpha = -2\gamma & ; & \lambda = 1 & \rightarrow & \alpha = \gamma \end{array}$$

$$\sigma_{11} = 2\gamma r^{-\frac{1}{2}} \cos(\tfrac{1}{2}\theta) \left[1 - \sin(\tfrac{3}{2}\theta) \sin(\tfrac{1}{2}\theta)\right] + \cdots$$

$$\sigma_{22} = 2\gamma r^{-\frac{1}{2}} \cos(\tfrac{1}{2}\theta) \left[1 + \sin(\tfrac{3}{2}\theta) \sin(\tfrac{1}{2}\theta)\right] + \cdots$$

$$\sigma_{12} = 2\gamma r^{-\frac{1}{2}} \left[\cos(\tfrac{1}{2}\theta) \cos(\tfrac{3}{2}\theta) \sin(\tfrac{1}{2}\theta)\right] + \cdots$$

Mode I : stress intensity factor

definition stress intensity factor K ("Kies")

$$\boxed{K_I = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{22}|_{\theta=0} \right) = 2\gamma\sqrt{2\pi}}$$

$[\ \mathrm{m}^{\frac{1}{2}}\ \mathrm{N}\ \mathrm{m}^{-2}\ ]$

Mode I : crack tip solution

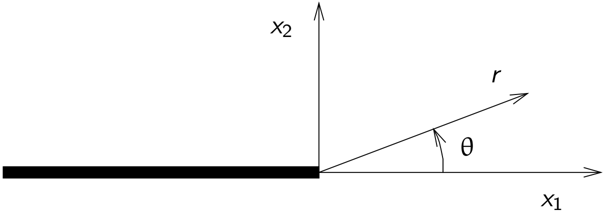
$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 - \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$
$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1 + \sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]$$
$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right]$$

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\cos(\tfrac{1}{2}\theta)\left\{\kappa - 1 + 2\sin^2(\tfrac{1}{2}\theta)\right\}\right]$$
$$u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta)\left\{\kappa + 1 - 2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

plane stress $\kappa = \dfrac{3-\nu}{1+\nu}$

plane strain $\kappa = 3 - 4\nu$

Mode II : displacement



displacements for Mode II

$$\left.\begin{array}{l} u_1(\theta > 0) = -u_1(\theta < 0) \\ u_2(\theta > 0) = u_2(\theta < 0) \end{array}\right\} \quad \rightarrow \quad \boxed{\alpha = \gamma = 0} \quad \longrightarrow$$

$$\Omega = i\beta z^{\lambda+1} = i\beta r^{\lambda+1} e^{i(\lambda+1)\theta}$$
$$\omega' = i\delta z^{\lambda+1} = i\delta r^{\lambda+1} e^{i(\lambda+1)\theta}$$

Mode II : stress intensity factor

definition stress intensity factor K ("Kies")

$$\boxed{K_{II} = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{12}|_{\theta=0} \right)} \qquad [\ \mathrm{m}^{\frac{1}{2}}\ \mathrm{N}\ \mathrm{m}^{-2}\ ]$$

Mode II : crack tip solution

$$
\begin{aligned}
\sigma_{11} &= \frac{K_{II}}{\sqrt{2\pi r}} \left[-\sin(\tfrac{1}{2}\theta)\left\{2+\cos(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right\}\right] \\
\sigma_{22} &= \frac{K_{II}}{\sqrt{2\pi r}} \left[\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)\cos(\tfrac{3}{2}\theta)\right] \\
\sigma_{12} &= \frac{K_{II}}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta)\left\{1-\sin(\tfrac{1}{2}\theta)\sin(\tfrac{3}{2}\theta)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta)\left\{\kappa+1+2\cos^2(\tfrac{1}{2}\theta)\right\}\right] \\
u_2 &= \frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}} \left[-\cos(\tfrac{1}{2}\theta)\left\{\kappa-1-2\sin^2(\tfrac{1}{2}\theta)\right\}\right]
\end{aligned}
$$

plane stress $\quad \kappa = \dfrac{3-\nu}{1+\nu}$

plane strain $\quad \kappa = 3-4\nu$

Mode III : Laplace equation

$$\left.\begin{array}{l} \varepsilon_{31} = \frac{1}{2} u_{3,1} \qquad ; \qquad \varepsilon_{32} = \frac{1}{2} u_{3,2} \\ \\ \text{Hooke's law} \end{array}\right\} \quad \rightarrow$$

$$\left.\begin{array}{l} \sigma_{31} = 2\mu\varepsilon_{31} = \mu u_{3,1} \qquad ; \qquad \sigma_{32} = 2\mu\varepsilon_{32} = \mu u_{3,2} \\ \\ \text{equilibrium} \end{array}\right\} \quad \rightarrow$$

$$\sigma_{31,1} + \sigma_{32,2} = \mu u_{3,11} + \mu u_{3,22} = 0 \quad \rightarrow$$

$$\nabla^2 u_3 = 0$$

Mode III : displacement

general solution $$u_3 = f + \bar{f}$$

specific choice $$f = (A + iB)z^{\lambda+1} \quad \longrightarrow \quad \bar{f} = (A - iB)\bar{z}^{\lambda+1}$$

Mode III : stress components

$$\sigma_{31} = 2(\lambda+1)r^{\lambda}\{A\cos(\lambda\theta) - B\sin(\lambda\theta)\}$$
$$\sigma_{32} = -2(\lambda+1)r^{\lambda}\{A\sin(\lambda\theta) + B\cos(\lambda\theta)\}$$

$$\sigma_{32}(\theta = \pm\pi) = 0 \quad \rightarrow$$

$$\begin{bmatrix} \sin(\lambda\pi) & \cos(\lambda\pi) \\ \sin(\lambda\pi) & -\cos(\lambda\pi) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \rightarrow$$

$$\det \begin{bmatrix} \sin(\lambda\pi) & \cos(\lambda\pi) \\ \sin(\lambda\pi) & -\cos(\lambda\pi) \end{bmatrix} = -\sin(2\pi\lambda) = 0 \quad \rightarrow \quad 2\pi\lambda = n\pi \quad \rightarrow$$

$$\lambda = -\tfrac{1}{2}, \frac{n}{2}, .. \qquad \text{with} \qquad n = 0, 1, 2, ..$$

crack tip solution $\quad \lambda = -\frac{1}{2} \quad \rightarrow \quad A = 0 \quad \rightarrow$

$$\sigma_{31} = Br^{-\frac{1}{2}}\{\sin(\tfrac{1}{2}\theta)\} \qquad ; \qquad \sigma_{32} = -Br^{-\frac{1}{2}}\{\cos(\tfrac{1}{2}\theta)\}$$

Mode III : Stress intensity factor

definition stress intensity factor

$$\boxed{K_{III} = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{32}|_{\theta=0} \right)}$$

Mode III : crack tip solution

stress components

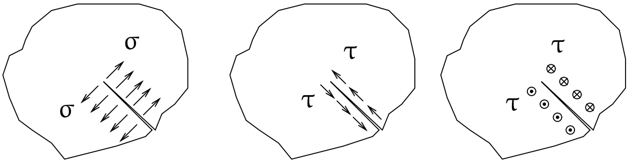
$$\sigma_{31} = \frac{K_{III}}{\sqrt{2\pi r}} \left[-\sin(\tfrac{1}{2}\theta) \right]$$

$$\sigma_{32} = \frac{K_{III}}{\sqrt{2\pi r}} \left[\cos(\tfrac{1}{2}\theta) \right]$$

displacement

$$u_3 = \frac{2K_{III}}{\mu} \sqrt{\frac{r}{2\pi}} \left[\sin(\tfrac{1}{2}\theta) \right]$$

Crack tip stress (mode I, II, III)



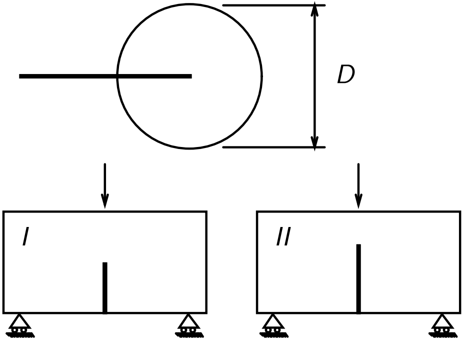
Mode I   Mode II   Mode III

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) \quad ; \quad \sigma_{ij} = \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta) \quad ; \quad \sigma_{ij} = \frac{K_{III}}{\sqrt{2\pi r}} f_{IIIij}(\theta)$$

crack intensity factors (SIF)

$$K_I = \beta_I \ \sigma\sqrt{\pi a} \quad ; \quad K_{II} = \beta_{II} \ \tau\sqrt{\pi a} \quad ; \quad K_{III} = \beta_{III} \ \tau\sqrt{\pi a}$$

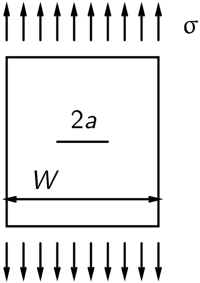
K-zone

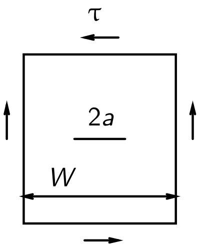


K-zone : D

$D_{II} \ll D_I$

SIF for specified cases

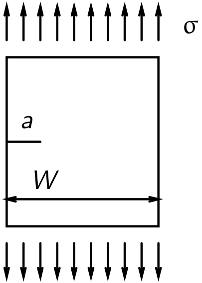




$$K_I = \sigma\sqrt{\pi a}\left(\sec\frac{\pi a}{W}\right)^{1/2}$$

$$K_{II} = \tau\sqrt{\pi a} \qquad \text{small } \frac{a}{W}$$

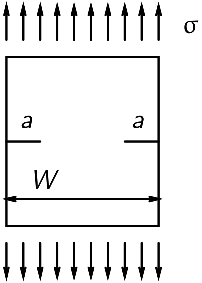
SIF for specified cases



$$K_I = \sigma\sqrt{a}\left[1.12\sqrt{\pi} - 0.41\frac{a}{W} + 18.7\left(\frac{a}{W}\right)^2 - 38.48\left(\frac{a}{W}\right)^3 + 53.85\left(\frac{a}{W}\right)^4\right]$$

$$\approx 1.12\sigma\sqrt{\pi a} \qquad \text{small } \frac{a}{W}$$

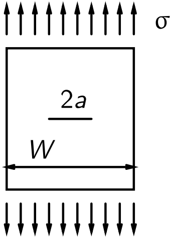
SIF for specified cases

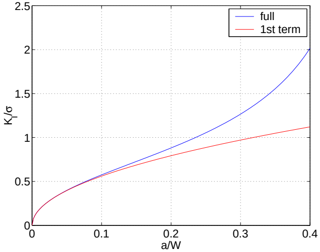


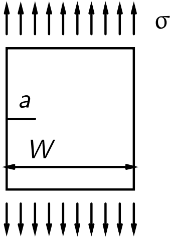
$$K_I = \sigma\sqrt{a}\left[\, 1.12\sqrt{\pi} + 0.76\frac{a}{W} - 8.48\left(\frac{a}{W}\right)^2 + 27.36\left(\frac{a}{W}\right)^3 \,\right]$$

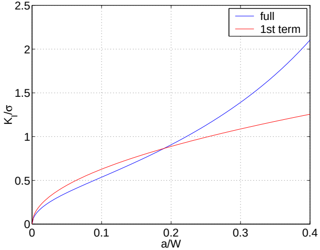
$$\approx 1.12\sigma\sqrt{\pi a}$$

SIF for specified cases



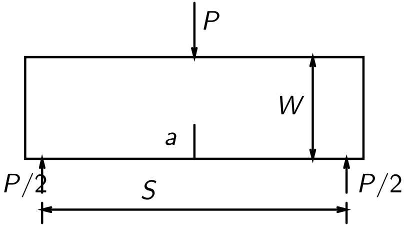






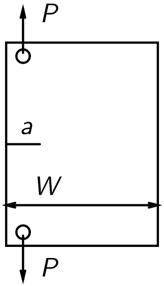
plots are made with 'Kfac.m'.

SIF for specified cases



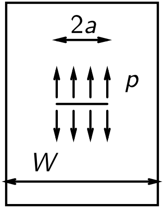
$$K_I = \frac{PS}{BW^{3/2}} \left[2.9 \left(\frac{a}{W} \right)^{\frac{1}{2}} - 4.6 \left(\frac{a}{W} \right)^{\frac{3}{2}} + 21.8 \left(\frac{a}{W} \right)^{\frac{5}{2}} - 37.6 \left(\frac{a}{W} \right)^{\frac{7}{2}} + 37.7 \left(\frac{a}{W} \right)^{\frac{9}{2}} \right]$$

SIF for specified cases



$$K_I = \frac{P}{BW^{1/2}}\left[29.6\left(\frac{a}{W}\right)^{\frac{1}{2}} - 185.5\left(\frac{a}{W}\right)^{\frac{3}{2}} + 655.7\left(\frac{a}{W}\right)^{\frac{5}{2}} - 1017\left(\frac{a}{W}\right)^{\frac{7}{2}} + 638.9\left(\frac{a}{W}\right)^{\frac{9}{2}}\right]$$

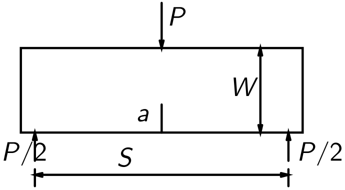
SIF for specified cases

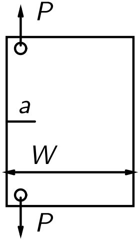


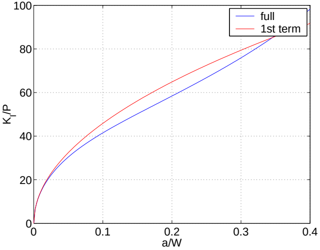
$$K_I = p\sqrt{\pi a}$$

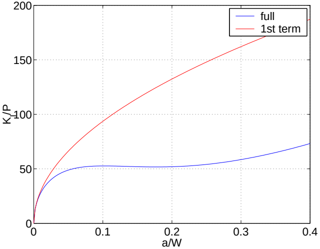
p per unit thickness

SIF for specified cases









plots are made with 'Kfac.m'.

K-based crack growth criteria

$$K_I = K_{Ic} \quad ; \quad K_{II} = K_{IIc} \quad ; \quad K_{III} = K_{IIIc}$$

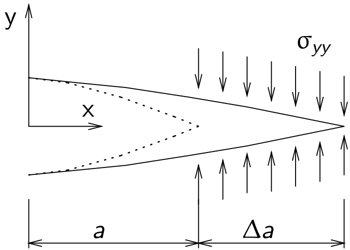
- K_{Ic} = *Fracture Toughness*

- calculate K_I, K_{II}, K_{III}
  - analytically
  - literature
  - relation $K - G$
  - numerically (EEM, BEM)

- experimental determination of $K_{Ic}, K_{IIc}, K_{IIIc}$

  - normalized experiments (exmpl. ASTM E399)
  - correlation with C_v (KAN p. 18 : $\dfrac{K_{Ic}^2}{E} = mC_v^n$)

Relation $G - K_I$



crack length $\quad a \qquad \sigma_{yy}(\theta = 0, r = x - a) = \dfrac{\sigma\sqrt{a}}{\sqrt{2(x-a)}} \quad ; \quad u_y = 0$

crack length $\quad a + \Delta a \qquad \sigma_{yy}(\theta = \pi, r = a + \Delta a - x) = 0$

$$u_y = \frac{(1+\nu)(\kappa+1)}{E}\;\frac{\sigma\sqrt{a+\Delta a}}{\sqrt{2}}\;\sqrt{a + \Delta a - x}$$

plane stress $\quad : \kappa = \dfrac{3-\nu}{1+\nu} \quad ;$ plane strain $\quad : \kappa = 3 - 4\nu$

Relation $G - K_I$ (continued)

accumulation of elastic energy

$$\Delta U = 2B \int_a^{a+\Delta a} \tfrac{1}{2}\sigma_{yy}\; dx\; u_y = B \int_a^{a+\Delta a} \sigma_{yy} u_y\; dx = B\, f(\Delta a)\; \Delta a \qquad \longrightarrow$$

energy release rate

$$G = \frac{1}{B} \lim_{\Delta a \longrightarrow 0} \left(\frac{\Delta U}{\Delta a} \right) = \lim_{\Delta a \longrightarrow 0} f(\Delta a) = \frac{(1+\nu)(\kappa+1)}{4E}\; \sigma^2 a\pi = \frac{(1+\nu)(\kappa+1)}{4E}\; K_I^2$$

plane stress $\qquad G = \dfrac{K_I^2}{E}$

plane strain $\qquad G = (1-\nu^2)\dfrac{K_I^2}{E}$

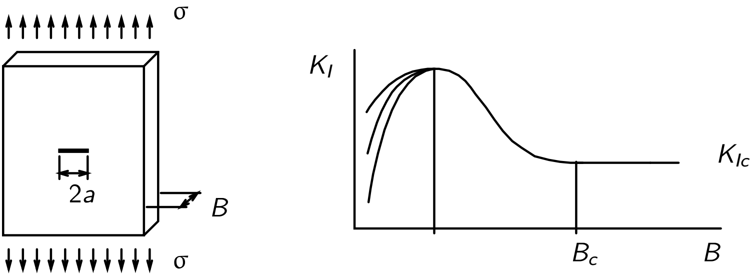
Multi mode load

$$G = \frac{1}{E}\left(c_1 K_I^2 + c_2 K_{II}^2 + c_3 K_{III}^2\right)$$

plane stress $\qquad G = \frac{1}{E}(K_I^2 + K_{II}^2)$

plane strain $\qquad G = \frac{(1-\nu^2)}{E}(K_I^2 + K_{II}^2) + \frac{(1+\nu)}{E}K_{III}^2$

The critical SIF value



$$K_{Ic} = \sigma_c\sqrt{\pi a}$$

$$B_c = 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

K_{Ic} values

Material	σ_v [MPa]	K_{Ic} [MPa$\sqrt{m}$]
steel, 300 maraging	1669	93.4
steel, 350 maraging	2241	38.5
steel, D6AC	1496	66.0
steel, AISI 4340	1827	47.3
steel, A533B reactor	345	197.8
steel, carbon	241	219.8
Al 2014-T4	448	28.6
Al 2024-T3	393	34.1
Al 7075-T651	545	29.7
Al 7079-T651	469	33.0
Ti 6Al-4V	1103	38.5
Ti 6Al-6V-2Sn	1083	37.4
Ti 4Al-4Mo-2Sn-0.5Si	945	70.3

MULTI-MODE LOADING

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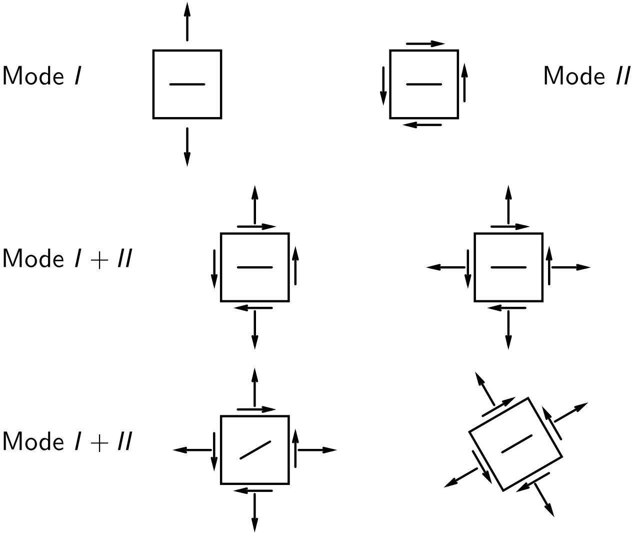
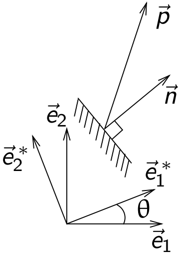
Multi-mode crack loading



Multi-mode crack loading

crack tip stresses	s_{ij}
Mode I	$s_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta)$
Mode II	$s_{ij} = \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$
Mode $I + II$	$s_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$

Stress component transformation



$$\vec{e}_1^{\,*} = \cos(\theta)\vec{e}_1 + \sin(\theta)\vec{e}_2 = c\vec{e}_1 + s\vec{e}_2$$
$$\vec{e}_2^{\,*} = -\sin(\theta)\vec{e}_1 + \cos(\theta)\vec{e}_2 = -s\vec{e}_1 + c\vec{e}_2$$

stress vector and normal unity vector

$$\vec{p} = p_1\vec{e}_1 + p_2\vec{e}_2 = p_1^*\vec{e}_1^{\,*} + p_2^*\vec{e}_2^{\,*} \quad \rightarrow$$
$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \quad \rightarrow$$

$$\underset{\sim}{p} = \underline{T}\,\underset{\sim}{p}^* \quad \rightarrow \quad \underset{\sim}{p}^* = \underline{T}^T\underset{\sim}{p}$$

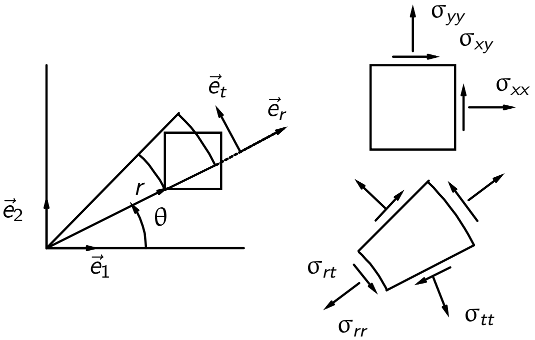
idem : $\underset{\sim}{n}^* = \underline{T}^T\underset{\sim}{n}$

Transformation stress matrix

$$
\begin{aligned}
&\underset{\sim}{p} = \underline{\sigma}\underset{\sim}{n} \quad \rightarrow \\
&\underline{T}\,\underset{\sim}{p}^* = \underline{\sigma}\,\underline{T}\underset{\sim}{n}^* \quad \rightarrow \quad \underset{\sim}{p}^* = \underline{T}^T\underline{\sigma}\,\underline{T}\,\underset{\sim}{n}^* = \underline{\sigma}^*\underset{\sim}{n}^* \quad \rightarrow \\
&\underline{\sigma}^* = \underline{T}^T\underline{\sigma}\,\underline{T} \quad \rightarrow \quad \underline{\sigma} = \underline{T}\,\underline{\sigma}^*\underline{T}^T
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} \sigma^*_{11} & \sigma^*_{12} \\ \sigma^*_{21} & \sigma^*_{22} \end{bmatrix} &= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \\
&= \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} c\sigma_{11} + s\sigma_{12} & -s\sigma_{11} + c\sigma_{12} \\ c\sigma_{21} + s\sigma_{22} & -s\sigma_{21} + c\sigma_{22} \end{bmatrix} \\
&= \begin{bmatrix} c^2\sigma_{11} + 2cs\sigma_{12} + s^2\sigma_{22} & -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} \\ -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} & s^2\sigma_{11} - 2cs\sigma_{12} + c^2\sigma_{22} \end{bmatrix}
\end{aligned}
$$

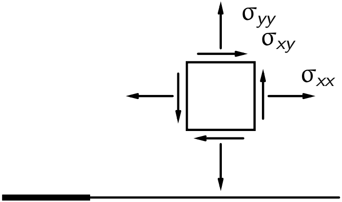
Cartesian to cylindrical transformation



$$\vec{e}_r = c\vec{e}_1 + s\vec{e}_2$$
$$\vec{e}_t = -s\vec{e}_1 + c\vec{e}_2$$

$$\begin{bmatrix} \sigma_{rr} & \sigma_{rt} \\ \sigma_{tr} & \sigma_{tt} \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$
$$= \begin{bmatrix} c^2\sigma_{xx} + 2cs\sigma_{xy} + s^2\sigma_{yy} & -cs\sigma_{xx} + (c^2 - s^2)\sigma_{xy} + cs\sigma_{yy} \\ -cs\sigma_{xx} + (c^2 - s^2)\sigma_{xy} + cs\sigma_{yy} & s^2\sigma_{xx} - 2cs\sigma_{xy} + c^2\sigma_{yy} \end{bmatrix}$$

Crack tip stresses : Cartesian

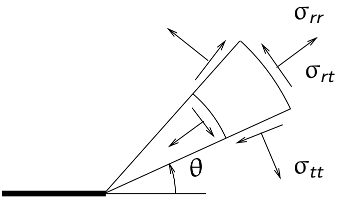


$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} f_{Ixx}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIxx}(\theta)$$
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} f_{Iyy}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIyy}(\theta)$$
$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} f_{Ixy}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIxy}(\theta)$$

$$f_{Ixx}(\theta) = \cos(\frac{\theta}{2}) \left[1 - \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right]$$
$$f_{Iyy}(\theta) = \cos(\frac{\theta}{2}) \left[1 + \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right]$$
$$f_{Ixy}(\theta) = \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \cos(\frac{3\theta}{2})$$

$$f_{IIxx}(\theta) = -\sin(\frac{\theta}{2}) \left[2 + \cos(\frac{\theta}{2}) \cos(\frac{3\theta}{2})\right]$$
$$f_{IIyy}(\theta) = \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \cos(\frac{3\theta}{2})$$
$$f_{IIxy}(\theta) = \cos(\frac{\theta}{2}) \left[1 - \sin(\frac{\theta}{2}) \sin(\frac{3\theta}{2})\right]$$

Crack tip stresses : cylindrical

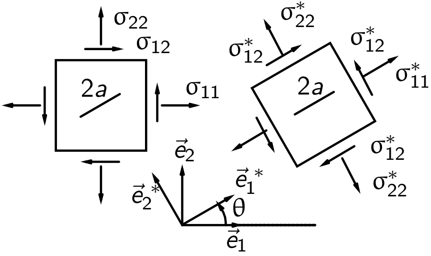


$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}} f_{Irr}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIrr}(\theta)$$
$$\sigma_{tt} = \frac{K_I}{\sqrt{2\pi r}} f_{Itt}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IItt}(\theta)$$
$$\sigma_{rt} = \frac{K_I}{\sqrt{2\pi r}} f_{Irt}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIrt}(\theta)$$

$$f_{Irr}(\theta) = \left[\frac{5}{4}\cos(\frac{\theta}{2}) - \frac{1}{4}\cos(\frac{3\theta}{2})\right]$$
$$f_{Itt}(\theta) = \left[\frac{3}{4}\cos(\frac{\theta}{2}) + \frac{1}{4}\cos(\frac{3\theta}{2})\right]$$
$$f_{Irt}(\theta) = \left[\frac{1}{4}\sin(\frac{\theta}{2}) + \frac{1}{4}\sin(\frac{3\theta}{2})\right]$$

$$f_{IIrr}(\theta) = \left[-\frac{5}{4}\sin(\frac{\theta}{2}) + \frac{3}{4}\sin(\frac{3\theta}{2})\right]$$
$$f_{IItt}(\theta) = \left[-\frac{3}{4}\sin(\frac{\theta}{2}) - \frac{3}{4}\sin(\frac{3\theta}{2})\right]$$
$$f_{IIrt}(\theta) = \left[\frac{1}{4}\cos(\frac{\theta}{2}) + \frac{3}{4}\cos(\frac{3\theta}{2})\right]$$

Multi-mode load



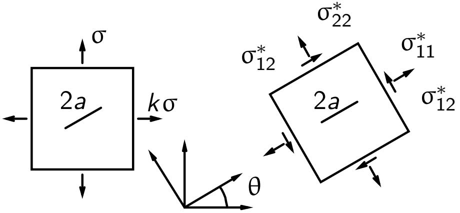
$$\begin{bmatrix} \sigma^*_{11} & \sigma^*_{12} \\ \sigma^*_{21} & \sigma^*_{22} \end{bmatrix} = \begin{bmatrix} c^2\sigma_{11} + 2cs\sigma_{12} + s^2\sigma_{22} & -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} \\ -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} & s^2\sigma_{11} - 2cs\sigma_{12} + c^2\sigma_{22} \end{bmatrix}$$

crack tip stresses $\quad s_{ij} = \dfrac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) + \dfrac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$

with $\quad K_I = \beta\ \sigma^*_{22}\sqrt{\pi a} \quad ; \quad K_{II} = \gamma\ \sigma^*_{12}\sqrt{\pi a}$

σ^*_{11} "does not do anything"

Example multi-mode load

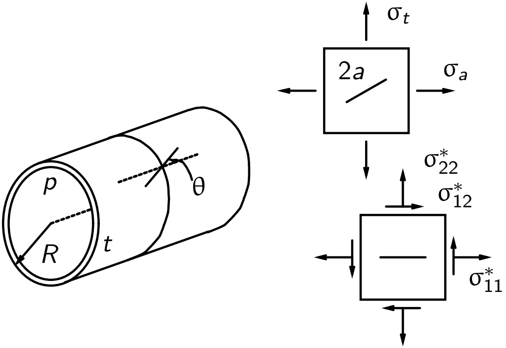


$$\sigma^*_{11} = c^2\sigma_{11} + 2cs\sigma_{12} + s^2\sigma_{22} = c^2k\sigma + s^2\sigma$$
$$\sigma^*_{22} = s^2\sigma_{11} - 2cs\sigma_{12} + c^2\sigma_{22} = s^2k\sigma + c^2\sigma$$
$$\sigma^*_{12} = -cs\sigma_{11} + (c^2 - s^2)\sigma_{12} + cs\sigma_{22} = cs(1-k)\sigma$$

crack tip stresses $$s_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{IIij}(\theta)$$

$$K_I = \beta_I \ \sigma^*_{22}\sqrt{\pi a} = \beta_I \ (s^2k + c^2)\sigma\sqrt{\pi a}$$
$$K_{II} = \beta_{II} \ \sigma^*_{12}\sqrt{\pi a} = \beta_{II} \ cs(1-k)\sigma\sqrt{\pi a}$$

Example multi-mode load



$$\sigma_t = \frac{pR}{t} = \sigma \quad ; \quad \sigma_a = \frac{pR}{2t} = \tfrac{1}{2}\sigma \quad \rightarrow \quad k = \frac{1}{2}$$

$$\sigma^*_{22} = s^2\,\tfrac{1}{2}\,\sigma + c^2\sigma \quad ; \quad \sigma^*_{12} = cs(1-\tfrac{1}{2})\sigma = \tfrac{1}{2}\,cs\,\sigma$$

$$K_I = \sigma^*_{22}\sqrt{\pi a} = (\tfrac{1}{2}s^2 + c^2)\sigma\sqrt{\pi a} = (\tfrac{1}{2}s^2 + c^2)\frac{pR}{t}\sqrt{\pi a}$$

$$K_{II} = \sigma^*_{12}\sqrt{\pi a} = \tfrac{1}{2}\,cs\,\sigma = \tfrac{1}{2}\,cs\,\frac{pR}{t}\sqrt{\pi a}$$

CRACK GROWTH DIRECTION

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Crack growth direction

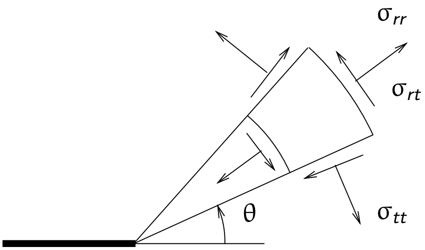
criteria for crack growth direction :

- maximum tangential stress (MTS) criterion
- strain energy density (SED) criterion

requirement : crack tip stresses in cylindrical coordinates

Maximum tangential stress criterion

Erdogan & Sih (1963)



Hypothesis : crack growth towards local maximum of σ_{tt}

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{\partial^2 \sigma_{tt}}{\partial \theta^2} < 0 \qquad \rightarrow \qquad \theta_c$$

$$\sigma_{tt}(\theta = \theta_c) = \sigma_{tt}(\theta = 0) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow \quad \text{crack growth}$$

Maximum tangential stress criterion

$$\frac{\partial \sigma_{tt}}{\partial \theta} = 0 \quad \rightarrow$$

$$\frac{3}{2}\frac{K_I}{\sqrt{2\pi r}}\left[-\frac{1}{4}\sin(\tfrac{\theta}{2}) - \frac{1}{4}\sin(\tfrac{3\theta}{2})\right] + \frac{3}{2}\frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{1}{4}\cos(\tfrac{\theta}{2}) - \frac{3}{4}\cos(\tfrac{3\theta}{2})\right] = 0 \quad \rightarrow$$

$$K_I \sin(\theta) + K_{II}\{3\cos(\theta) - 1\} = 0$$

$$\frac{\partial^2 \sigma_{tt}}{\partial \theta^2} < 0 \quad \rightarrow$$

$$\frac{3}{4}\frac{K_I}{\sqrt{2\pi r}}\left[-\frac{1}{4}\cos(\tfrac{\theta}{2}) - \frac{3}{4}\cos(\tfrac{3\theta}{2})\right] + \frac{3}{4}\frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin(\tfrac{\theta}{2}) + \frac{9}{4}\sin(\tfrac{3\theta}{2})\right] < 0$$

$$\sigma_{tt}(\theta = \theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \quad \rightarrow$$

$$\frac{1}{4}\frac{K_I}{K_{Ic}}\left[3\cos(\tfrac{\theta_c}{2}) + \cos(\tfrac{3\theta_c}{2})\right] + \frac{1}{4}\frac{K_{II}}{K_{Ic}}\left[-3\sin(\tfrac{\theta_c}{2}) - 3\sin(\tfrac{3\theta_c}{2})\right] = 1$$

Mode I load

$$K_{II} = 0$$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_I \sin(\theta) = 0 \qquad \rightarrow \qquad \theta_c = 0$$

$$\left.\frac{\partial^2 \sigma_{tt}}{\partial \theta^2}\right|_{\theta_c} < 0$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \qquad \rightarrow \qquad K_I = K_{Ic}$$

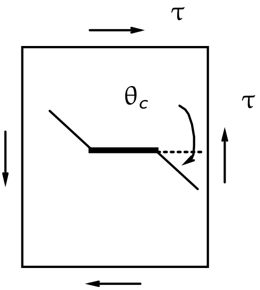
Mode II load

$$K_I = 0$$

$$\frac{\partial \sigma_{tt}}{\partial \theta} = K_{II}(3\cos(\theta_c) - 1) = 0 \qquad \rightarrow \qquad \theta_c = \pm\arccos(\tfrac{1}{3}) = \pm 70.6^o$$

$$\left.\frac{\partial^2 \sigma_{tt}}{\partial \theta^2}\right|_{\theta_c} < 0 \qquad \rightarrow \qquad \theta_c = -70.6^o$$

$$\sigma_{tt}(\theta_c) = \frac{K_{Ic}}{\sqrt{2\pi r}} \qquad \rightarrow \qquad K_{IIc} = \sqrt{\tfrac{3}{4}} K_{Ic}$$

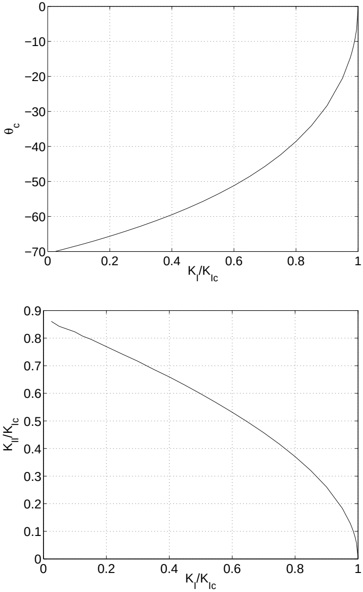


Multi-mode load

$$K_I[-\sin(\tfrac{\theta}{2}) - \sin(\tfrac{3\theta}{2})] + K_{II}[-\cos(\tfrac{\theta}{2}) - 3\cos(\tfrac{3\theta}{2})] = 0$$
$$K_I[-\cos(\tfrac{\theta}{2}) - 3\cos(\tfrac{3\theta}{2})] + K_{II}[\sin(\tfrac{\theta}{2}) + 9\sin(\tfrac{3\theta}{2})] < 0$$
$$K_I[3\cos(\tfrac{\theta}{2}) + \cos(\tfrac{3\theta}{2})] + K_{II}[-3\sin(\tfrac{\theta}{2}) - 3\sin(\tfrac{3\theta}{2})] = 4K_{Ic}$$

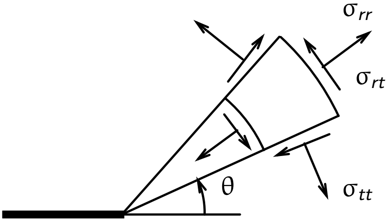
$$\left.\begin{array}{l} -K_I f_1 - K_{II} f_2 = 0 \\ -K_I f_2 + K_{II} f_3 < 0 \\ K_I f_4 - 3K_{II} f_1 = 4K_{Ic} \end{array}\right\} \rightarrow \left.\begin{array}{c} -\left(\dfrac{K_I}{K_{Ic}}\right) f_1 - \left(\dfrac{K_{II}}{K_{Ic}}\right) f_2 = 0 \\ -\left(\dfrac{K_I}{K_{Ic}}\right) f_2 + \left(\dfrac{K_{II}}{K_{Ic}}\right) f_3 < 0 \\ \left(\dfrac{K_I}{K_{Ic}}\right) f_4 - 3\left(\dfrac{K_{II}}{K_{Ic}}\right) f_1 = 4 \end{array}\right\}$$

Multi-mode load



Strain energy density (SED) criterion

Sih (1973)



$$U_i = \text{ Strain Energy Density (Function) } = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}$$

$$S = \text{ Strain Energy Density Factor } = rU_i = S(K_I, K_{II}, \theta)$$

Hypothesis : crack growth towards local minimum of SED

$$\frac{\partial S}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \quad \rightarrow \quad \theta_c$$

$$S(\theta = \theta_c) = S(\theta = 0, \text{pl.strain}) = S_c \quad \rightarrow \quad \text{crack growth}$$

SED

$$U_i = \frac{1}{2E}(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2) - \frac{\nu}{E}(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + \frac{1}{2G}(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2)$$

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos(\tfrac{\theta}{2})\left[1 - \sin(\tfrac{\theta}{2})\sin(\tfrac{3\theta}{2})\right] - \frac{K_{II}}{\sqrt{2\pi r}}\sin(\tfrac{\theta}{2})\left[2 + \cos(\tfrac{\theta}{2})\cos(\tfrac{3\theta}{2})\right]$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos(\tfrac{\theta}{2})\left[1 + \sin(\tfrac{\theta}{2})\sin(\tfrac{3\theta}{2})\right] + \frac{K_{II}}{\sqrt{2\pi r}}\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})\cos(\tfrac{3\theta}{2})$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})\cos(\tfrac{3\theta}{2}) + \frac{K_{II}}{\sqrt{2\pi r}}\cos(\tfrac{\theta}{2})\left[1 - \sin(\tfrac{\theta}{2})\sin(\tfrac{3\theta}{2})\right]$$

SED factor

$$S = rU_i = S(K_I, K_{II}, \theta) = a_{11}k_I^2 + 2a_{12}k_Ik_{II} + a_{22}k_{II}^2$$

with
$$\begin{aligned}
a_{11} &= \tfrac{1}{16G}(1+\cos(\theta))(\kappa - \cos(\theta)) \\
a_{12} &= \tfrac{1}{16G}\sin(\theta)\{2\cos(\theta) - (\kappa - 1)\} \\
a_{22} &= \tfrac{1}{16G}\{(\kappa+1)(1-\cos(\theta)) + (1+\cos(\theta))(3\cos(\theta)-1)\} \\
k_i &= K_i/\sqrt{\pi}
\end{aligned}$$

$$\frac{\partial S}{\partial \theta} = 0 \quad \rightarrow$$

$$\frac{k_I^2}{16G}\{2\sin(\theta)\cos(\theta) - (\kappa-1)\sin(\theta)\} + \frac{k_Ik_{II}}{16G}\{2 - 4\sin^2(\theta) - (\kappa-1)\cos(\theta)\} + \frac{k_{II}^2}{16G}\{-6\sin(\theta)\cos(\theta) + (\kappa-1)\sin(\theta)\} = 0$$

$$\frac{\partial^2 S}{\partial \theta^2} > 0 \quad \rightarrow$$

$$\frac{k_I^2}{16G}\{2 - 4\sin^2(\theta) - (\kappa-1)\cos(\theta)\} + \frac{k_Ik_{II}}{16G}\{-8\sin(\theta)\cos(\theta) + (\kappa-1)\sin(\theta)\} + \frac{k_{II}^2}{16G}\{-6 + 12\sin^2(\theta) + (\kappa-1)\cos(\theta)\} > 0$$

Mode I load

$$S = a_{11} k_I^2 = \frac{\sigma^2 a}{16G}\{1+\cos(\theta)\}\{\kappa - \cos(\theta)\}$$

$$\frac{\partial S}{\partial \theta} = \sin(\theta)\{2\cos(\theta) - (\kappa - 1)\} = 0 \quad \rightarrow$$

$$\theta_c = 0 \qquad \text{or} \qquad \arccos\left(\tfrac{1}{2}(\kappa - 1)\right)$$

$$\frac{\partial^2 S}{\partial \theta^2} = 2\cos(2\theta) - (\kappa - 1)\cos(\theta) > 0 \qquad \rightarrow \qquad \theta_c = 0$$

$$S(\theta_c) = \frac{\sigma^2 a}{16G}\{2\}\{\kappa - 1\} = \frac{\sigma^2 a}{8G}(\kappa - 1)$$

$$S_c = S(\theta_c, \text{pl.strain}) = \frac{(1+\nu)(1-2\nu)}{2\pi E} K_{Ic}^2$$

Mode II load

$$
\begin{aligned}
S &= a_{22} k_{II}^2 \\
&= \frac{\tau^2 a}{16G}\left[(\kappa+1)\{1-\cos(\theta)\} + \{1+\cos(\theta)\}\{3\cos(\theta)-1\}\right]
\end{aligned}
$$

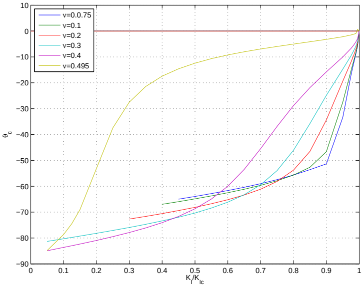
$$
\left.
\begin{aligned}
&\frac{\partial S}{\partial \theta} = \sin(\theta)\left[-6\cos(\theta) + (\kappa-1)\right] = 0 \\
&\frac{\partial^2 S}{\partial \theta^2} = 6 - \cos^2(\theta) + (\kappa-1)\cos(\theta) > 0
\end{aligned}
\right\} \quad \rightarrow
$$

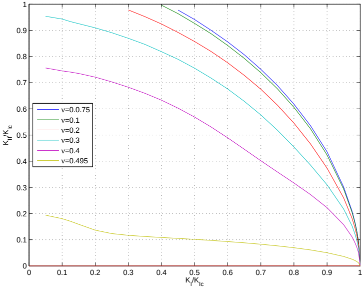
$$
\theta_c = \pm \arccos\left(\tfrac{1}{6}(\kappa-1)\right)
$$

$$
S(\theta_c) = \frac{\tau^2 a}{16G}\{\tfrac{1}{12}(-\kappa^2 + 14\kappa - 1)\}
$$

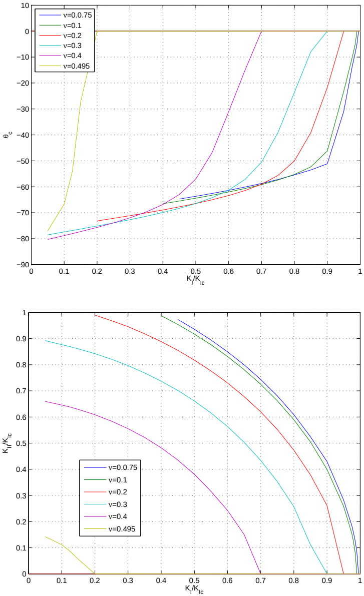
$$
S(\theta_c) = S_c \quad \rightarrow \quad \tau_c = \frac{1}{\sqrt{a}}\sqrt{\frac{192 G S_c}{-\kappa^2 + 14\kappa - 1}}
$$

Multi-mode load; plane strain





Multi-mode load; plane stress

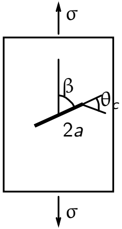


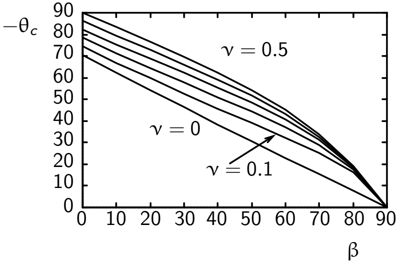
Multi-mode load; plane strain

$$k_I = \sigma\sqrt{a}\sin^2(\beta) \qquad ; \qquad k_{II} = \sigma\sqrt{a}\sin(\beta)\cos(\beta)$$

$$S = \sigma^2 a \sin^2(\beta)\left\{a_{11}\sin^2(\beta) + 2a_{12}\sin(\beta)\cos(\beta) + a_{22}\cos^2(\beta)\right\}$$

$$\frac{\partial S}{\partial \theta} = (\kappa - 1)\sin(\theta_c - 2\beta) - 2\sin\{2(\theta_c - \beta)\} - \sin(2\theta_c) = 0$$
$$\frac{\partial^2 S}{\partial \theta^2} = (\kappa - 1)\cos(\theta_c - 2\beta) - 4\cos\{2(\theta_c - \beta)\} - 2\cos(2\theta_c) > 0$$





From Gdoutos

DYNAMIC FRACTURE MECHANICS

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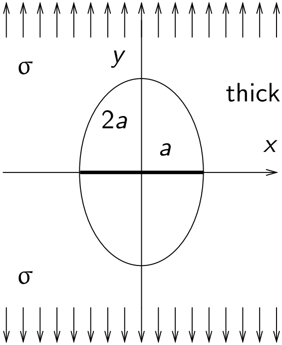
Dynamic fracture mechanics

- impact load
- (quasi)static load $\longrightarrow$ fast fracture
  - kinetic approach
  - static approach

Crack growth rate   Mott (1948)

$$\frac{dU_e}{da} - \frac{dU_i}{da} = \frac{dU_a}{da} + \frac{dU_d}{da} + \frac{dU_k}{da}$$



thickness B

$$\frac{dU_e}{da} = 0 \qquad ; \qquad \frac{dU_d}{da} = 0$$

$$U_a = 4aB\gamma \quad \rightarrow \quad \frac{dU_a}{da} = 4\gamma B$$

$$U_i = 2\pi a^2 B \tfrac{1}{2} \frac{\sigma^2}{E} \quad \rightarrow \quad -\frac{dU_i}{da} = \frac{2\pi a B \sigma^2}{E}$$

Kinetic energy

$$\left.\begin{array}{l} U_k = \frac{1}{2}\rho B \int_\Omega (\dot{u}_x^2 + \dot{u}_y^2)\, dxdy \\ \text{material velocity} \qquad \dot{u}_x \ll \dot{u}_y = \frac{du_y}{dt} = \frac{du_y}{da}\frac{da}{dt} = \frac{du_y}{da}\, s \end{array}\right\}$$

$$\left.\begin{array}{l} U_k = \frac{1}{2}\rho s^2 B \int_\Omega \left(\frac{du_y}{da}\right)^2 dxdy \\ \text{assumption} \qquad \frac{ds}{da} = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} \frac{dU_k}{da} = \frac{1}{2}\rho s^2 B \int_\Omega \frac{d}{da}\left(\frac{du_y}{da}\right)^2 dxdy \\ u_y = 2\sqrt{2}\frac{\sigma}{E}\sqrt{a^2 - ax} \qquad \rightarrow \qquad \frac{du_y}{da} = \sqrt{2}\,\frac{\sigma}{E}\,\frac{2a - x}{\sqrt{a^2 - ax}} \end{array}\right\}$$

$$\frac{dU_k}{da} = \rho s^2 B \left(\frac{\sigma}{E}\right)^2 a \int_\Omega \frac{1}{a^3}\,\frac{x^2(x - 2a)}{(a - x)^2}\, dxdy = \rho s^2 B \left(\frac{\sigma}{E}\right)^2 a\, k(a)$$

Energy balance

$$\frac{2\pi a \sigma^2}{E} = 4\gamma + \rho s^2 \left(\frac{\sigma}{E}\right)^2 ak \qquad \rightarrow$$

$$\left.\begin{array}{l} s = \left(\dfrac{E}{\rho}\right)^{\frac{1}{2}} \left(\dfrac{2\pi}{k}\right)^{\frac{1}{2}} \left(1 - \dfrac{2\gamma E}{\pi a \sigma^2}\right)^{\frac{1}{2}} \qquad \left(\rightarrow \quad \dfrac{ds}{da} \neq 0 \text{ !!}\right) \\[2em] \sqrt{\dfrac{2\pi}{k}} \approx 0.38 \qquad ; \qquad a_c = \dfrac{2\gamma E}{\pi\sigma^2} \qquad ; \qquad c = \sqrt{\dfrac{E}{\rho}} \end{array}\right\}$$

$$\left.\begin{array}{l} s = 0.38\, c \left(1 - \dfrac{a_c}{a}\right)^{\frac{1}{2}} \\[2em] a \gg a_c \end{array}\right\} \qquad \rightarrow \qquad \boxed{s \approx 0.38\ c}$$

Experimental crack growth rates

	steel	copper	aluminum	glass	rubber
E [GPa]	210	120	70	70	20
ρ [kg/m^2]	7800	8900	2700	2500	900
ν	0.29	0.34	0.34	0.25	0.5
c [m/sec]	5190	3670	5090	5300	46
s [m/sec]	1500			2000	
s/c	0.29			0.38	

$$\boxed{0.2 < \frac{s}{c} < 0.4}$$

Elastic wave speeds

$$
\begin{aligned}
C_0 &= \text{elongational wave speed} &&= \sqrt{\frac{E}{\rho}} \\
C_1 &= \text{dilatational wave speed} &&= \sqrt{\frac{\kappa+1}{\kappa-1}}\sqrt{\frac{\mu}{\rho}} \\
C_2 &= \text{shear wave speed} &&= \sqrt{\frac{\mu}{\rho}} \\
C_R &= \text{Rayleigh velocity} &&= 0.54\ C_0 \quad \text{á} \quad 0.62\ C_0
\end{aligned}
$$

Corrections

Dulancy & Brace (1960) $\quad s = 0.38\ C_0 \left(1 - \frac{a_c}{a}\right)$

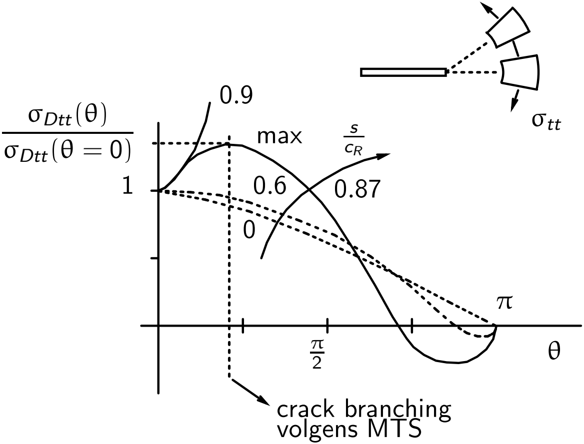
Freund (1972) $\quad s = C_R \left(1 - \frac{a_c}{a}\right)$

Crack tip stress

Yoffe (1951) : $$\sigma_{Dij} = \frac{K_D}{\sqrt{2\pi r}}\, f_{ij}(\theta, r, s, E, \nu)$$

Crack branching Yoffe (1951)

$$\sigma_{Dij} = \frac{K_{ID}}{\sqrt{2\pi r}} f_{ij}(\theta, r, s, E, \nu)$$



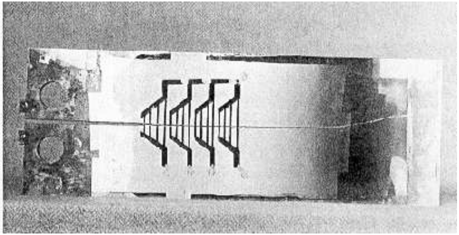
Source: Gdoutos (1993) p.245

Fast fracture and crack arrest

$$K_D \geq K_{Dc}(s, T) \qquad \longrightarrow \quad \text{crack growth}$$

$$K_D < \min_{0<s<C_R} K_{Dc}(s, T) = K_A \qquad \longrightarrow \quad \text{crack arrest}$$

Experiments



Source: KAN1985 p.210

- High Speed Photography : 10^6 frames/sec
- Robertson : CA Temperature (CAT) test (KAN1985 p.258)

PLASTIC CRACK TIP ZONE

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Von Mises and Tresca yield criteria

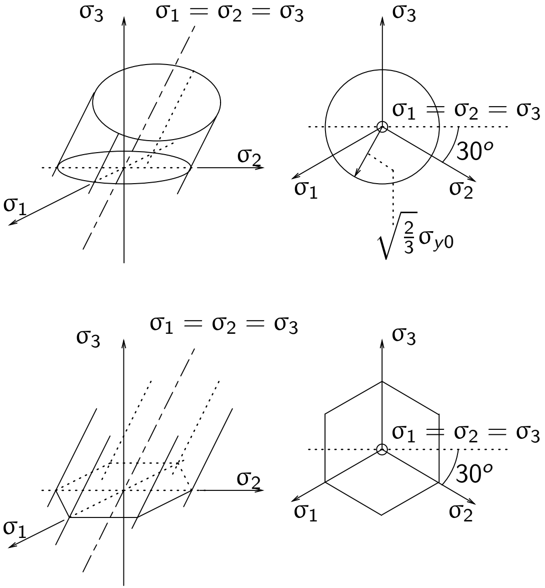
Von Mises $$W^d = W_c^d$$

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

Tresca $$\tau_{max} = \tau_{max_c}$$

$$\sigma_{max} - \sigma_{min} = \sigma_y$$

Yield surfaces in principal stress space



Principal stresses at the crack tip

plane stress state $\qquad \sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & 0 \\ \sigma_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \rightarrow \qquad \det(\underline{\sigma} - \sigma\underline{I}) = 0 \qquad \rightarrow$$

characteristic equation

$$\sigma\left[\sigma^2 - \sigma(\sigma_{xx} + \sigma_{yy}) + (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)\right] = 0 \qquad \rightarrow$$

$$\begin{aligned}
\sigma_1 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \left\{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2\right\}^{1/2} \\
\sigma_2 &= \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \left\{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \sigma_{xy}^2\right\}^{1/2} \\
\sigma_3 &= 0
\end{aligned}$$

plane strain state $\qquad \sigma_3 = \nu(\sigma_1 + \sigma_2)$

Principal stresses at crack tip

crack tip stresses $$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{Iij}(\theta)$$

$$\sigma_{1(+),2(-)} = \frac{K_I}{\sqrt{2\pi r}} \left[\cos(\tfrac{\theta}{2}) \pm \sqrt{\tfrac{1}{4}\left\{-2\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})\sin(\tfrac{3\theta}{2})\right\}^2 + \left\{\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})\cos(\tfrac{3\theta}{2})\right\}^2}\right]$$

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}} \cos(\tfrac{\theta}{2})\{1 + \sin(\tfrac{\theta}{2})\}$$

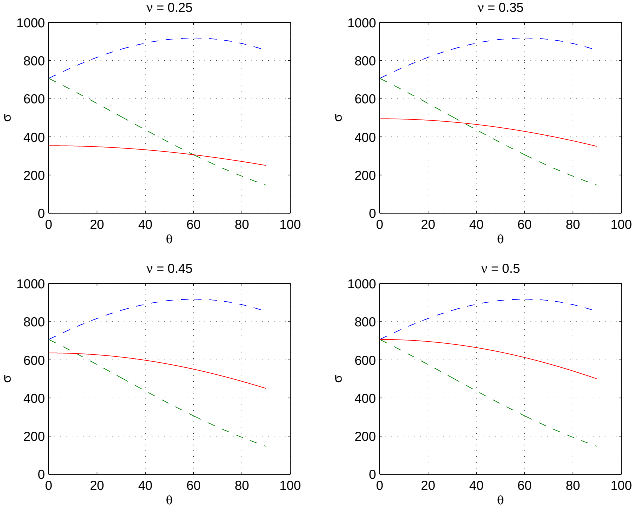
$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}} \cos(\tfrac{\theta}{2})\{1 - \sin(\tfrac{\theta}{2})\}$$

$\sigma_3 = 0$ or $\sigma_3 = \dfrac{2\nu K_I}{\sqrt{2\pi r}} \cos(\tfrac{\theta}{2})$

Principal stresses at crack tip

plane stress $\quad \sigma_1 > \sigma_2 > \sigma_3$

plane strain $\quad \sigma_1 > \sigma_2 > \sigma_3 \quad$ or $\quad \sigma_1 > \sigma_3 > \sigma_2$



Von Mises plastic zone

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

plane stress $\qquad \sigma_3 = 0$

$$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = 2\sigma_y^2$$

$$\frac{K_I^2}{2\pi r_y} \cos^2(\tfrac{\theta}{2}) \left[6 \sin^2(\tfrac{\theta}{2}) + 2\right] = 2\sigma_y^2$$

$$r_y = \frac{K_I^2}{2\pi\sigma_y^2} \cos^2(\tfrac{\theta}{2}) \left[1 + 3\sin^2(\tfrac{\theta}{2})\right] = \frac{K_I^2}{4\pi\sigma_y^2} \left[1 + \cos(\theta) + \tfrac{3}{2}\sin^2(\theta)\right]$$

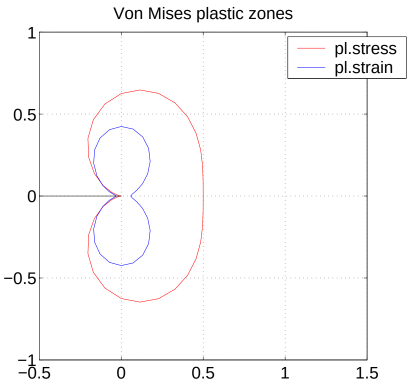
plane strain $\qquad \sigma_3 = \nu(\sigma_1 + \sigma_2)$

$$(\nu^2 - \nu + 1)(\sigma_1^2 + \sigma_2^2) + (2\nu^2 - 2\nu - 1)\sigma_1\sigma_2 = \sigma_y^2$$

$$\frac{K_I^2}{2\pi r_y} \cos^2(\tfrac{\theta}{2}) \left[6 \sin^2(\tfrac{\theta}{2}) + 2(1 - 2\nu)^2\right] = 2\sigma_y^2$$

$$r_y = \frac{K_I^2}{4\pi\sigma_y^2} \left[(1 - 2\nu)^2 \{1 + \cos(\theta)\} + \tfrac{3}{2}\sin^2(\theta)\right]$$

Von Mises plastic zone



Plot made with 'plazone.m'.

Tresca plastic zone

$$\sigma_{max} - \sigma_{min} = \sigma_y$$

plane stress

$$\{\sigma_{max}, \sigma_{min}\} = \{\sigma_1, \sigma_3\}$$

$$\frac{K_I}{\sqrt{2\pi r_y}} \left[\cos(\tfrac{\theta}{2}) + \left|\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})\right|\right] = \sigma_y$$

$$r_y = \frac{K_I^2}{2\pi\sigma_y^2} \left[\cos(\tfrac{\theta}{2}) + \left|\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})\right|\right]^2$$

plane strain I

$$\sigma_1 > \sigma_2 > \sigma_3 \quad \rightarrow \quad \{\sigma_{max}, \sigma_{min}\} = \{\sigma_1, \sigma_3\}$$

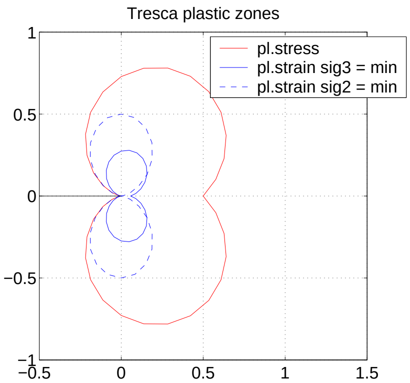
$$r_y = \frac{K_I^2}{2\pi\sigma_y^2} \left[(1-2\nu)\cos(\tfrac{\theta}{2}) + \left|\cos(\tfrac{\theta}{2})\sin(\tfrac{\theta}{2})\right|\right]^2$$

plane strain II

$$\sigma_1 > \sigma_3 > \sigma_2 \quad \rightarrow \quad \{\sigma_{max}, \sigma_{min}\} = \{\sigma_1, \sigma_2\}$$

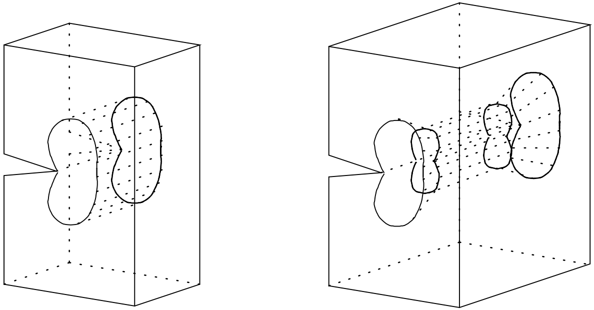
$$r_y = \frac{K_I^2}{2\pi\sigma_y^2} \sin^2(\theta)$$

Tresca plastic zone



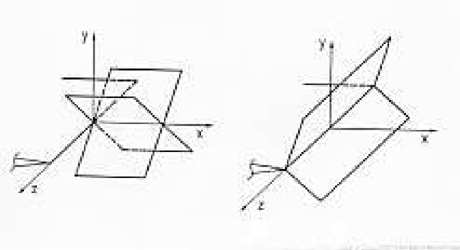
Plot made with 'plazone.m'.

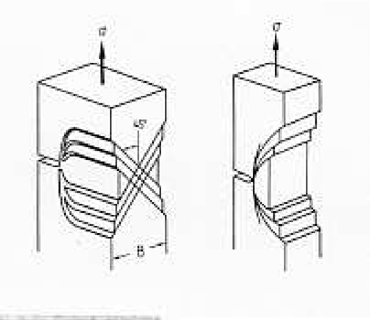
Influence of the plate thickness

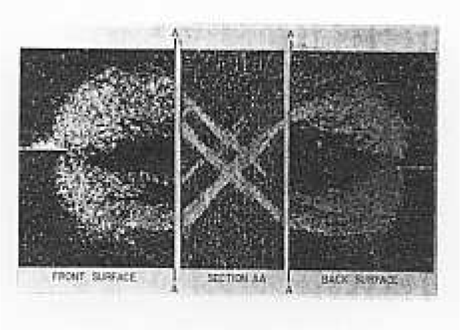


$$B_c > \frac{25}{3\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2 > 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

Shear planes

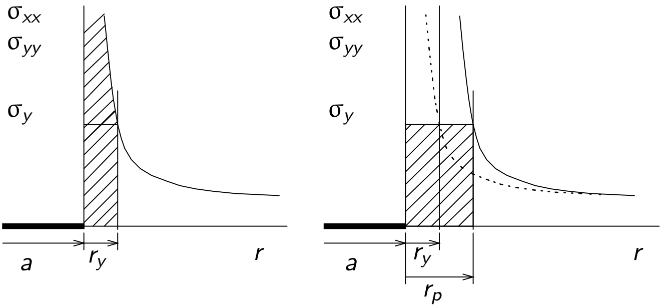






Source: Gdoutos p.60/61/62; Kanninen p.176

Irwin plastic zone correction

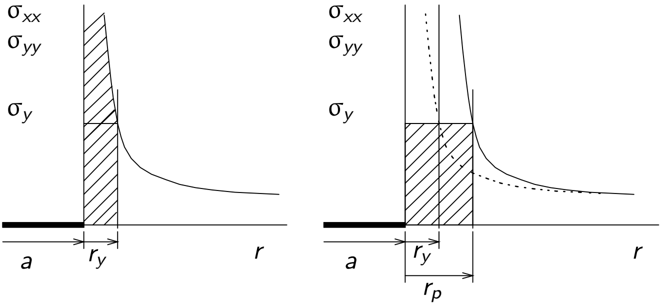


$$\theta = 0 \qquad \rightarrow \qquad \sigma_{xx} = \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}$$

$$\text{yield} \qquad \sigma_{xx} = \sigma_{yy} = \sigma_y \qquad \rightarrow \qquad r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_y}\right)^2$$

equilibrium not satisfied $\rightarrow$ **correction** required $\rightarrow$ shaded area equal

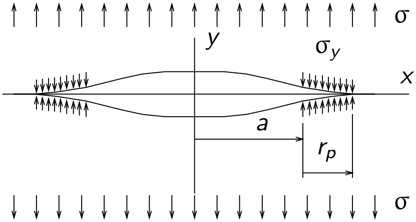
Irwin plastic zone correction



$$\sigma_y r_p = \int_0^{r_y} \sigma_{yy}(r)\, dr = \frac{K_I}{\sqrt{2\pi}} \int_0^{r_y} r^{-\frac{1}{2}}\, dr = \frac{2K_I}{\sqrt{2\pi}} \sqrt{r_y} \quad \rightarrow$$

$$r_p = \frac{2K_I}{\sqrt{2\pi}} \frac{\sqrt{r_y}}{\sigma_y} \qquad \rightarrow \qquad r_p = \frac{1}{\pi} \left(\frac{K_I}{\sigma_y} \right)^2 = 2\ r_y$$

Dugdale-Barenblatt plastic zone correction



load σ $\qquad K_I(\sigma) = \sigma\sqrt{\pi(a+r_p)}$

load σ_y $\qquad K_I(\sigma_y) = 2\sigma_y\sqrt{\dfrac{a+r_p}{\pi}}\,\arccos\left(\dfrac{a}{a+r_p}\right)$

singular term $= 0$ $\quad \rightarrow \quad K_I(\sigma) = K_I(\sigma_y) \quad \rightarrow$

$$\frac{a}{a+r_p} = \cos\left(\frac{\pi\sigma}{2\sigma_y}\right) \qquad \rightarrow \qquad r_p = \frac{\pi K_I^2}{8\sigma_y^2}$$

Plastic constraint factor

$$\sqrt{\tfrac{1}{2}\{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\}} =$$
$$\left[\sqrt{1-n-m+n^2+m^2-mn}\,\right]\sigma_{max} = \sigma_y$$
$$\text{PCF} = \frac{\sigma_{max}}{\sigma_y} = \frac{1}{\sqrt{1-n-m+n^2+m^2-mn}}$$

PCF at the crack tip

pl.sts $\quad n = \left[1-\sin(\tfrac{\theta}{2})\right] / \left[1+\sin(\tfrac{\theta}{2})\right] \quad ; \quad m = 0$

pl.stn $\quad n = \left[1-\sin(\tfrac{\theta}{2})\right] / \left[1+\sin(\tfrac{\theta}{2})\right] \quad ; \quad m = 2\nu / \left[1+\sin(\tfrac{\theta}{2})\right]$

PCF at the crack tip in the crack plane

pl.sts $\quad n = 1\ ;\ m = 0 \quad \rightarrow \quad \text{PCF} = 1$

pl.stn $\quad n = 1\ ;\ m = 2\nu \quad \rightarrow \quad \text{PCF} = \dfrac{1}{\sqrt{1-4\nu+4\nu^2}}$

Plastic zones in the crack plane

criterion	state	r_y or r_p	$\frac{r_y \mid r_p}{(K_I/\sigma_y)^2}$
Von Mises	plane stress	$\frac{1}{2\pi}\left(\frac{K_I}{\sigma_y}\right)^2$	0.1592
Von Mises	plane strain	$\frac{1}{18\pi}\left(\frac{K_I}{\sigma_y}\right)^2$	0.0177
Tresca	plane stress	$\frac{1}{2\pi}\left(\frac{K_I}{\sigma_y}\right)^2$	0.1592
Tresca	plane strain $\sigma_1 > \sigma_2 > \sigma_3$	$\frac{1}{18\pi}\left(\frac{K_I}{\sigma_y}\right)^2$	0.0177
Tresca	plane strain $\sigma_1 > \sigma_3 > \sigma_2$	0	0
Irwin	plane stress	$\frac{1}{\pi}\left(\frac{K_I}{\sigma_y}\right)^2$	0.3183
Irwin	plane strain (PCF $=$ 3)	$\frac{1}{\pi}\left(\frac{K_I}{3\sigma_y}\right)^2$	0.0354
Dugdale	plane stress	$\frac{\pi}{8}\left(\frac{K_I}{\sigma_y}\right)^2$	0.3927
Dugdale	plane strain (PCF $=$ 3)	$\frac{\pi}{8}\left(\frac{K_I}{3\sigma_y}\right)^2$	0.0436

Small Scale Yielding

- LEFM & SSY
- correction $\rightarrow$ effective crack length a_{eff}
- Irwin / Dugdale-Barenblatt correction

- SSY : outside plastic zone : $K_I(a_{eff})$-stress

$$a_{eff} = a + (r_y | r_p) \quad \leftrightarrow \quad K_I = \beta_I(a_{eff})\sigma\sqrt{\pi a_{eff}}$$

NONLINEAR FRACTURE MECHANICS

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Crack-tip opening displacement

crack tip displacement

$$u_y = \frac{\sigma\sqrt{\pi a}}{2\mu}\sqrt{\frac{r}{2\pi}}\left[\sin(\tfrac{1}{2}\theta)\left\{\kappa + 1 - 2\cos^2(\tfrac{1}{2}\theta)\right\}\right]$$

displacement in crack plane $\quad \theta = \pi;\ r = a - x$

$$u_y = \frac{(1+\nu)(\kappa+1)}{E}\,\frac{\sigma}{2}\,\sqrt{2a(a-x)}$$

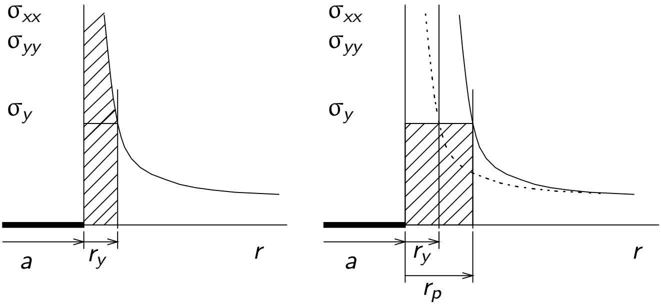
Crack Opening Displacement (COD)

$$\delta(x) = 2u_y(x) = \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2a(a-x)}$$

Crack Tip Opening Displacement (CTOD)

$$\delta_t = \delta(x = a) = 0$$

CTOD by Irwin



effective crack length

$$a_{eff} = a + r_y = a + \frac{1}{2\pi} \left(\frac{K_I}{\sigma_y} \right)^2$$

CTOD by Irwin

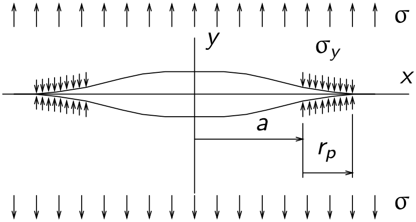
$$\delta(x) = \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2a_{eff}(a_{eff}-x)}$$
$$= \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2(a+r_y)(a+r_y-x)}$$

$$\delta_t = \delta(x=a) = \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2(a+r_y)r_y}$$
$$= \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2ar_y+2r_y^2}$$
$$\approx \frac{(1+\nu)(\kappa+1)}{E}\,\sigma\sqrt{2ar_y}$$

plane stress : $\delta_t = \frac{4}{\pi}\frac{K_I^2}{E\sigma_y} = \frac{4}{\pi}\frac{G}{\sigma_y}$

plane strain : $\delta_t = \left[\frac{1}{\sqrt{3}}\right]\frac{4(1-\nu^2)}{\pi}\frac{K_I^2}{E\sigma_y}$

CTOD by Dugdale



effective crack length

$$a_{eff} = a + r_p = a + \frac{\pi}{8}\left(\frac{K_I}{\sigma_y}\right)^2$$

CTOD by Dugdale

displacement from requirement "singular term = 0" : $\bar{u}_y(x)$

$$\bar{u}_y(x) = \frac{(a+r_p)\sigma_y}{\pi E}\left[\frac{x}{a+r_p}\ln\left\{\frac{\sin^2(\hat{\gamma}-\gamma)}{\sin^2(\hat{\gamma}+\gamma)}\right\} + \cos(\hat{\gamma})\ \ln\left\{\frac{\sin(\hat{\gamma})+\sin(\gamma)}{\sin(\hat{\gamma})-\sin(\gamma)}\right\}^2\right]$$

$$\gamma = \arccos\left(\frac{x}{a+r_p}\right) \quad ; \quad \hat{\gamma} = \frac{\pi}{2}\,\frac{\sigma}{\sigma_y}$$

Crack Tip Opening Displacement

$$\left.\begin{array}{l} \delta_t = \lim\limits_{x \longrightarrow a} 2\bar{u}_y(x) = \dfrac{8\sigma_v a}{\pi E}\ \ln\left\{\sec\left(\dfrac{\pi}{2}\,\dfrac{\sigma}{\sigma_y}\right)\right\} \\ \text{series expansion} \quad \& \quad \sigma \ll \sigma_y \end{array}\right\} \rightarrow$$

plane stress : $\delta_t = \dfrac{K_I^2}{E\sigma_y} = \dfrac{G}{\sigma_y}$

plane strain : $\delta_t = \left[\dfrac{1}{2}\right]\ (1-\nu^2)\ \dfrac{K_I^2}{E\sigma_y}$

CTOD crack growth criterion

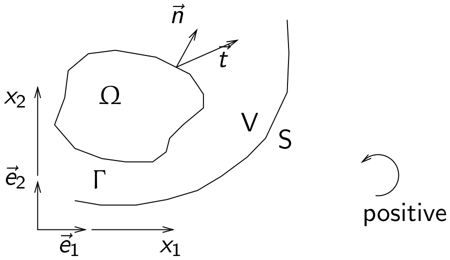
- $\delta_t \sim (G, K_I)$ at LEFM
- $\delta_t =$ measure for deformation at crack tip (LEFM)
- $\delta_t =$ measure for (large) plastic deformation at crack tip (NLFM)

- criterion

$$\boxed{\delta_t = \delta_{tc}(\dot{\varepsilon}, T)}$$

- δ_t calculate or measure
- δ_{tc} experimental determination (ex. BS 5762)

J-integral



$$J_k = \int_\Gamma \left(W n_k - t_i \frac{\partial u_i}{\partial x_k} \right) \, d\Gamma \qquad ; \qquad W = \text{specific energy} = \int_0^{\varepsilon_{pq}} \sigma_{ij} \, d\varepsilon_{ij}$$

$$J = J_1 = \int_\Gamma \left(W n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, d\Gamma \qquad \left[\frac{\text{N}}{\text{m}} \right]$$

Integral along closed curve

$$J_k = \int\limits_{\Gamma} \left(W\delta_{jk} - \sigma_{ij} u_{i,k} \right) n_j \, d\Gamma$$

inside Γ no singularities $\quad \rightarrow \quad$ Stokes (Gauss in 3D)

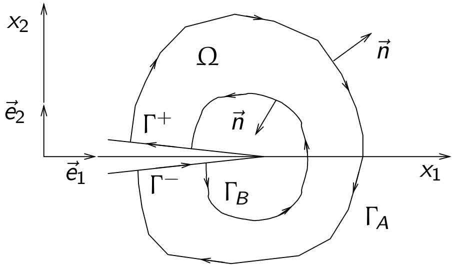
$$\int\limits_{\Omega} \left(\frac{dW}{d\varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial x_j} \delta_{jk} - \sigma_{ij,j} u_{i,k} - \sigma_{ij} u_{i,kj} \right) d\Omega$$

homogeneous hyper-elastic	$\sigma_{mn} = \dfrac{\partial W}{\partial \varepsilon_{mn}}$
linear strain	$\varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m})$
equilibrium equations	$\sigma_{ij,j} = 0$

$$\int\limits_{\Omega} \left\{ \tfrac{1}{2}\sigma_{mn}(u_{m,nk} + u_{n,mk}) - \sigma_{ij} u_{i,kj} \right\} d\Omega =$$

$$\int\limits_{\Omega} \left(\sigma_{mn} u_{m,nk} - \sigma_{ij} u_{i,kj} \right) d\Omega = 0$$

Path independency



$$\int_{\Gamma_A} f_1 \, d\Gamma + \int_{\Gamma_B} f_1 \, d\Gamma + \int_{\Gamma^-} f_1 \, d\Gamma + \int_{\Gamma^+} f_1 \, d\Gamma = 0$$

no loading of crack faces : $\quad n_1 = 0 \quad ; \quad t_i = 0 \quad$ on $\quad \Gamma^+$ and Γ^-

$$\left.\begin{array}{l} \displaystyle\int_{\Gamma_A} f_1 \, d\Gamma + \int_{\Gamma_B} f_1 \, d\Gamma = 0 \\ \displaystyle\int_{\Gamma_A} f_1 \, d\Gamma = J_{1_A} \quad ; \quad \int_{\Gamma_B} f_1 \, d\Gamma = -J_{1_B} \end{array}\right\} \rightarrow J_{1_A} - J_{1_B} = 0 \rightarrow \boxed{J_{1_A} = J_{1_B}}$$

Relation $J \sim K$

lin. elast. material : $\quad W = \frac{1}{2}\sigma_{mn}\varepsilon_{mn} = \frac{1}{4}\sigma_{mn}(u_{m,n} + u_{n,m})$

$$
\begin{aligned}
J_k &= \int\limits_{\Gamma} \left(\tfrac{1}{4}\sigma_{mn}(u_{m.n} + u_{n,m})\delta_{jk} - \sigma_{ij}u_{i,k} \right) n_j \, d\Gamma \\
&= \int\limits_{\Gamma} \left(\tfrac{1}{2}\sigma_{mn}u_{m,n}\delta_{jk} - \sigma_{ij}u_{i,k} \right) n_j \, d\Gamma
\end{aligned}
$$

Mode I + II + III

$$
\begin{aligned}
\sigma_{ij} &= \frac{1}{\sqrt{2\pi r}} \left[K_I f_{Iij} + K_{II} f_{IIij} + K_{III} f_{IIIij} \right] \\
u_i &= u_{Ii} + u_{IIi} + u_{IIIi}
\end{aligned}
$$

substitution and integration over $\Gamma =$ circle

$$
\begin{aligned}
J_1 &= \frac{(\kappa + 1)(1 + \nu)}{4E} \left(K_I^2 + K_{II}^2 \right) + \frac{(1 + \nu)}{E} K_{III}^2 \\
J_2 &= -\frac{(\kappa + 1)(1 + \nu)}{2E} K_I K_{II}
\end{aligned}
$$

Relation $J \sim G$

Mode I $$J_1 = J = \frac{(\kappa+1)(1+\nu)}{4E}\ K_I^2 = G$$

plane stress $$\kappa + 1 = \frac{3-\nu}{1+\nu} + \frac{1+\nu}{1+\nu} = \frac{4}{1+\nu} \qquad \rightarrow \qquad J = \frac{1}{E}\ K_I^2$$

plane strain $$\kappa + 1 = 4 - 4\nu \qquad \rightarrow \qquad J = \frac{(1-\nu^2)}{E}\ K_I^2$$

Relation $J \sim \delta_t$

plane stress	Irwin	$J = \dfrac{\pi}{4}\,\sigma_y \delta_t$
	Dugbale	$J = \sigma_y \delta_t$
plane strain	Irwin	$J = \dfrac{\pi}{4}\,\sqrt{3}\,\sigma_y \delta_t$
	Dugbale	$J = 2\sigma_y \delta_t$

Plastic constraint factor

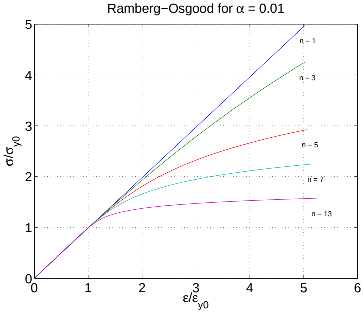
$$J = m\,\sigma_y \delta_t$$

$$m = -0.111 + 0.817\,\frac{a}{W} + 1.36\,\frac{\sigma_u}{\sigma_y}$$

Ramberg-Osgood material law

$$\frac{\varepsilon}{\varepsilon_{y0}} = \frac{\sigma}{\sigma_{y0}} + \alpha \left(\frac{\sigma}{\sigma_{y0}} \right)^n$$

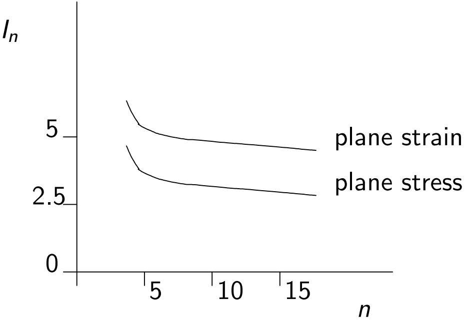
n	strain hardening parameter	$(n \geq 1)$
$n = 1$	linear elastic	
$n \to \infty$	ideal plastic	



HRR-solution

$$\sigma_{ij} = \sigma_{y0}\beta\, r^{-\frac{1}{n+1}}\, \tilde{\sigma}_{ij}(\theta) \qquad ; \qquad u_i = \alpha\varepsilon_{y0}\beta^n\, r^{\frac{1}{n+1}}\, \tilde{u}_i(\theta)$$

with : $$\beta = \left[\frac{J}{\alpha\sigma_{y0}\varepsilon_{y0}I_n}\right]^{\frac{1}{n+1}}$$ (I_n from num. anal.)



J-integral crack growth criterion

- LEFM : $\quad J_k \sim G \sim (K_I, K_{II}, K_{III})$
- NLFM : Ramberg-Osgood : $\quad J$ determines crack tip stress

- criterion

$$\boxed{J = J_c}$$

- calculate J
- J_{Ic} from experiments e.g. ASTM E813

NUMERICAL FRACTURE MECHANICS

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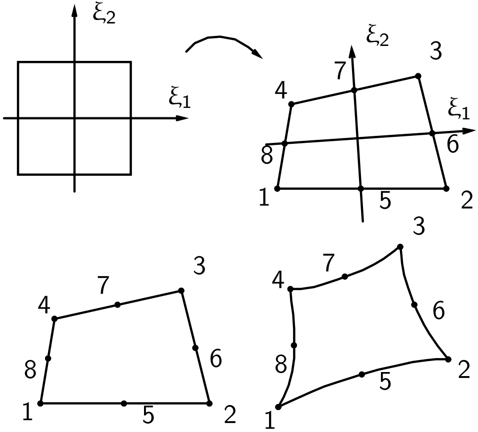
Numerical fracture mechanics

- Methods EEM ; BEM
- Calculations
  - G
  - K
  - δ_t
  - J
- Simulation crack growth

Quadratic elements

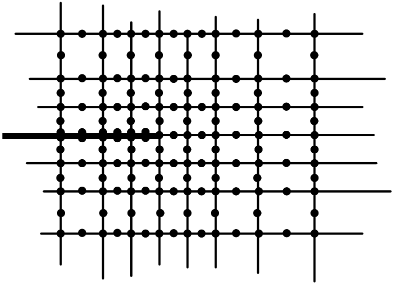


isoparametric coordinates : $-1 \leq \xi_i \leq 1$

shape functions for each node n

$\psi_n(\xi_1, \xi_2) =$ quadratic in ξ_1 and ξ_2

Crack tip mesh

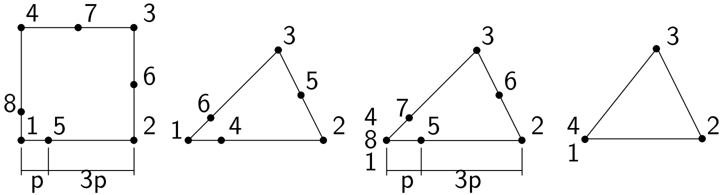


- bad approximation stress field $1/\sqrt{r}$
- results are mesh-dependent

Special elements

- enriched elements
  - crack tip field added to element displacement field
  - structure $\underline{K}$ and $\underset{\sim}{f}$ changes
  - transition elements for compatibility
- hybrid elements
  - modified variational principle

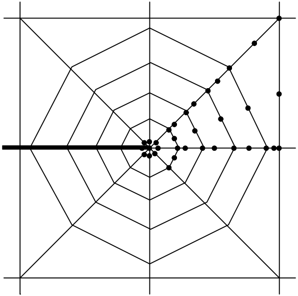
Quarter point elements



- Distorted Quadratic Quadrilateral $(1/\sqrt{r})$
- Distorted Quadratic Triangle $(1/\sqrt{r})$
- Collapsed Quadratic Quadrilateral $(1/\sqrt{r})$
- Collapsed Distorted Linear Quadrilateral $(1/r)$

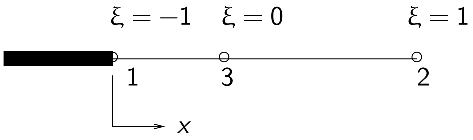
- good approximation stress field $(1/\sqrt{r}$ or $1/r)$
- bad approximation non-singular stress field
- standard FEM-programs can be used

Crack tip rozet



- Quarter Point Elements : 8x
- Transition Elements : number is problem dependent
- Buffer Elements

One-dimensional case



position

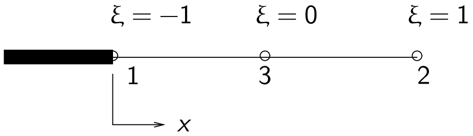
$$
\begin{aligned}
x &= \tfrac{1}{2}\xi(\xi-1)x_1 + \tfrac{1}{2}\xi(\xi+1)x_2 - (\xi^2-1)x_3 \\
&= \tfrac{1}{2}\xi(\xi+1)L - (\xi^2-1)x_3
\end{aligned}
$$

displacement and strain

$$
\begin{aligned}
u &= \tfrac{1}{2}\xi(\xi-1)u_1 + \tfrac{1}{2}\xi(\xi+1)u_2 - (\xi^2-1)u_3 \\
\frac{du}{d\xi} &= (\xi - \tfrac{1}{2})u_1 + (\xi + \tfrac{1}{2})u_2 - 2\xi u_3 \qquad \rightarrow \\
\frac{du}{dx} &= \frac{du}{d\xi}\,\frac{d\xi}{dx} = \frac{du}{d\xi}\Big/\frac{dx}{d\xi}
\end{aligned}
$$

Mid point element

mid-point element : $x_3 = \frac{1}{2}L$



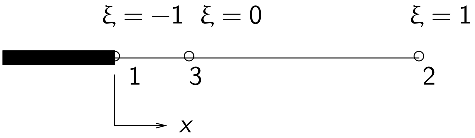
$$x = \tfrac{1}{2}\xi(\xi+1)L - (\xi^2-1)\tfrac{1}{2}L = \tfrac{1}{2}(\xi+1)L \qquad \Rightarrow$$

$$\frac{dx}{d\xi} = \tfrac{1}{2}L$$

$$\frac{du}{dx} = \frac{\frac{du}{d\xi}}{\frac{1}{2}L} \quad \rightarrow \quad \left.\frac{du}{dx}\right|_{\substack{x=0\\ \xi=-1}} = \left(\frac{2}{L}\right)\left\{\left(-\tfrac{3}{2}\right)u_1 + \left(\tfrac{1}{2}\right)u_2 + 2u_3\right\}$$

Quarter point element

quarter-point element : $x_3 = \frac{1}{4}L$



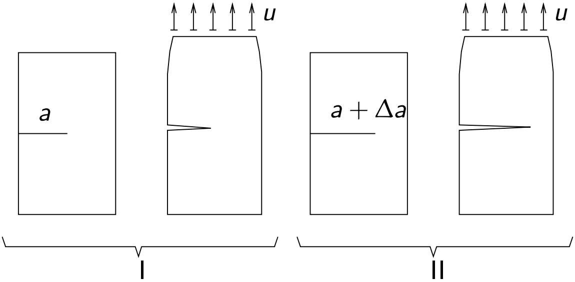
$$x = \tfrac{1}{2}\xi(\xi+1)L - (\xi^2-1)\tfrac{1}{4}L = \tfrac{1}{4}(\xi+1)^2 L \quad \rightarrow \quad \xi + 1 = \sqrt{\frac{4x}{L}} \quad \Rightarrow$$

$$\frac{dx}{d\xi} = \tfrac{1}{2}(\xi+1)L = \sqrt{xL}$$

$$\frac{du}{dx} = \frac{\frac{du}{d\xi}}{\sqrt{xL}} \quad \rightarrow \quad \left.\frac{du}{dx}\right|_{\substack{x=0 \\ \xi=-1}} = \infty$$

singularity $\quad \dfrac{1}{\sqrt{x}}$

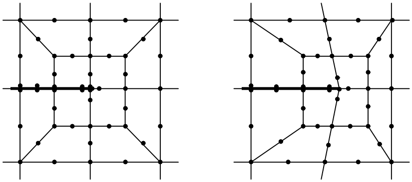
Virtual crack extension method (VCEM)



$$\text{fixed grips} \quad \rightarrow \quad \frac{dU_e}{da} = 0 \quad \Rightarrow$$

$$G = -\frac{1}{B}\frac{dU_i}{da} \approx -\frac{1}{B}\frac{U_i(a+\Delta a) - U_i(a)}{\Delta a}$$

VCEM : stiffness matrix variation



$$B\,G = -\frac{dU_i}{da} = -\tfrac{1}{2}\,\underset{\sim}{u}^T\,\frac{\Delta\underline{C}}{\Delta a}\,\underset{\sim}{u} \qquad \text{with} \qquad \Delta\underline{C} = \underline{C}(a+\Delta a) - \underline{C}(a)$$

- G from analysis crack tip mesh only
- nodal point displacement : $\pm$ 0.001 $*$ element size
- not possible with crack tip in interface
- unloaded crack plane
- no thermal stresses

Stress intensity factor

- calculate G_I and G_{II} with VCEM
- calculate K_I and K_{II} from

$$K_I^2 = E'G_I \qquad ; \qquad K_{II}^2 = E'G_{II}$$

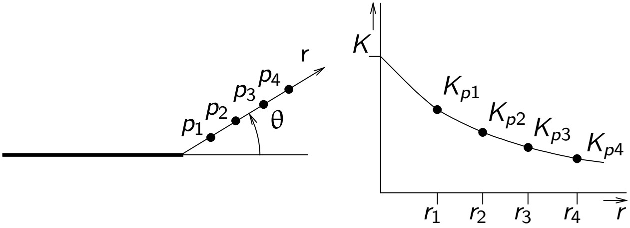
plane stress	$E' = E$
plane strain	$E' = E/(1-\nu^2)$

- difficult for crack propagation study

SIF : stress field

$$K_I = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{22}|_{\theta=0} \right) \quad ; \quad K_{II} = \lim_{r \longrightarrow 0} \left(\sqrt{2\pi r}\ \sigma_{12}|_{\theta=0} \right)$$

extrapolation to crack tip



questions :

- which elements ?
- how much elements ?
- which integration points ?

SIF : displacement field

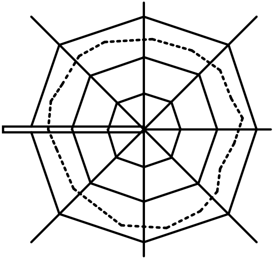
crack tip displacement y-component

$$u_y = \frac{4(1-\nu^2)}{E}\sqrt{\frac{r}{2\pi}}\,K_I\,g_{ij}(\theta) \quad \rightarrow$$

$$K_I = \lim_{r \longrightarrow 0}\left[\frac{E}{4(1-\nu^2)}\sqrt{\frac{2\pi}{r}}\,u_y(\theta=0)\right]$$

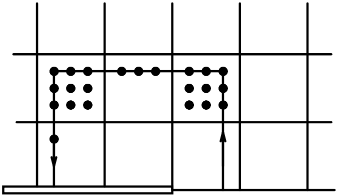
more accurate than SIF from stress field

J-integral



$$J = \int_{\Gamma} \left(W n_1 - t_i \frac{\partial u_i}{\partial x_1} \right) \, d\Gamma \qquad \text{with} \quad W = \int_0^{\varepsilon} \sigma_{ij} \, d\varepsilon_{ij}$$

J-integral : Direct calculation

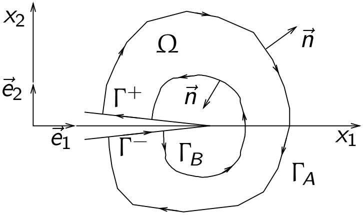


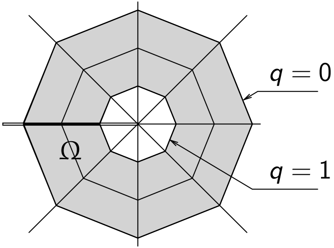
$$J = 2\int_y \left[W - \left(\sigma_{xx}\frac{\partial u_x}{\partial x} + \sigma_{yx}\frac{\partial u_y}{\partial x} \right) \right] dy - 2\int_x \left[\left(\sigma_{xy}\frac{\partial u_x}{\partial x} + \sigma_{yy}\frac{\partial u_y}{\partial x} \right) \right] dx$$

$$W = \frac{E}{2(1-\nu^2)}(\varepsilon_{xx}^2 + 4\nu\varepsilon_{xx}\varepsilon_{yy} + 2(1-\nu)\varepsilon_{xy}^2 + \varepsilon_{yy}^2)$$

$\Rightarrow$ path through integration points
$\Rightarrow$ no need for quarter point elements

J-integral : Domain integration

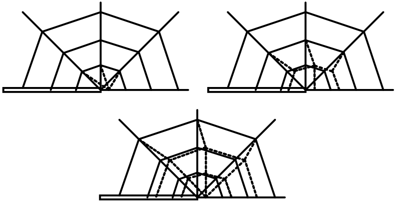




$$J = \int_{\Omega} \frac{\partial q}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \, d\Omega$$

interpolation $\qquad q^e = \underset{\sim}{N}^T \left(\underset{\sim}{\xi} \right) \underset{\sim}{q}^e$

De Lorenzi J-integral : VCE technique



$$J = \int_\Omega \frac{\partial q}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) d\Omega -$$

$$\int_{\Gamma_s} q p_i \frac{\partial u_i}{\partial x_1} \, d\Gamma - \int_\Omega q(\rho q_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial x_1} \, d\Omega + \int_\Omega q \sigma_{ij} \frac{\partial \varepsilon_{ij}^o}{\partial x_1} \, d\Omega$$

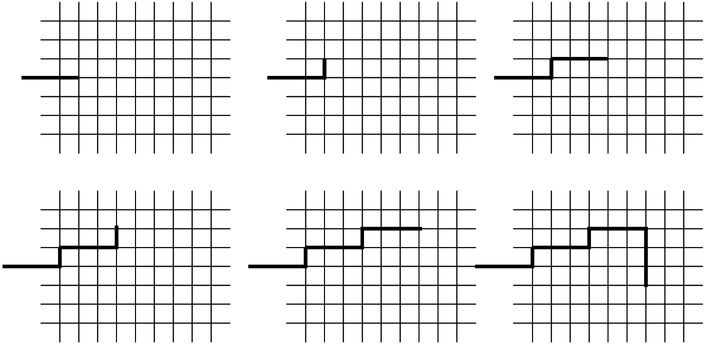
- rigid region
  - elongation Δa of crack
  - translation δx_1 of internal nodes
  - fixed position of boundary
- $q = \dfrac{\delta x_1}{\Delta a}$ = shift function $(0 < q < 1)$

Crack growth simulation

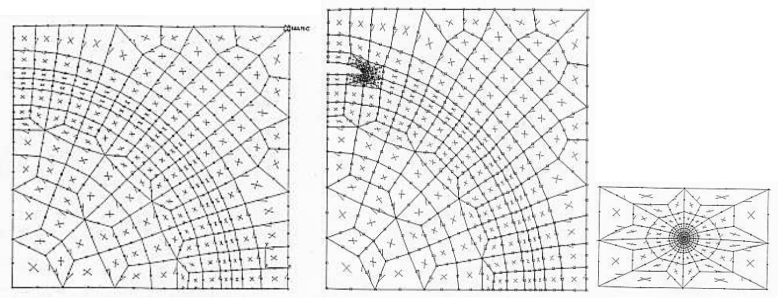
- Node release
- Moving Crack Tip Mesh
- Element splitting
- Smeared crack approach

Node release

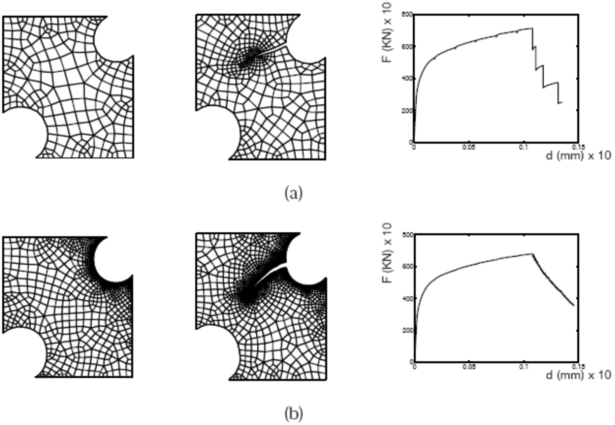
node collocation technique



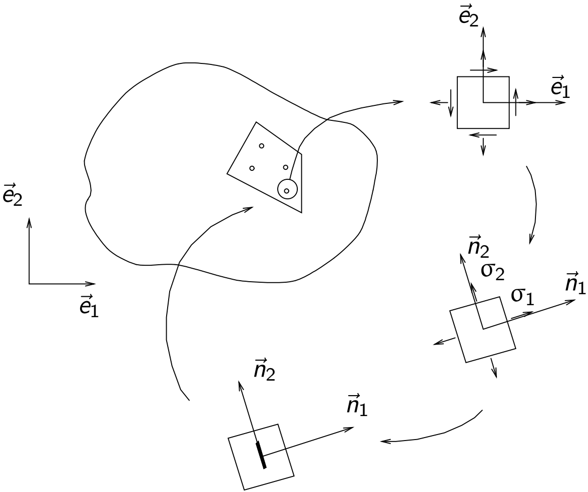
Moving Crack Tip Mesh



Element splitting



Smeared crack approach



FATIGUE

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Teletekst Wo 3 oktober 2007

Van de 274 stalen bruggen in ons land kampen er 25 met metaalmoeheid. Dat is de uitkomst van een groot onderzoek van het ministerie van Verkeer. Bij twaalf bruggen zijn de problemen zo groot dat noodmaatregelen nodig zijn.

Ook de meer dan 2000 betonnen bruggen en viaducten zijn onderzocht. De helft daarvan moet nog nader worden bekeken. Ze gaan mogelijk minder lang mee dan was berekend, maar de veiligheid komt volgens het ministerie niet in gevaar.

Verkeersbeperkende maatregelen zijn dan ook niet nodig. Die werden in april wel getroffen voor het vrachtverkeer over de Hollandse Brug bij Almere.

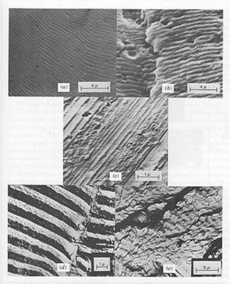
Fatigue

- $\pm$ 1850 (before Griffith !) :
  cracks at diameter-jumps in axles carriages / trains
- failure due to cyclic loading with small amplitude
- Wöhler : systematic experimental examination

cyclic loading :

- variable mechanical loads
- vibrations
- pressurization / depressurization
- thermal loads (heating / cooling)
- random external loads

Crack surface

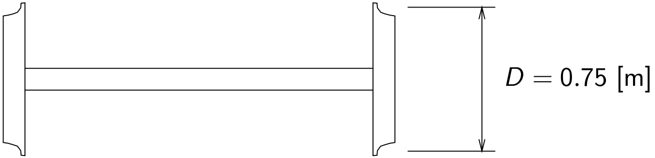



- clam shell markings (beach marks)
  - irregular crack growth
  - crack growth under changing conditions
- striations
  - sliding of slip planes
  - plastic blunting / sharpening of crack tip
  - regular crack growth

Experiments

- full-scale testing
  a.o.
  - train axles
  - airplanes
- laboratory testing
  - harmonic loading
  - constant force/moment
  - strain/deflection
  - SIF

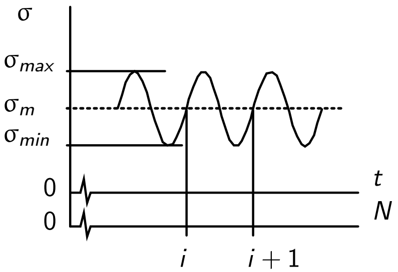
Train axle



1 rev $= \pi D = \pi \times 0.75 \approx 2.25$ [m]
1 km $= 1000$ m $= \frac{1000}{2.25} = \frac{4000}{9} \approx 445$ [c(ycles)]
1 day Maastricht - Groningen $= 2.5 \times 333$ [km] $= 1000$ [km]
1 day Maastricht - Groningen $= 445 \times 10^3$ [c]
1 year $= 300 \times 445 \times 10^3$ [c] $= 1335 \times 10^5$ [c] $\approx 1.5 \times 10^8$ [c]

frequency :
100 [km/h] $= 445 \times 10^2$ [c/h] $= \frac{44500}{3600} = 12.5$ [c/sec] $= 12.5$ [Hz]

Fatigue load (stress controlled)



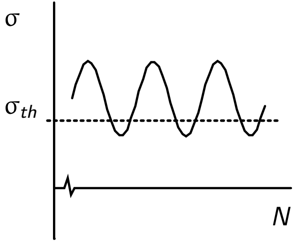
$\Delta\sigma = \sigma_{max} - \sigma_{min}$; $\sigma_a = \frac{1}{2}\Delta\sigma$

$\sigma_m = \frac{1}{2}(\sigma_{max} + \sigma_{min})$; $R_\sigma = \sigma_{min}/\sigma_{max}$; $\dfrac{\sigma_a}{\sigma_m} = \dfrac{1-R}{1+R}$

-	frequency	bending		30 - 80 Hz
		tensile	electric	50 - 300 Hz
			mechanic	$<$ 50 Hz
			hydraulic	1 - 50 Hz

- no influence frequency for $\pm$ 5000 [c/min] (metals)

Fatigue limit (σ_{th})



$\sigma < \sigma_{th}$: no increase of damage

materials with fatigue limit

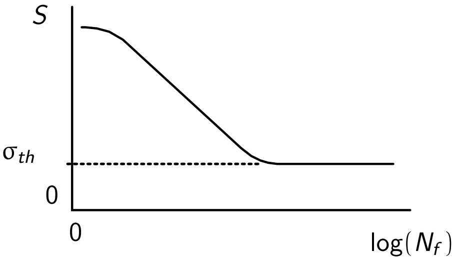
- mild steel
- low strength steels
- Ti / Al / Mg -alloys

materials without fatigue limit

- some austenitic steels
- high strength steels
- most non-ferro alloys
- Al / Mg-alloys

(S-N)-curve

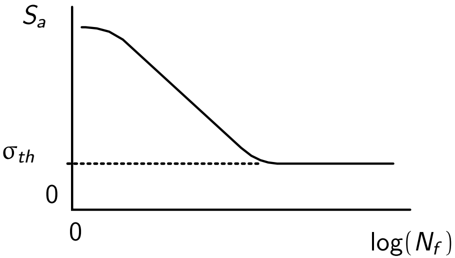
B.S. 3518 part I 1984 : $S = \sigma_{max}$



- reference : $R = -1$ and $\sigma_m = 0 \quad \rightarrow \quad \sigma_{max} = \frac{1}{2}\Delta\sigma$
- fatigue life : N_f at $\sigma_{max}(= S)$
- fatigue limit : $\sigma_{th}(= \sigma_{fat}) \quad \rightarrow \quad N_f = \infty(\pm 10^9)$
- fatigue strength : $\sigma_e = \sigma_{max}$ when $N_f \approx 50 \times 10^6$
- steels : $\sigma_{th} \approx \frac{1}{2}\sigma_b$

$(S_a$-N)-curve

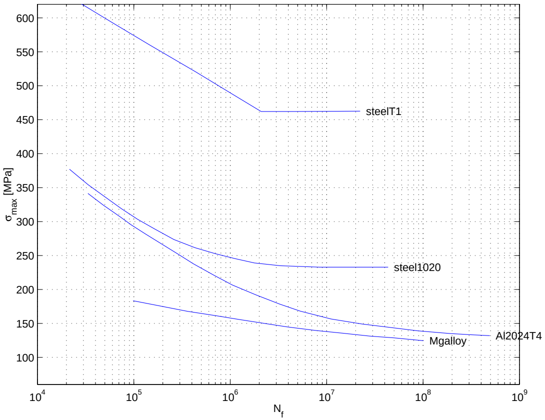
B.S. 3518 part I 1984 : $S_a = \frac{1}{2}\Delta\sigma = \sigma_a$



- reference : $R = -1$ and $\sigma_m = 0 \quad \rightarrow \quad \sigma_a = \sigma_{max}$

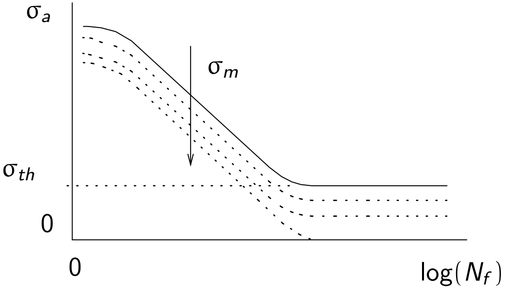
$$(S_a - N) \ \text{curve} = (S - N) \ \text{curve}$$

Examples



600
550
500
450
400
350
300
250
200
150
100
σ_max [MPa]
10^4
10^5
10^6
10^7
10^8
10^9
N_f
steelT1
steel1020
Al2024T4
Mgalloy

Influence of average stress



Correction for average stress

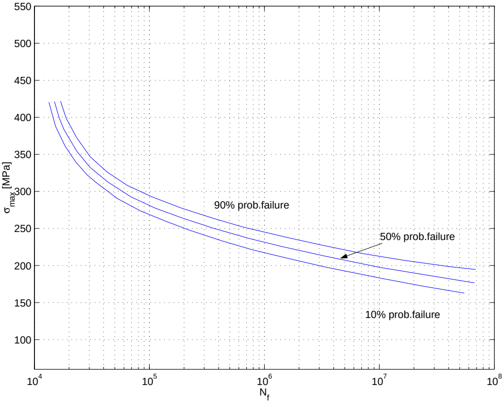
Gerber (1874) $$\frac{\sigma_a^*}{\sigma_a} = 1 - \left(\frac{\sigma_m}{\sigma_u}\right)^2$$

Goodman (1899) $$\frac{\sigma_a^*}{\sigma_a} = 1 - \frac{\sigma_m}{\sigma_u}$$

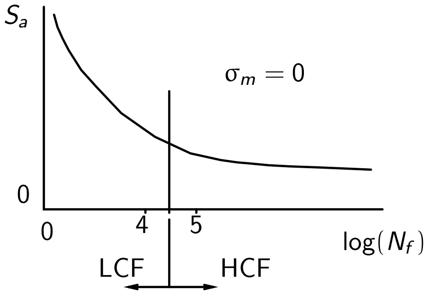
Soderberg (1939) $$\frac{\sigma_a^*}{\sigma_a} = 1 - \frac{\sigma_m}{\sigma_{y0}}$$

σ_u	:	tensile strength
σ_{y0}	:	initial yield stress

(P-S-N)-curve


550
500
450
400
350
300
250
200
150
100
σ_{max} [MPa]
90% prob.failure
50% prob.failure
10% prob.failure
10^4
10^5
10^6
10^7
10^8
N_f

High/low cycle fatigue

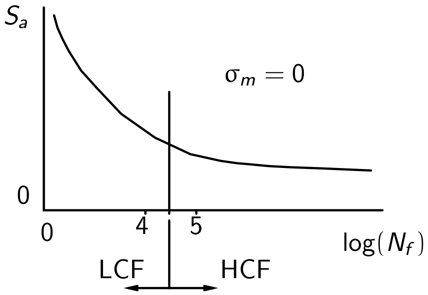


high cycle fatigue

- $N_f > \pm 50000$
- low stresses $\quad \rightarrow \quad$ LEFM + SSY
- stress-life curve
- Basquin relation

$$K_{max} = \beta \sigma_{max} \sqrt{\pi a} \quad ; \quad K_{min} = \beta \sigma_{min} \sqrt{\pi a} \quad ; \quad \Delta K = \beta \Delta \sigma \sqrt{\pi a}$$

High/low cycle fatigue



low cycle fatigue

- $N_f < \pm 50000$
- high stresses $\quad \rightarrow \quad$ EPFM
- strain-life curve
- Manson-Coffin relation

Basquin relation

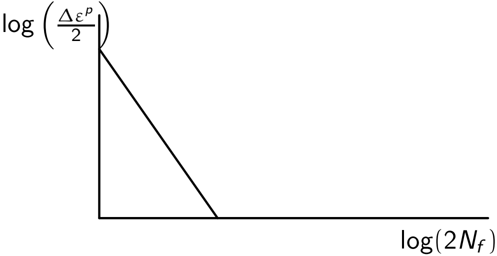
$$\frac{1}{2}\Delta\sigma = \sigma_a = \sigma_f'(2N_f)^b \quad \rightarrow \quad \Delta\sigma N_f^{-b} = \text{constant}$$

$$\begin{aligned} \sigma_f' &= \text{fatigue strength coefficient} \\ &\approx \sigma_b \quad \text{(monotonic tension)} \\ b &= \text{fatigue strength exponent} \\ & \text{(Basquin exponent)} \end{aligned}$$

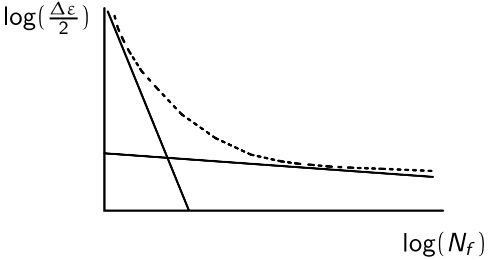
Manson-Coffin relation

$$\tfrac{1}{2}\Delta\varepsilon^p = \varepsilon_f'(2N_f)^c \quad \rightarrow \quad \Delta\varepsilon^p N_f^{-c} = \text{constant}$$

$$\begin{aligned} \varepsilon_f' &= \text{fatigue ductility coefficient} \\ &\approx \varepsilon_b \quad \text{(monotonic tension)} \\ c &= \text{fatigue ductility exponent} \qquad (-0.5 < c < -0.7) \end{aligned}$$



Total strain-life curve



$$
\begin{aligned}
\frac{\Delta\varepsilon}{2} &= \frac{\Delta\varepsilon^e}{2} + \frac{\Delta\varepsilon^p}{2} \\
&= \frac{1}{E}\sigma_f'(2N_f)^b + \varepsilon_f'(2N_f)^c
\end{aligned}
$$

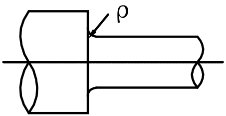
Influence factors

- load spectrum
- stress concentrations
- stress gradients
- material properties
- surface quality
- environment

Load spectrum

- sign / magnitude / rate / history
- multi-axial $\rightarrow$ lower f.limit than uni-axial

Stress concentrations



$$\Delta\sigma_{th}(\text{notched}) = \frac{1}{K_f}\,\Delta\sigma_{th}(\text{unnotched}) \quad ; \qquad 1 < K_f < K_t$$

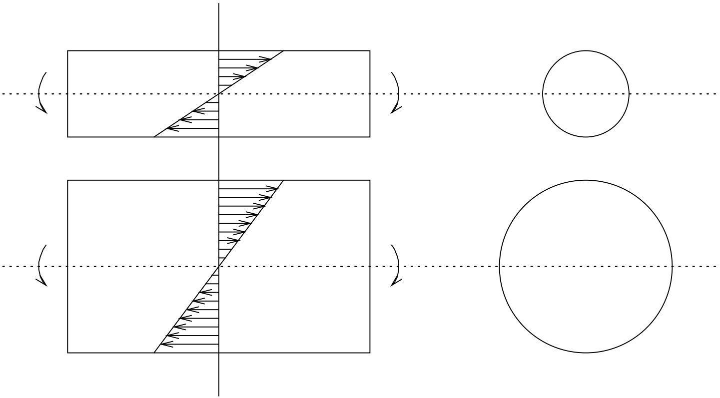
K_f : fatigue strength reduction factor
(effective stress concentration factor)

$K_f = 1 + q(\rho)(K_t - 1)$ $\qquad$ $q(\rho)$ = notch sensitivity factor

Peterson : $q = \dfrac{1}{1 + \frac{a}{\rho}}$ with a = material parameter

Neuber : $q = \dfrac{1}{1 + \sqrt{\frac{b}{\rho}}}$ with b = grain size parameter

Stress gradients

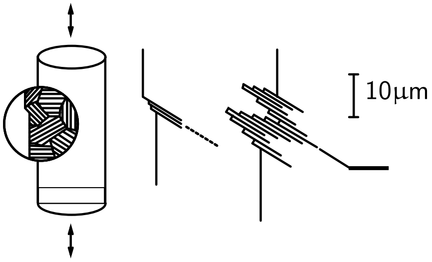


full-scale experiments necessary

Material properties

- grain size/structure :
  small grains $\rightarrow$ higher f.limit at low temp.
  large grains $\rightarrow$ higher f.limit at high temp.
  (less grain boundaries $\rightarrow$ less creep)
- texture
- inhomogeneities and flaws
- residual stresses
- fibers and particles

Surface quality

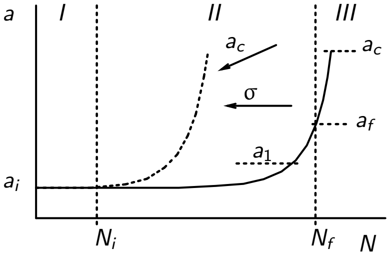


- surface $\rightarrow$ extrusions & intrusions $\rightarrow$ notch + inclusion of O_2 etc.
- bulk defect $\rightarrow$ internal surfaces
- internal grain boundaries / triple points (high T) $\rightarrow$ voids
- manufacturing $\rightarrow$ minimize residual tensile stresses
- surface finish $\rightarrow$ minimize defects (roughness)
- surface treatment (mech/temp) $\rightarrow$ residual pressure stresses
- coating $\rightarrow$ environmental protection
- high σ_{y0} $\rightarrow$ more resistance to slip band formation

Environment

- temperature $\longrightarrow$ creep - fatigue
- low temperature : ships / liquefied gas storage
- elevated temperature ($T > 0.5\,T_m$) : turbine blades
- creep mechanism :
  diffusion / dislocation movement / migration of vacancies / grain boundary sliding $\longrightarrow$
  grain boundary voids / wedge cracks
- chemical influence $\longrightarrow$ corrosion-fatigue

Crack growth

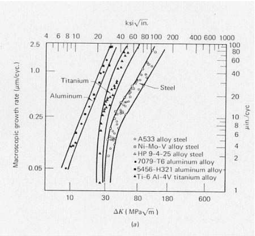


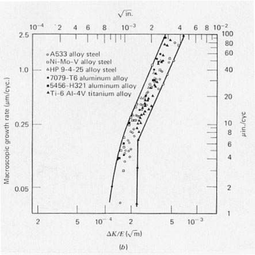
I :	$N < N_i$	-	N_i = fatigue crack initiation life
		-	a_i = initial fatigue crack
II :	$N_i < N < N_f$	-	slow stable crack propagation
		-	a_1 = non-destr. inspection detection limit
III :	$N_f < N$	-	global instability
		-	towards catastrophic failure
		-	$a = a_c$: failure

$$\frac{N_r}{N_f} = 1 - \frac{N}{N_f} \qquad N_r = \text{rest life}$$

Crack growth models

- $$\frac{da}{dN} \sim \text{striation spacing } \sim 6\left(\frac{\Delta K}{E}\right)^2 \qquad \text{(Bates, Clark (1969))}$$
- $$\frac{da}{dN} \sim f(\sigma, a) \sim \sigma^m a^n \qquad ; \qquad m \approx 2-7 \qquad ; \qquad n \approx 1-2$$
- $$\frac{da}{dN} \sim \delta_t \sim \frac{(\Delta K)^2}{E\sigma_y} \qquad (BRO263)$$
- $$\frac{da}{dN} \sim \Delta K \quad \rightarrow \quad \frac{da}{dN} \sim \frac{\Delta K}{E}$$

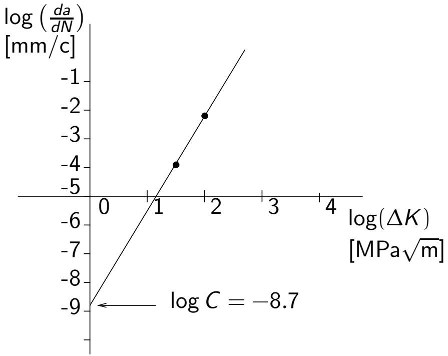




Source: HER1976a p515

- Paris law : $$\frac{da}{dN} = C(\Delta K)^m$$

Paris law

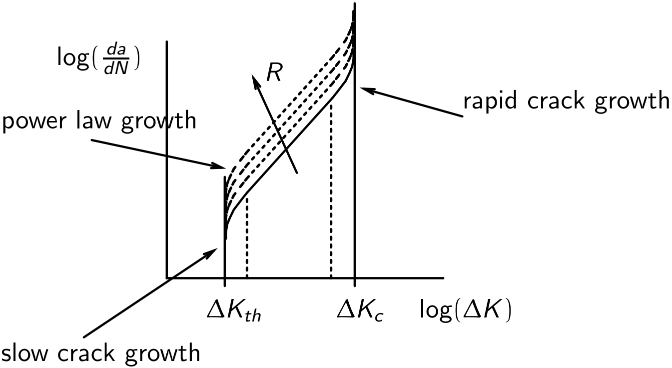


$$\frac{da}{dN} = C(\Delta K)^m \qquad \rightarrow \qquad \log\left(\frac{da}{dN}\right) = \log(C) + m\log(\Delta K)$$

$$\log(\Delta K) = 0 \rightarrow \log(C) = \log\left(\frac{da}{dN}\right) = -8.7 \quad \rightarrow \quad C = 2 \times 10^{-9} \quad \frac{[\text{mm}]}{[\text{MPa}\sqrt{\text{m}}]^m}$$

$$m = \frac{(-2)-(-4)}{(2)-(1.5)} = 4$$

Limits of Paris law



- $\Delta K \approx \Delta K_{th} \quad \Rightarrow \quad$ roughness induced crack closure
- $\Delta K < \Delta K_{th} \quad \Rightarrow \quad$ growth very short cracks $\quad (10^{-8}$ mm/cycle$)$
  $\rightarrow$ dangerous overestimation of fatigue life
- $\sigma_m \uparrow \quad \rightarrow \quad R \uparrow (\frac{7}{9} \rightarrow \frac{10}{12} \rightarrow \frac{100}{102} \rightarrow 1)$

Paris law parameters

material	ΔK_{th} [$\mathrm{MNm}^{-3/2}$]	m[-]	$C \times 10^{-11}$[!]
mild steel	3.2 - 6.6	3.3	0.24
structural steel	2.0 - 5.0	3.85 - 4.2	0.07 - 0.11
idem in sea water	1.0 - 1.5	3.3	1.6
aluminium	1.0 - 2.0	2.9	4.56
aluminium alloy	1.0 - 2.0	2.6 - 3.9	3 - 19
copper	1.8 - 2.8	3.9	0.34
titanium	2.0 - 3.0	4.4	68.8

Conversion

$$\frac{da}{dN} = C\,(\Delta\sigma\,\sqrt{\pi a})^m \qquad \rightarrow \qquad C = \frac{\frac{da}{dN}}{(\Delta\sigma\,\sqrt{\pi a})^m}$$

[in] and [ksi] $\quad\rightarrow\quad$ [m] and [MPa]

$$1\,\frac{[\,\mathrm{in}\,]}{[\,\mathrm{ksi}\,\sqrt{\mathrm{in}}\,]^m} = \frac{0.0254\,[\,\mathrm{m}\,]}{\{6.86\;[\,\mathrm{MPa}\,]\,\sqrt{0.0254\,[\,\mathrm{m}\,]}\,\}^m} = \left(\frac{0.0254}{(1.09)^m}\right) \quad \frac{[\,\mathrm{m}\,]}{[\,\mathrm{MPa}\,\sqrt{\mathrm{m}}\,]^m}$$

[m] and [MPa] $\quad\rightarrow\quad$ [mm] and [MPa]

$$1\,\frac{[\,\mathrm{m}\,]}{[\,\mathrm{MPa}\,\sqrt{\mathrm{m}}\,]^m} = \frac{10^3\,[\,\mathrm{mm}\,]}{\{[\,\mathrm{MPa}\,]\,\sqrt{10^3}\,[\,\sqrt{\mathrm{mm}}\,]\}^m} = \left(\frac{10^3}{\{\sqrt{10^3}\}^m}\right) \quad \frac{[\,\mathrm{mm}\,]}{[\,\mathrm{MPa}\,\sqrt{\mathrm{mm}}\,]^m}$$

Fatigue life : analytical integration

integration Paris law $\quad\rightarrow\quad$ fatigue life N_f

$$N_f - N_i = \frac{(\Delta\sigma)^{-m}}{\beta^m C(\sqrt{\pi})^m(1-\frac{m}{2})}\; a_f^{(1-\frac{m}{2})}\left[1-\left(\frac{a_i}{a_f}\right)^{(1-\frac{m}{2})}\right]$$

numerical procedure

set $\Delta\sigma,\ \Delta N,\ a_c$

initialize $N = 0,\ a = a_0$

while $a < a_c$

$$\Delta K = \beta\,\Delta\sigma\sqrt{\pi * a}$$

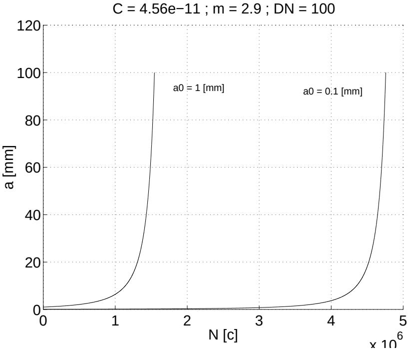
$$\frac{da}{dN} = C * (\Delta K)^m \quad\rightarrow\quad \Delta a = \frac{da}{dN} * \Delta N$$

$$a = a + \Delta a$$

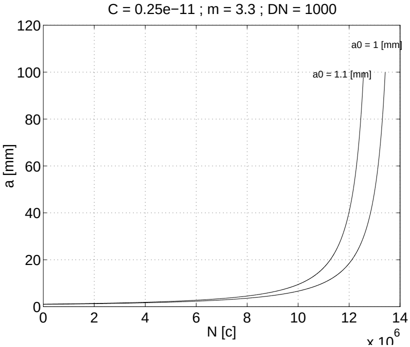
$$N = N + \Delta N$$

end

Initial crack length



aluminum ; $\Delta\sigma = 50$ [MPa]



mild steel ; $\Delta\sigma = 50$ [MPa]

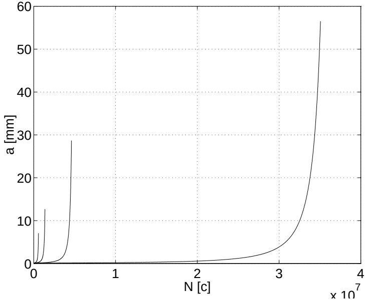
Fatigue load

fatigue life at $\qquad a_f = a_c = \dfrac{2\gamma}{\pi} \dfrac{E}{\Delta\sigma^2} \qquad \rightarrow \qquad N_f$

aluminum $\qquad C = 4.56e-11 \quad ; \quad m = 2.9$

$E = 70$ [GPa] $\quad ; \quad \gamma = 1$ [J/m^2]

$\Delta\sigma$ [MPa]	25	50	75	100
a_0 [mm]	0.1	0.1	0.1	0.1
a_c [mm]	56	28	12.5	7
N_f [c]	35070000	4610000	1366000	572000



Other crack grow laws

Erdogan (1963) (general empirical law)

$$\frac{da}{dN} = \frac{C(1+\beta)^m(\Delta K - \Delta K_{th})^n}{K_{I_c} - (1+\beta)\Delta K} \qquad \text{with} \quad \beta = \frac{K_{max} + K_{min}}{K_{max} - K_{min}}$$

Broek & Schijve (1963)

$$\frac{da}{dN} = CK_{max}^2 \, \Delta K$$

Other crack grow laws

Forman (1967) ($K_{max} \rightarrow K_c$)

$$\frac{da}{dN} = \frac{C(\Delta K)^n}{(1-R)K_c - \Delta K} \qquad \text{with} \quad R = \frac{K_{min}}{K_{max}}$$

Donahue (1972) ($\Delta K \rightarrow \Delta K_{th}$)

$$\frac{da}{dN} = C(\Delta K - \Delta K_{th})^m \qquad \text{with} \quad \Delta K_{th} = (1-R)^{\gamma} \Delta K_{th}(R=0)$$

Walker (1970) (influence R)

$$\frac{da}{dN} = C \left\{ \frac{\Delta K}{(1-R)^n} \right\}^m \qquad \text{with} \quad m = 0.4 \quad ; \quad n = 0.5$$

Other crack grow laws

Priddle (1976) ($\Delta K \rightarrow \Delta K_{th}$ & $K_{max} \rightarrow K_c$)

$$\frac{da}{dN} = C\left(\frac{\Delta K - \Delta K_{th}}{K_{I_c} - K_{max}}\right)^m$$

with $\Delta K_{th} = A(1-R)^{\gamma}$ and $\frac{1}{2} \leq \gamma \leq 1$ [Schijve (1979)]

McEvily & Gröger (1977) (theoretical)

$$\frac{da}{dN} = \frac{A}{E\sigma_v}(\Delta K - \Delta K_{th})^2\left(1 + \frac{\Delta K}{K_{I_c} - K_{max}}\right)$$

with $\Delta K_{th} = \sqrt{\frac{1-R}{1+R}}\Delta K_0$

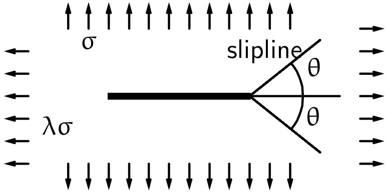
$A, \Delta K_0 \quad \sim \quad$ influence environment

Other crack grow laws

NASA / FLAGRO program (1989)

$$
\begin{aligned}
\frac{da}{dN} &= \frac{C(1-R)^m \Delta K^n (\Delta K - \Delta K_{th})^p}{[(1-R)K_{Ic} - \Delta K]^q} \\
m = p = q = 0 \qquad &\rightarrow \qquad \text{Paris} \\
m = p = 0, q = 1 \qquad &\rightarrow \qquad \text{Forman} \\
p = q = 0, m = (m_w - 1)n \qquad &\rightarrow \qquad \text{Walker}
\end{aligned}
$$

Crack growth at low cycle fatigue



$$\frac{da}{dN} = \frac{3 - \sin^{-2}(\theta)\cos^{-2}(\frac{\theta}{2})}{9\sin(\theta)} \frac{K}{E\sigma_v} \left(1 - \beta\gamma^{-\frac{1}{2}}\right) \frac{K_{max}^2}{\{1 - (1-\lambda)\frac{\sigma_{max}}{\sigma_v}\}}$$

$$\left.\begin{array}{l} \theta = \cos^{-1}\left(\frac{1}{3}\right) \\ \dfrac{\beta}{\sqrt{\gamma}} = 0.5 + 0.1R + 0.4R^2 \end{array}\right\} \quad \rightarrow$$

$$\frac{da}{dN} = \frac{7}{64\sqrt{2}} \frac{K}{E\sigma_v} (1 - 0.2R - 0.8R^2) \frac{K_{max}^2}{\{1 - (1-\lambda)\frac{\sigma_{max}}{\sigma_v}\}}$$

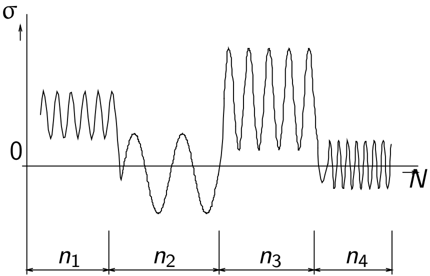
Crack growth at low cycle fatigue

J-integral based Paris law

$$\frac{da}{dN} = C^* \left(\Delta J\right)^{m^*}$$

$$\text{with} \qquad \Delta J = \int_\Gamma \left\{ W^* n_1 - \Delta t_i \frac{\partial \Delta u_i}{\partial x_1} \right\} d\Gamma \quad ; \qquad W^* = \int_{\varepsilon_{pq_{min}}}^{\varepsilon_{pq_{max}}} \Delta\sigma_{ij}\, d\varepsilon_{ij}$$

Load spectrum



$$\sum_{i=1}^{L} \frac{n_i}{N_{if}} = 1 \qquad \text{Palmgren-Miner (1945) law}$$

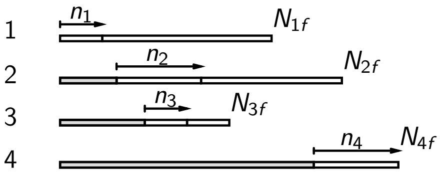
$\Rightarrow$ life time by piecewise integration $\frac{da}{dN} \sim f(\Delta K, K_{max})$

$\Rightarrow$ no interaction

$\Rightarrow$ interaction $\rightarrow$ Palmgren-Miner no longer valid :

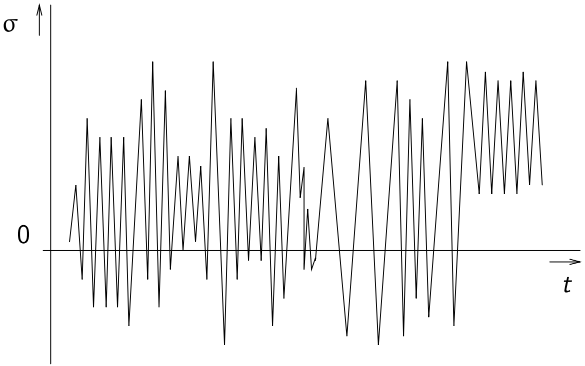
$$\sum_{i=1}^{L} \frac{n_i}{N_{if}} = 0.6 - 2.0$$

Miner's rule



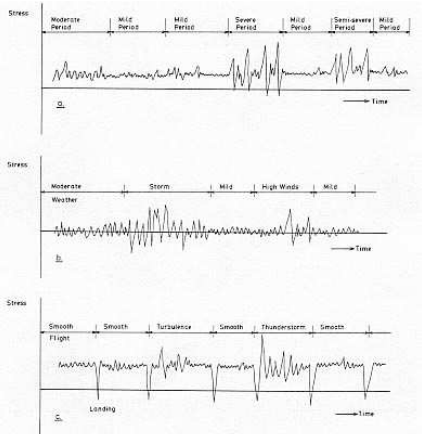
$$
\begin{aligned}
1 \quad &\longrightarrow \quad 1 - \frac{n_1}{N_{1f}} \\
2 \quad &\longrightarrow \quad \left(1 - \frac{n_1}{N_{1f}}\right) - \frac{n_2}{N_{2f}} \\
3 \quad &\longrightarrow \quad \left(1 - \frac{n_1}{N_{1f}} - \frac{n_2}{N_{2f}}\right) - \frac{n_3}{N_{3f}} \\
4 \quad &\longrightarrow \quad \left(1 - \frac{n_1}{N_{1f}} - \frac{n_2}{N_{2f}} - \frac{n_3}{N_{3f}}\right) - \frac{n_4}{N_{4f}} = 0
\end{aligned}
$$

Random load



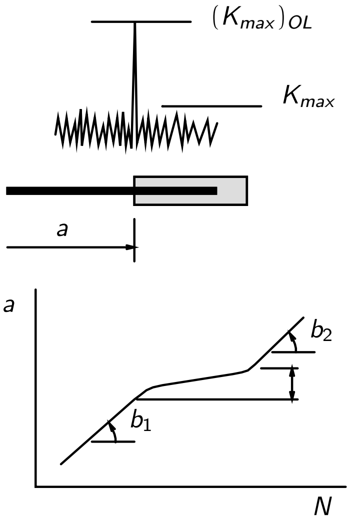
- cyclic counting procedure
  - (mean crossing) peak count
  - range pair (mean) count
  - rain flow count
- statistical representation $\longrightarrow$ load spectrum

Measured load histories



- instrumentation with strain gages at critical locations
- measure load history
- continuous monitoring during service $\longrightarrow$ update spectrum
- standard spectra

Tensile overload

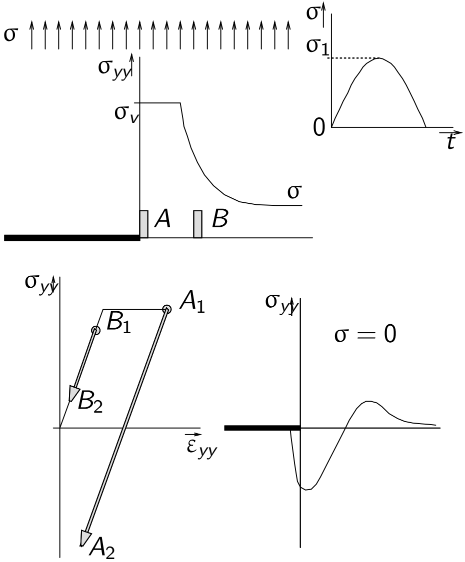


Crack retardation

Al 2024-T3 (Hertzberg, 1976)

ΔK [MPa$\sqrt{\text{m}}$]	% P_{max} [-]	nr. P_{max} [-]	delay [10^3 cycles]
15	53	1	6
15	82	1	16
15	109	1	59
16.5	50	1	4
16.5	50	10	5
16.5	50	100	9.9
16.5	50	450	10.5
16.5	50	2000	22
16.5	50	9000	44
23.1	50	1	9
23.1	75	1	55
23.1	100	1	245

Plastic zone residual stress



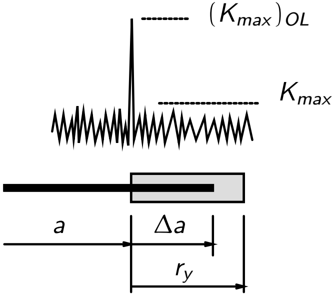
Crack retardation models

Willenborg (1971)

$$K_R = \phi \left[(K_{max})_{OL} \left[\sqrt{1 - \frac{\Delta a}{r_y}} \right] - K_{max} \right] \qquad ; \quad \Delta a < r_y$$

$K_R =$ residual SIF ; $K_R = 0 \quad \rightarrow \quad$ delay distance

$\phi = [1 - (K_{th}/K_{max})](S-1)^{-1}$; $S =$ shut-off ratio

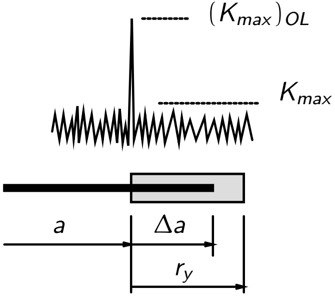


Crack retardation models

Johnson (1981)

$$R^{eff} = \frac{K_{min} - K_R}{K_{max} - K_R} \quad ; \quad r_y = \frac{1}{\beta\pi}\left(\frac{(K_{max})_{OL}}{\sigma_v}\right)^2$$

$\beta =$ plastic constraint factor



Crack retardation models

Elber (1971)

$$\Delta K_{eff} = U\,\Delta K \quad ; \quad U = 0.5 + 0.4R \quad \text{with} \quad -0.1 \leq R \leq 0.7$$

Schijve (1981)

$$U = 0.55 + 0.33R + 0.12R^2 \quad \text{with} \quad -1.0 < R < 0.54$$

Design against fatigue

- infinite life design
- safe life design
- damage tolerant design
- fail safe design

Infinite life design

$$\sigma < \sigma_{th} \qquad (\sigma < \sigma_e)$$

$\Rightarrow$ no fatigue damage
$\Rightarrow$ sometimes economically undesirable

Safe life design

$\Rightarrow$ determine load spectra

$\Rightarrow$ empirical rules / numerical analysis / laboratory tests $\longrightarrow$
fatigue life : $(S - N)$-curves

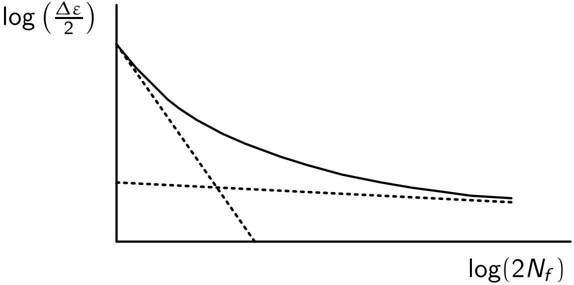
$\Rightarrow$ apply safety factors

$\Rightarrow$ sometimes safety factors are undesirable (weight)

$\Rightarrow$ stress-life design or strain-life design

Stress/strain life design

Basquin	$\frac{1}{2}\Delta\sigma = \sigma_f'(2N_f)^b$	$\rightarrow$	$\frac{1}{2}\Delta\varepsilon^e = \dfrac{1}{E}\sigma_f'(2N_f)^b$
Manson-Coffin	$\frac{1}{2}\Delta\varepsilon^p = \varepsilon_f'(2N_f)^c$		
combination	$\Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^p \quad \rightarrow$		
	$\frac{1}{2}\Delta\varepsilon = \frac{1}{2}\sigma_f'(2N_f)^b + \varepsilon_f'(2N_f)^c$		



Damage tolerant design

$\Rightarrow$ dangerous situations not acceptable
safety factors undesirable

$\Rightarrow$ determine load spectra

$\Rightarrow$ periodic inspection (insp. schedules) $\rightarrow$ monitor cracks

$\Rightarrow$ NDT important

$\Rightarrow$ calculate safe rest life
(integrate appropriate $\dfrac{da}{dN}$-growth law)

$\Rightarrow$ repair when necessary

Fail safe design

$\Rightarrow$ design for safety : crack arrest / etc.

ENGINEERING PLASTICS

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Engineering plastics (polymers)

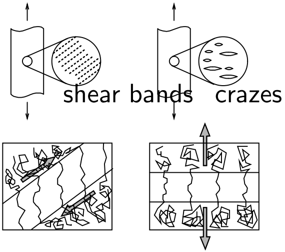
ABS	acrylonitrilbutadieenstyreen	EM
HIPS	high-impact polystyrene	TP
LDPE	low-density polythene	TP
Nylon		
PC	polycarbonate	TP
PMMA	polymethylemethacrylate (plexiglas)	TP
PP	polypropylene	TP
PPO	polyfenyleneoxide	TP
PS	polystyrene	TP
PSF	polysulfone	TP
PTFE	polytetrafluorethene (teflon)	TP
PVC	polyvinylchloride	TP
PVF	polyvinylfluoride	TP
PVF2	polyvinylidieenfluoride	TP

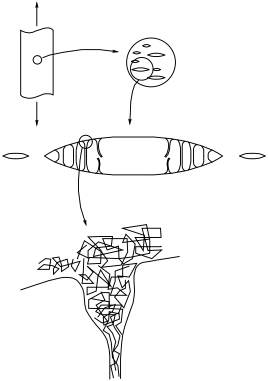
Mechanical properties

- (nonlinear) elastic
- visco-elastic
- thermal influences
- anisotropy

Damage

- shearing (shear yielding)
  no change in density
- crazing (normal yielding)
  change in density : 40 - 60 % decrease





Properties of engineering plastics

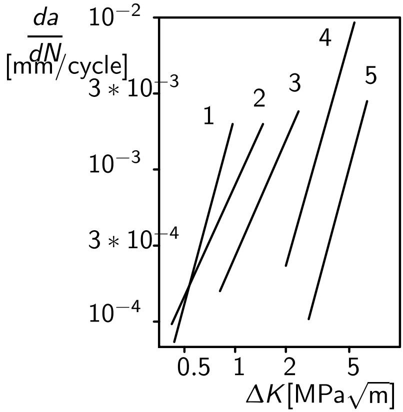
	AC	CZ	K_{Ic}	
PMMA	a	+	13.2	
PS	a	+	17.6	
PSF	a	-	low	
PC	a	-	high	main chain segmental motions $\rightarrow$ energy dissipation
Nylon 66	cr	-		main chain segmental motions $\rightarrow$ energy dissipation crystalline regions $\rightarrow$ crack retardation
PVF2	sc			
PET	sc			amorphous $\rightarrow$ strain induced crystallization at crack tip
CPLS		-		cross-linked $\rightarrow$ suppressed crazing
HIPS		+		μ-sized rubber spheres $\rightarrow$ enhanced crazing
ABS				blending

AC : a	=	amorphous
AC : c	=	crystalline
AC : sc	=	semicrystalline
AC : cr	=	crystalline regions
CZ	=	crazing
K_{Ic}	=	fracture toughness in $\mathrm{MPa}\sqrt{\mathrm{m}}$

Fatigue

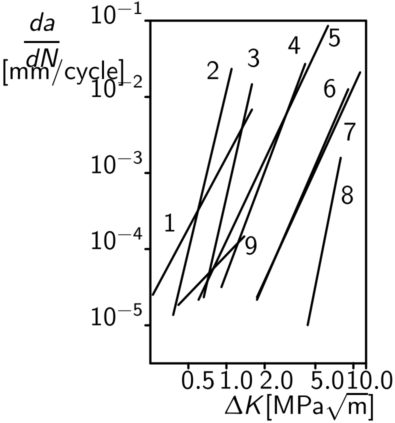
- amorphous $\leftrightarrow$ crystalline
- high $\leftrightarrow$ low molecular weight
- main chain motions
- toughening

FCP for polymers



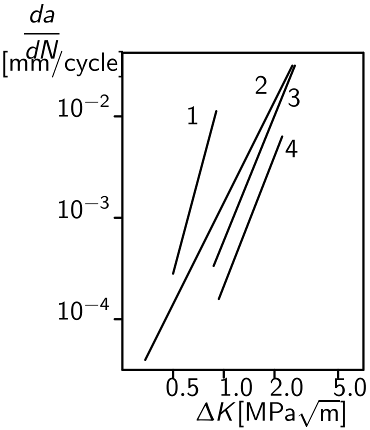
1	:	PMMA	5 Hz crazing
2	:	LDPE	1 Hz
3	:	ABS	10 Hz
4	:	PC	10 Hz no crazing
5	:	Nylon	10 Hz crystalline regions

FCP for polymers : crystalline versus amorphous



1	:	PS	5	:	PC
2	:	PMMA	6	:	Nylon 6.6
3	:	PSF	7	:	PVF2
4	:	PPO	8	:	PET
9	:	PVC			

FCP polymers : toughening



1	:	CLPS
2	:	PS
3	:	HIPS
4	:	ABS